# 目　　录

# 第 I 部分　引　论

在这一部分中，我们介绍若干常见的重要的随机变量弱相依性的定义，建立各种混合序列的协方差的界，并讨论各个不同定义间的关系，这些将给出于第一章中.

在第二章中，我们给出混合序列部分和的某些矩的估计，它在极限定理中扮演重要的角色，在许多定理的证明中常常是必不可少的关键所在.

# 第一章　定义和基本不等式

在本书中，我们总设$\{X_n,n\geqslant 1\}$是定义在概率空间$(\Omega,\mathscr{F},\mathscr{P})$上的随机变量序列．描述$\{X_n\}$的弱相依性或渐近独立性的方法很多．在§1.1 中，我们给出若干常见且重要的定义，在§1.2 中，建立若干关于$\{X_n\}$的协方差的不等式．它在$\{X_n\}$的极限性质的研究中是十分有用的．在这二节中，我们也讨论了各个定义间的关系．

## §1.1　定　义

设$\mathscr{A}$和$\mathscr{B}$是$\mathscr{F}$的两个子$\sigma$-域，记$L_p(\mathscr{A})$为所有$\mathscr{A}$可测且$p$阶矩有限的随机变量全体．定义

$$\alpha(\mathscr{A},\mathscr{B})=\sup_{A\in\mathscr{A},B\in\mathscr{B}}|P(AB)-P(A)P(B)|,$$

$$\rho(\mathscr{A},\mathscr{B})=\sup_{X\in L_2(\mathscr{A}),Y\in L_2(\mathscr{B})}\frac{|EXY-EXEY|}{\sqrt{\mathrm{Var}X\ \mathrm{Var}Y}},$$

$$\varphi(\mathscr{A},\mathscr{B})=\sup_{A\in\mathscr{A},B\in\mathscr{B},P(A)>0}|P(B|A)-P(B)|,$$

$$\psi(\mathscr{A},\mathscr{B})=\sup_{A\in\mathscr{A},B\in\mathscr{B},P(A)P(B)>0}\frac{|P(AB)-P(A)P(B)|}{P(A)P(B)},$$

$$\beta(\mathscr{A},\mathscr{B})=E(\mathrm{tvar}_{B\in\mathscr{B}}|P(B|\mathscr{A})-P(B)|),$$

$$\lambda(\mathscr{A},\mathscr{B})=\sup_{X\in L_{1/\alpha}(\mathscr{A}),Y\in L_{1/\beta}(\mathscr{B})}\frac{|EXY-EXEY|}{\|X\|_{1/\alpha}\|Y\|_{1/\beta}},$$

其中 tvar 是指全变差，$\|X\|_p=(E|X|^p)^{1/p}$．记$\mathscr{F}_a^b=\sigma(X_i,a\leqslant i\leqslant b)$，$\mathbb{Z}$是全体整数集，$\mathbb{Z}^+$是全体非负整数集，$\mathbb{N}$是全体正整数集．若干常见且重要的混合序列的定义如下．

**定义 1.1.1**　序列$\{X_n,n\geqslant 1\}$说是$\alpha$混合或强混合的，若

$$\alpha(n)=\sup_{k\in\mathbb{N}}\alpha(\mathscr{F}_1^k,\mathscr{F}_{k+n}^\infty)\to 0,n\to\infty.$$

**定义 1.1.2**　序列$\{X_n,n\geqslant 1\}$说是$\rho$混合的，若

$$\rho(n) = \sup_{k\in N} \rho(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) \to 0, n \to \infty.$$

**定义 1.1.3** 序列$\{X_n, n\geqslant 1\}$说是$\varphi$混合或一致强混合的，若

$$\varphi(n) = \sup_{k\in N} \varphi(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) \to 0, n \to \infty.$$

**定义 1.1.4** 序列$\{X_n, n\geqslant 1\}$说是$\psi$混合或$*$混合的，若

$$\psi(n) = \sup_{k\in N} \psi(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) \to 0, n \to \infty.$$

**定义 1.1.5** 序列$\{X_n, n\geqslant 1\}$说是绝对正则的，若

$$\beta(n) = \sup_{k\in N} \beta(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) \to 0, n \to \infty.$$

**定义 1.1.6** 设$0\leqslant\alpha,\beta\leqslant 1, \alpha+\beta=1$. 序列$\{X_n, n\geqslant 1\}$说是$(\alpha, \beta)$混合的，若

$$\lambda(n) = \sup_{k\in N} \lambda(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) \to 0, n \to \infty.$$

**注 1.1.1** 对具有参数集为$R^+$或$R$或$\mathbb{Z}$的序列，上述定义的修正是平凡的.

**注 1.1.2** $\alpha$混合的概念由 Rosenblatt(1956)所引入. $\rho$混合概念由 Kolmogorov 和 Rozanov(1960)所引入. Dobrushin(1956)首先对马氏过程引入了$\varphi$混合的定义. 对于平稳过程这一定义是由 Ibragimov(1959)及 Rozanov 和 Volconski(1959)分别陈述的(我们也可追溯到 Hirschfeld(1935)和 Gebelein(1941)). 绝对正则是由 Kolmogorov 提出的(参见 Rozanov 和 Volconski 1959). Blum, Hanson 和 Koopmans(1963)给出了$\psi$混合的概念. $(\alpha,\beta)$混合概念是由 Bradley(1985a)和邵启满(1989a)独立地给出的.

**注 1.1.3** Doob(1953)指出，一个 Döeblin 不可约马氏链是$\varphi$混合的且$\varphi(n)\leqslant ab^n$(某$a>0, 0\leqslant b<1$)；Rosenblatt(1971)证明一个纯非确定马氏链是$\alpha$混合的；Davydov(1973)给出一类$\beta$混合的马氏链.

**注 1.1.4** 为简单计，我们总设混合系数$\alpha(n), \rho(n), \cdots, \lambda(n)$等都是非增的.

显然地，由定义可见

$$\rho(n) = \lambda_{1/2,1/2}(n), \qquad \lambda_{1,0}(n) = \varphi(n) \leqslant \psi(n).$$

且进一步，若在$\rho$混合定义中取$X=I(A), Y=I(B)$，则有

$$\alpha(n) \leqslant \rho(n).$$

Kolmogorov 和 Rozanov(1960)对于 Gauss 序列研究了 $\alpha$ 混合和 $\rho$ 混合间的关系.

**定理 1.1.1** 对 Gauss 序列 $\{X_n, n\geqslant 1\}$ 我们有

$$\alpha(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) \leqslant \rho(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) \leqslant 2\pi\alpha(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty).$$

**证** 前一不等式是显然的.

对任给 $\varepsilon>0$,存在随机变量 $X\in L_2(\mathscr{F}_1^k)$, $Y\in L_2(\mathscr{F}_{k+n}^\infty)$使得 $EX=EY=0$, $\mathrm{Var}X= \mathrm{Var}Y=1$ 且

$$r := EXY \geqslant \rho(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) - \varepsilon.$$

注意到 $A:=\{X>0\}\in\mathscr{F}_1^k$, $B:=\{Y>0\}\in\mathscr{F}_{k+n}^\infty$,我们有

$$P(A)P(B) = 1/4,$$

$$P(AB) = \frac{1}{2\pi\sqrt{1-r^2}}\int_{-\infty}^0\int_{-\infty}^0 \cdot \exp\left\{-\frac{1}{2(1-r^2)}(x^2-2rxy+y^2)\right\}dxdy.$$

通过一个初等的计算(参见 Cramer 1946 p. 290),得

$$P(AB)=\frac{1}{4}+\frac{1}{2\pi}\arcsin r. \tag{1.1.1}$$

若 $\alpha(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty)>1/4$,显然地

$$2\pi\alpha(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) > \frac{\pi}{2} \geqslant \rho(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty);$$

若 $\alpha(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty)\leqslant 1/4$,由(1.1.1)我们得

$$\alpha(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) \geqslant P(AB) - P(A)P(B) = \frac{1}{2\pi}\arcsin r,$$

由此即得

$$\rho(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty) - \varepsilon \leqslant r \leqslant \sin 2\pi\alpha \leqslant 2\pi\alpha.$$

由 $\varepsilon$ 的任意性得证定理.

Kolmogorov 和 Rozanov(1960)也研究了一个弱平稳序列的谱函数与 $\rho$ 混合性之间的关系. 首先,我们给出一些记号及有关平稳序列的概念. 记 $\{X_n\}$ 的协方差函数为

$$R(n) = EX_mX_{m+n}.$$

由 Herglotz 定理，对 $R(n)$ 存在着谱表示如下：

$$R(n) = \int_{-\pi}^{\pi} e^{in\lambda} dF(\lambda),$$

其中 $F(\lambda)$ 称为平稳序列的谱函数. 当谱函数是绝对连续时，它的导数 $f(\lambda)=F'(\lambda)$ 称为平稳序列的谱密度.

**定理 1.1.2** 若平稳序列的谱函数不是绝对连续的，那么 $\rho(n)\equiv 1$，即序列不是 $\rho$ 混合的. 反之，若谱函数是绝对连续的，那么

$$\rho(n) = \inf_h \operatorname{ess}\sup_\lambda |f(\lambda) - e^{i\lambda n}h(e^{-i\lambda})|/f(\lambda),$$

其中 inf 是对在单位圆中解析连续的 $h$ 来取的；进一步若存在单位圆中解析函数 $h_0(z)$ 具有边界值 $h_0(e^{-i\lambda})$ 使得 $|f(\lambda)/h_0(e^{-i\lambda})|\geqslant\varepsilon>0$ 且 $(f(\lambda)/h_0(e^{-i\lambda}))^{(k)}$ 一致地有界，那么对某 $c>0$

$$\rho(n) \leqslant cn^{-k}.$$

特别地，当 $f(\lambda)$ 是 $e^{i\lambda}$ 的有理函数时，对某 $c>0$

$$\rho(n) = e^{-cn}.$$

定理 1.1.2 的证明不在此陈述(参见 Kolmogorov, Rozanov 1960).

## §1.2 基本不等式

设 $X$ 为 $\mathscr{F}^k_{-\infty}$ 可测，$Y$ 为 $\mathscr{F}^\infty_{k+n}$ 可测.

在本节中，对各种不同的混合序列我们来建立协方差 $\mathrm{Cov}(X,Y)=EXY-EXEY$ 的界，首先，我们考察 $\alpha$ 混合情形.

**引理 1.2.1** 设 $\{X_n, n\geqslant 1\}$ 是 $\alpha$ 混合序列，$X\in\mathscr{F}^k_{-\infty}$，$Y\in\mathscr{F}^\infty_{k+n}$ 且 $|X|\leqslant C_1$，$|Y|\leqslant C_2$. 那么

$$|EXY-EXEY|\leqslant 4C_1C_2\alpha(n). \tag{1.2.1}$$

**证** 由条件期望的性质，我们有

$$|EXY - EXEY| = |E\{X(E(Y|\mathscr{F}^k_{-\infty}) - EY)\}|$$

$$\leqslant C_1E|E(Y|\mathscr{F}^k_{-\infty}) - EY| = C_1|E\xi\{E(Y|\mathscr{F}^k_{-\infty}) - EY\}|,$$

其中 $\xi=\mathrm{sgn}(E(Y|\mathscr{F}^k_{-\infty})-EY)\in\mathscr{F}^k_{-\infty}$，即

$$|EXY - EXEY| \leqslant C_1 |E\xi Y - E\xi EY|.$$

同样讨论可得

$$|E\xi Y - E\xi EY| \leqslant C_2 |E\xi\eta - E\xi E\eta|,$$

其中 $\eta = \mathrm{sgn}(E(\xi | \mathscr{F}_{k+n}^{\infty}) - E\xi)$. 所以

(1.2.2) $$|EXY - EXEY| \leqslant C_1 C_2 |E\xi\eta - E\xi E\eta|.$$

令 $A = \{\xi = 1\}, B = \{\eta = 1\}$. 显然地 $A \in \mathscr{F}_{-\infty}^{k}, B \in \mathscr{F}_{k+n}^{\infty}$. 利用 $\alpha$ 混合的定义,我们得

$$\begin{aligned} &|E\xi\eta - E\xi E\eta| \\ &= |P(AB) + P(A^c B^c) - P(AB^c) - P(A^c B) \\ &\quad - (P(A) - P(A^c))(P(B) - P(B^c))| \leqslant 4\alpha(n). \end{aligned}$$

代入(1.2.2)得(1.2.1).

**引理 1.2.2** 设$\{X_n, n \in \mathbb{Z}\}$是 $\alpha$ 混合序列,$X \in \mathscr{F}_{-\infty}^{k}, Y \in \mathscr{F}_{k+n}^{\infty}$且对某 $p > 1, E|X|^p < \infty, |Y| \leqslant C$. 那么

(1.2.3) $$|EXY - EXEY| \leqslant 6C \| X \|_p (\alpha(n))^{1/q},$$

其中 $1/p + 1/q = 1$.

**证** 设 $X_N = XI(|X| \leqslant N), X'_N = X - X_N$. 写

$$|EXY - EXEY| \leqslant |EX_N Y - EX_N EY| + |EX'_N Y - EX'_N EY|.$$

由引理 1.2.1, $|EX_N Y - EX_N EY| \leqslant 4CN\alpha(n)$. 对上式右边第二项,我们有

$$|EX'_N Y - EX'_N EY| \leqslant 2CE|X'_N| \leqslant 2CN^{-p+1} E|X|^p.$$

取 $N = \| X \|_p (\alpha(n))^{-1/p}$就得(1.2.3).

对随机变量 $X$ 和 $R^+$上连续不减函数 $f(x), f(0) = 0$,但不恒等于 0,定义

$$\| X \|_f = \inf\{t > 0, Ef(|X|/t) \leqslant 1\}.$$

由此定义,容易知道

(1.2.4) $$\| X \|_f = 0 \Leftrightarrow X = 0 \quad \text{a.s.}$$

且当 $0 < \| X \|_f < \infty$时,有 $Ef(|X| / \| X \|_f) \leqslant 1$;此外,若$|X_1| \leqslant |X_2|$ a.s.,那么$\| X_1 \|_f \leqslant \| X_2 \|_f$.

**引理 1.2.3** 设$\{X_n, n \in \mathbb{Z}\}$是 $\alpha$ 混合序列,$X \in \mathscr{F}_{-\infty}^{k}, Y \in \mathscr{F}_{k+n}^{\infty}$, $f(x)$和 $g(x)$是 $R^+$上两个连续函数,$f(0) = g(0) = 0$,且对

某 $r>0, s>0, f(x)/x^{\frac{r+s}{r}}\nearrow\infty, g(x)/x^{\frac{r+s}{s}}\nearrow\infty, \|X\|_f<\infty, \|Y\|_g<\infty$. 那么

(1.2.5)
$$|EXY - EXEY| \leqslant 10 \text{ inv } f\left(\frac{1}{\alpha(n)}\right) \cdot \text{inv } g\left(\frac{1}{\alpha(n)}\right)\alpha(n) \|X\|_f \|Y\|_g.$$

**证** 从引理的条件容易看出 $E|X|^{1+s/r}<\infty$ 且 $E|Y|^{1+r/s}<\infty$. 若 $\|X\|_f=0$ 或 $\|Y\|_g=0$, 从(1.2.4)即得(1.2.5)成立. 若 $\alpha(n)=0$, 由 $X$ 和 $Y$ 的独立性得(1.2.5)是平凡的. 现在我们假设 $\|X\|_f>0$, $\|Y\|_g>0$ 和 $\alpha(n)>0$. 存在 $M>0$ 和 $N>0$ 使得

$$\alpha(n) = 1/f(M/\|X\|_f) = 1/g(N/\|Y\|_g).$$

设

$$X_M = XI(|X| \leqslant M), X'_M = X - X_M,$$
$$Y_N = YI(|Y| \leqslant N), Y'_N = Y - Y_N.$$

我们有

(1.2.6)
$$\begin{aligned}|EXY - EXEY| &\leqslant |EX_MY_N - EX_MEY_N| \\ &+ |EX'_MY_N - EX'_MEY_N| \\ &+ |EX_MY'_N - EX_MEY'_N| \\ &+ |EX'_MY'_N - EX'_MEY'_N| \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

由引理 1.2.1, $I_1 \leqslant 4MN\alpha(n)$. 注意到 $f(x)/x\nearrow\infty, g(x)/x\nearrow\infty$, 我们有

$$\begin{aligned}E|X'_M| &= E(|X'_M|/\|X'_M\|_f)\|X'_M\|_f \\ &\leqslant Ef(|X'_M|/\|X'_M\|_f)M/f(M/\|X'_M\|_f) \\ &\leqslant M/f(M/\|X\|_f).\end{aligned}$$

所以

$$I_2 \leqslant 2MN/f(m/\|X\|_f) = 2 \text{ inv } f\left(\frac{1}{\alpha(n)}\right) \cdot \text{inv } g\left(\frac{1}{\alpha(n)}\right)\alpha(n)\|X\|_f\|Y\|_g.$$

类似地，对 $I_3$ 有同样估计.

进一步，注意到 $f(x)/x^{\frac{r+s}{r}}\nearrow\infty$ 和 $g(x)/x^{\frac{r+s}{s}}\nearrow\infty$，我们有

$$
\begin{aligned}
EX'_MY'_N &\leqslant \left(E(|X'_M|/\|X'_M\|_f)^{\frac{r+s}{r}}\right)^{\frac{r}{r+s}} \\
&\quad\cdot\left(E(|Y'_N|/\|Y'_N\|_g)^{\frac{r+s}{s}}\right)^{\frac{s}{r+s}}\|X'_M\|_f\|Y'_N\|_g \\
&\leqslant (Ef(|X'_M|/\|X'_M\|_f))^{\frac{r}{r+s}}(Eg(|Y'_N|/\|Y'_N\|_g))^{\frac{s}{r+s}} \\
&\quad\cdot MN/(f(M/\|X'_M\|_f))^{\frac{r}{r+s}}(g(N/\|Y'_N\|_g))^{\frac{s}{r+s}} \\
&\leqslant MN/(f(M/\|X\|_f))^{\frac{r}{r+s}}(g(N/\|Y\|_g))^{\frac{s}{r+s}}
\end{aligned}
$$

因此

$$
\begin{aligned}
I_4 &\leqslant 2MN/(f(M/\|X\|_f))^{\frac{r}{r+s}}(g(N/\|Y\|_g))^{\frac{s}{r+s}} \\
&\leqslant 2\ \mathrm{inv}\ f\left(\frac{1}{\alpha(n)}\right)\mathrm{inv}\ g\left(\frac{1}{\alpha(n)}\right)\alpha(n)\|X\|_f\|Y\|_g.
\end{aligned}
$$

现在把这些估计代入(1.2.6)即得(1.2.5).

作为这一引理的推论，我们有

**引理 1.2.4** 设 $\{X_n, n\in\mathbb{Z}\}$ 是 $\alpha$ 混合序列，$X\in\mathscr{F}_{-\infty}^{k}$，$Y\in\mathscr{F}_{k+n}^{\infty}$，且 $E|X|^p<\infty$，$E|Y|^q<\infty$，$1/p+1/q<1$. 那么

(1.2.7) $\quad |EXY-EXEY|\leqslant 10\|X\|_p\|Y\|_q(\alpha(n))^{1-\frac{1}{p}-\frac{1}{q}}$.

**引理 1.2.5** 设 $\{X_n, n\in\mathbb{Z}\}$ 是 $\alpha$ 混合序列，$X\in\mathscr{F}_{-\infty}^{k}$，$Y\in\mathscr{F}_{k+n}^{\infty}$，且 $E|X|^{2+\delta}<C_1$，$E|Y|^{2+\delta}\leqslant C_2$. 那么

(1.2.8) $\quad |EXY-EXEY|\leqslant 10(C_1C_2)^{\frac{1}{2+\delta}}(\alpha(n))^{\frac{\delta}{2+\delta}}$.

对 $(\alpha,\beta)$ 混合序列和 $\rho$ 混合序列，我们有下列引理.

**引理 1.2.6** 设 $\{X_n, n\in\mathbb{Z}\}$ 是 $(\alpha,\beta)$ 混合序列，$X\in L_p(\mathscr{F}_{-\infty}^{k})$，$Y\in L_q(\mathscr{F}_{k+n}^{\infty})$，$p,q\geqslant 1$ 且 $1/p+1/q=1$. 那么

(1.2.9) $\quad |EXY-EXEY|\leqslant 4\lambda(n)^{\frac{1}{\alpha p}\wedge\frac{1}{\beta q}}\|X\|_p\|Y\|_q$.

**证** 不失一般性假设 $\alpha p\geqslant 1$，它蕴含着 $\beta q\leqslant 1$. 令

$$Y_1 = YI(|Y|\leqslant C),\ Y_2 = Y - Y_1,$$

其中 $C$ 是正的常数，在下面确定. 写

(1.2.10) $\quad |EXY - EXEY|\leqslant |EXY_1 - EXEY_1| + |EXY_2 - EXEY_2|$.

由$(\alpha,\beta)$混合的定义和Hölder不等式

$$|EXY_1-EXEY_1|\leqslant\lambda(n)\|X\|_{1/\alpha}\|Y\|_{1/\beta}$$

$$\leqslant\lambda(n)C^{1-\beta q}\|X\|_p\|Y\|_q^{\beta q},$$

$$|EXY_2|\leqslant(E|Y_2|^q)^{1-\frac{1}{\alpha p}}(E|X|^{\alpha p}|Y_2|^{\beta q})^{\frac{1}{\alpha p}}$$

$$\leqslant(E|Y_2|^q)^{1-\frac{1}{\alpha p}}(E|X|^{\alpha p}E|Y_2|^{\beta q}$$

$$+\lambda(n)(E|X|^p)^\alpha(|Y_2|^q)^\beta)^{\frac{1}{\alpha p}}$$

$$\leqslant(E|Y|^q)^{1-\frac{1}{\alpha p}}(E|X|^{\alpha p}E|Y|^qC^{-\alpha q}$$

$$+\lambda(n)(E|X|^p)^\alpha(E|Y|^q)^\beta)^{\frac{1}{\alpha p}}$$

$$\leqslant\|X\|_p\|Y\|_q^qC^{-\frac{q}{p}}+\lambda^{\frac{1}{\alpha p}}(n)\|X\|_p\|Y\|_q$$

且

$$|EXEY_2|\leqslant\|X\|_p\|Y\|_q^qC^{-q/p}$$

把这些估计代入(1.2.10)并取$C=\|Y\|_q(\lambda(n))^{-1/\alpha q}$我们得(1.2.9).

令(1.2.9)中$p=q=2$.容易看到

(1.2.11) $$\rho(n)\leqslant4\lambda(n)^{\frac{1}{2\alpha}\wedge\frac{1}{2\beta}}.$$

作为引理1.2.6的一个推论,注意到$\rho(n)=\lambda_{1/2,1/2}(n)$,我们有

**引理1.2.7** 设$\{X_n,n\in\mathbb{Z}\}$是$\rho$混合序列,$X\in L_p(\mathscr{F}_{-\infty}^k)$且$Y\in L_q(\mathscr{F}_{k+n}^\infty)$,$p,q\geqslant1$,$1/p+1/q=1$.那么

$$|EXY-EXEY|\leqslant4\rho(n)^{\frac{2}{p}\wedge\frac{2}{q}}\|X\|_p\|Y\|_q.$$

对于$\varphi$混合情形,我们有如下三个结果.

**引理1.2.8** 设$\{X_n,n\in\mathbb{Z}\}$是$\varphi$混合序列,$X\in L_p(\mathscr{F}_{-\infty}^k)$且$Y\in L_q(\mathscr{F}_{k+n}^\infty)$,$p,q\geqslant1$,$1/p+1/q=1$.那么

(1.2.12) $$|EXY-EXEY|\leqslant2(\varphi(n))^{\frac{1}{p}}\|X\|_p\|Y\|_q.$$

**证** 首先,我们假设$X$和$Y$是简单函数,即

$$X=\sum_i a_iI_{A_i},Y=\sum_j b_jI_{B_j},$$

其中$\sum_i$和$\sum_j$是有限和且$A_i\cap A_k=\phi(i\neq k)$,$B_j\cap B_l=\phi(j\neq l)$,

$A_i \in \mathscr{F}_{-\infty}^{k}, B_j \in \mathscr{F}_{k+n}^{\infty}$. 所以

$$EXY - EXEY = \sum_{i,j} a_i b_j P(A_i B_j) - \sum_{i,j} a_i b_j P(A_i) P(B_j).$$

由 Hölder 不等式我们有

$$(1.2.13) \qquad \begin{aligned} |EXY - EXEY| &= \Big| \sum_i a_i (P(A_i))^{1/p} \\ &\quad \cdot \sum_j (P(B_j | A_i) - P(B_j)) b_j (P(A_i))^{1/q} \Big| \\ &\leqslant \Big( \sum_i |a_i|^p P(A_i) \Big)^{1/p} \Big( \sum_i P(A_i) \Big| \\ &\quad \cdot \sum_j b_j (P(B_j | A_i) - P(B_j)) \Big|^q \Big)^{1/q} \\ &\leqslant \| X \|_p \Big| \sum_i P(A_i) \Big( \sum_j |b_j|^q (P(B_j | A_i) \\ &\quad + P(B_j)) \Big) \Big( \sum_j | P(B_j | A_i) - P(B_j) | \Big)^{\frac{q}{p}} \Big|^{\frac{1}{2}} \\ &\leqslant 2^{1/q} \| X \|_p \| Y \|_q \max_i \Big( \sum_j | P(B_j | A_i) \\ &\quad - P(B_j) | \Big)^{1/p}. \end{aligned}$$

注意到

$$(1.2.14) \qquad \begin{aligned} & \sum_j | P(B_j | A_i) - P(B_j) | \\ &= (P(\cup_j^+ B_j | A_i) - P(\cup_j^+ B_j)) \\ &\quad - (P(\cup_j^- B_j | A_i) - P(\cup_j^- B_j)) \\ &\leqslant 2\varphi(n), \end{aligned}$$

其中$\cup_j^+$($\cup_j^-$)是对$P(B_j|A_i)-P(B_j)>0$($P(B_j|A_i)-P(B_j)<0$)的所有 $j$ 上求并. 把(1.2.14)代入(1.2.13),对简单函数得证(1.2.12).

为完成引理的证明,设

$$X_N = \begin{cases} 0 & \text{if } |X| > N, \\ k/N & \text{if } k/N < X \leqslant (k+1)/N, |X| \leqslant N; \end{cases}$$

$$Y_N = \begin{cases} 0 & \text{if } |Y| > N, \\ k/N & \text{if } k/N < Y \leqslant (k+1)/N, |Y| \leqslant N. \end{cases}$$

我们已证对 $X_N$ 和 $Y_N$(1.2.12)成立. 此外,注意到

$$E|X - X_N|^p \to 0, E|Y - Y_N|^q \to 0, N \to \infty.$$

让 $N \to \infty$,对一般情形得(1.2.12)成立.

设(1.2.12)中 $p=q=2$. 容易看到

(1.2.15) $$\rho(n) \leqslant 2\varphi^{1/2}(n).$$

从引理 1.2.8 的证明,我们可得到

**引理 1.2.9** 设$\{X_n, n \in \mathbb{Z}\}$是 $\varphi$ 混合序列, $X \in \mathscr{F}_{-\infty}^k$ 且 $Y \in \mathscr{F}_{k+n}^{\infty}$, $|X| \leqslant C_1$, $|Y| \leqslant C_2$. 那么

(1.2.16) $$|EXY - EXEY| \leqslant 2C_1C_2\varphi(n).$$

令(1.2.12)中 $p=1$ 且 $q=\infty$. 从引理 1.2.8,我们又有

**引理 1.2.10** 设$\{X_n, n \in \mathbb{Z}\}$是 $\varphi$ 混合序列, $X \in \mathscr{F}_{-\infty}^k$ 且 $Y \in \mathscr{F}_{k+n}^{\infty}$, $E|X| < \infty$, $|Y| \leqslant C$. 那么

(1.2.17) $$|EXY - EXEY| \leqslant 2C\varphi(n)E|X|.$$

最后,我们考察 $\psi$ 混合情形.

**引理 1.2.11** 设$\{X_n, n \in \mathbb{Z}\}$是 $\psi$ 混合序列, $X \in \mathscr{F}_{-\infty}^k$ 且 $Y \in \mathscr{F}_{k+n}^{\infty}$, $E|X| < \infty$, $E|Y| < \infty$. 那么 $E|XY| < \infty$ 且

(1.2.18) $$|EXY - EXEY| \leqslant \psi(n)E|X|E|Y|.$$

**证** 首先,假设 $X$ 和 $Y$ 是非负简单函数. 我们有

$$\begin{aligned} |EXY - EXEY| &= \Big|\sum_{i,j} a_i b_j (P(A_iB_j) - P(A_i)P(B_j))\Big| \\ &\leqslant \sum_{i,j} a_i b_j \psi(n) P(A_i)P(B_j) \\ &= \psi(n) EXEY. \end{aligned}$$

由此,(1.2.18)对非负随机变量 $X$ 和 $Y$ 成立.

对一般情形,写 $X = X^+ - X^-$, $Y = Y^+ - Y^-$. 我们有

$$\begin{aligned} |EXY - EXEY| \leqslant &\ |EX^+Y^+ - EX^+EY^+| \\ &+ |EX^+Y^- - EX^+EX^-| + |EX^-Y^+ \\ &- EX^-EY^+| + |EX^-Y^- - EX^-EY^-| \end{aligned}$$

$$\leqslant \psi(n)(EX^{+} + EX^{-})(EY^{+} + EY^{-})$$
$$\leqslant \psi(n)E|X|E|Y|.$$

最后，我们综合诸混合性质间关系. 容易验证

$$2\alpha(n)\leqslant\beta(n)\leqslant\varphi(n). \tag{1.2.19}$$

从马氏过程为 $\psi$ 混合的充要条件，我们可指出一个 $\varphi$ 混合(Markov)序列不是 $\psi$ 混合的例(见 Blum，Hanson 和 Koopmans 1963). Ibragimov 和 Solv(1969)给出一个平稳 $\alpha$ 混合 Causs 过程但不是 $\beta$ 混合的例子；这一过程也是 $\rho$ 混合而非 $\beta$ 混合的例子. Davydov(1973)构造了一个平稳 $\alpha$ 混合 Markov 过程，它的混合系数以几何速度趋向零而它不是 $\rho$ 混合的. 一个几何地遍历 Markov 过程，它不是 Doeblin 常返的，它是 $\beta$ 混合而不是 $\varphi$ 混合的(见 Andrews 1984). 综合这些结果，并回顾注 1.1.4，(1.2.11)和(1.2.15)我们有

$$\psi\text{混合} \underset{\not\Leftarrow}{\Rightarrow} \varphi\text{混合} \underset{\not\Leftarrow}{\Rightarrow} \begin{cases} \beta\text{混合} \underset{\not\Leftarrow}{\Rightarrow} \alpha\text{混合} \\ \not\Downarrow \;\; \not\Uparrow \\ \rho\text{混合} \underset{\not\Leftarrow}{\Rightarrow} \alpha\text{混合} \end{cases}$$
$$\Uparrow$$
$$\lambda\text{混合}$$

# 第二章　部分和的矩估计

混合随机变量序列的部分和的矩的各种估计在极限理论中扮演了重要角色. 在 §2.1 中，我们给出了各种混合序列部分和的方差的若干表示形式，§2.2 专门导出部分和的矩的若干不等式，顺便也在该节中给出了某些概率不等式.

## §2.1　部分和的方差

设 $\{X_n, n \geq 1\}$ 是(弱)平稳序列，$EX_1 = 0$，$EX_1^2 < \infty$. 记 $S_n = \sum_{i=1}^{n} X_i$. 我们研究它的方差 $\mathrm{Var}\, S_n$. 设平稳序列 $\{X_n\}$ 的相关函数为 $R(n)$，谱函数为 $F(\lambda)$.

首先，我们通过 $R(n)$ 或 $F(\lambda)$ 来给出 $\mathrm{Var}\, S_n$ 的表示式.

**定理 2.1.1**

$$\mathrm{Var}\, S_n = \sum_{|j|<n} (n - |j|) R(j), \tag{2.1.1}$$

$$\mathrm{Var}\, S_n = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{n\lambda}{2}}{\sin^2 \frac{\lambda}{2}} dF(\lambda). \tag{2.1.2}$$

若谱函数 $F(\lambda)$ 是绝对连续的，即存在谱密度 $f(\lambda)$，且进一步 $f(\lambda)$ 在 $\lambda=0$ 连续，那么

$$\mathrm{Var}\, S_n = 2\pi f(0) n + o(n) \qquad n \to \infty. \tag{2.1.3}$$

**证**　由相关函数的定义

$$\mathrm{Var}\, S_n = \sum_{i,k=1}^{n} EX_i X_k = \sum_{i,k=1}^{n} R(i-k) = \sum_{|j|<n} (n - |j|) R(j).$$

又由相关函数的谱分解得

$$\mathrm{Var}\ S_n = \int_{-\pi}^{\pi} \sum_{|j|<n} (n - |j|) e^{ij\lambda} dF(\lambda),$$

从初等计算可知

$$\sum_{|j|<n} (n - |j|) e^{ij\lambda} = \sin^2 \frac{n\lambda}{2} \Big/ \sin^2 \frac{\lambda}{2},$$

代入得(2.1.1)和(2.1.2).

若 $f(\lambda)$ 存在且在 $\lambda=0$ 连续. 注意到

$$\int_{-\pi}^{\pi} \left( \sin^2 \frac{n\lambda}{2} \Big/ \sin^2 \frac{\lambda}{2} \right) d\lambda = 2n\pi,$$

我们有

$$\begin{aligned}
\mathrm{Var}\ S_n - 2\pi f(0) n &= \int_{-\pi}^{\pi} \left( \sin^2 \frac{n\lambda}{2} \Big/ \sin^2 \frac{\lambda}{2} \right) (f(\lambda) - f(0)) d\lambda \\
&\leqslant \max_{|\lambda| \leqslant n^{-\frac{1}{4}}} |f(\lambda) - f(0)| \int_{-n^{-\frac{1}{4}}}^{n^{-\frac{1}{4}}} \frac{\sin^2 \frac{n\lambda}{2}}{\sin^2 \frac{\lambda}{2}} d\lambda \\
&\quad + \frac{1}{\sin^2 (n^{-\frac{1}{4}}/2)} \int_{n^{-\frac{1}{4}} \leqslant |\lambda| \leqslant \pi} \sin^2 \frac{n\lambda}{2} (f(\lambda) - f(0)) d\lambda \\
&\leqslant 2\pi n \max_{|\lambda| \leqslant n^{-\frac{1}{4}}} |f(\lambda) - f(0)| + 0(n^{1/2}) = o(n).
\end{aligned}$$

得(2.1.3)成立.

当平稳序列满足某种混合条件时，方差 $\mathrm{Var}\ S_n$ 具有显明形式.

**定理 2.1.2** 设平稳序列 $\{X_n\}$ 是 $\varphi$-混合的且当 $n\to\infty$ 时 $\mathrm{Var}\ S_n\to\infty$. 那么

$$\mathrm{Var}\ S_n = nh(n), \tag{2.1.4}$$

其中 $h(n)$ 是 $n$ 的缓变函数且它的定义域可延拓于 $R$ 使得 $h(x)$ 也是 $R$ 上缓变函数.

**证** 首先，我们来证 $h(n)$ 是缓变的. 记 $\sigma_n^2=\mathrm{Var}\ S_n$. 等价地，我们来证对每一正整数 $k$

$$\lim_{n\to\infty} \sigma_{kn}^2/\sigma_n^2 = k. \tag{2.1.5}$$

设

$$\xi_j = \sum_{s=1}^{n} X_{(j-1)n+(j-1)r+s}, \qquad j = 1,2,\cdots,k;$$

$$\eta_j = \sum_{s=1}^{r} X_{jn+(j-1)r+s}, \qquad j = 1,2\cdots,k-1;$$

$$\eta_k = -\sum_{s=1}^{(k-1)r} X_{nk+s},$$

其中 $r=[\log\sigma_n^2]$. 由定理 1.1.2，$\{X_n\}$ 具有谱密度 $f(\lambda)$. 利用(2.1.2)我们得

(2.1.6) $$\sigma_n^2 = \mathrm{Var}\ S_n = \int_{-\pi}^{\pi}\left(\sin^2\frac{n\lambda}{2}\right)\Big/\left(\sin^2\frac{\lambda}{2}\right)f(\lambda)d\lambda$$

$$\leqslant n^2\int_{-\pi}^{\pi} f(\lambda)d\lambda.$$

因此 $r=[\log\sigma_n^2]=O(\log n)$. 且进一步

(2.1.7)

$$\sigma_{kn}^2 = \mathrm{Var}\ S_{kn} = \sum_{j=1}^{K} E\xi_j^2 + 2\sum_{i\neq j} E\xi_i\xi_j + \sum_{i,j} E\xi_i\eta_j + \sum_{i,j} E\eta_i\eta_j.$$

由序列的平稳性，$E\xi_j^2=\sigma_n^2=\mathrm{Var}\ S_n$. 从引理 1.2.8，对 $i\neq j$ 我们有

(2.1.8) $$|E\xi_i\xi_j| \leqslant 2\varphi(|i-j|r)^{1/2}\|\xi_i\|_2\|\xi_j\|_2 \leqslant 2\varphi(r)^{1/2}\sigma_n^2.$$

利用 Schwarz 不等式和(1.2.6)，我们有

(2.1.9) $$|E\xi_i\eta_j| \leqslant \|\xi_i\|_2\|\eta_j\|_2 = \sigma_n\sigma_r = O(\sigma_n\log\sigma_n),$$

(2.1.10) $$|E\eta_i\eta_j| \leqslant \sigma_r^2 = O((\log\sigma_n)^2).$$

把(2.1.8)，(2.1.9)和(2.1.10)代入(2.1.7)中并注意到当 $n\to\infty$ 时 $\varphi(r)=o(1)$，我们得

$$\sigma_{kn}^2 = k\sigma_n^2 + o(\sigma_n^2),$$

由此即得(2.1.5).

其次，我们来证 $h(n)$ 的定义域可延拓于 $R$ 使得 $h(x)$ 也是 $R$ 上缓变函数，回顾(2.1.6)，我们定义

$$\phi(x) = \int_{-\pi}^{\pi}\left(\sin^2\frac{x\lambda}{2}\right)\Big/\left(\sin^2\frac{\lambda}{2}\right)f(\lambda)d\lambda,$$

$$h(x) = \phi(x)/x.$$

为证明 $h(x)$ 是缓变的，只需验证对任何 $a>0$

(2.1.11) $$\lim_{x\to\infty}\psi(ax)/\psi(x)=a.$$

不难从 $\psi$ 的定义知道当 $x\to\infty$ 时

$$\psi(x)=\psi([x])(1+o(1)).$$

当(2.1.11)式中的 $a$ 为整数时,我们有

$$\frac{\psi(ax)}{\psi(x)}=\frac{[ax]h([ax])}{[x]h([x])}(1+o(1))=a(1+o(1)).$$

所以,对 $a=p/q$,其中 $p$ 和 $q$ 是整数,我们有

$$\lim_{x\to\infty}\frac{\psi(ax)}{\psi(x)}=\lim_{x\to\infty}\frac{\psi\left(p\dfrac{x}{q}\right)}{\psi\left(\dfrac{x}{q}\right)}\frac{\psi\left(\dfrac{x}{q}\right)}{\psi\left(q\dfrac{x}{q}\right)}=\frac{p}{q}=a.$$

对任一正实数 $a$,令

$$\psi_1(a)=\varliminf_{x\to\infty}\frac{\psi(ax)}{\psi(x)},\psi_2(a)=\varlimsup_{x\to\infty}\frac{\psi(ax)}{\psi(x)}.$$

对任何有理数 $a$,由上面的证明知 $\psi_1(a)=\psi_2(a)$. 因此,只需证明 $\psi_1(x)$和 $\psi_2(x)$都是连续的. 因为

$$\left|\frac{\psi((a+\varepsilon)x)-\psi(ax)}{\psi(x)}\right|$$

$$\leqslant\frac{1}{\psi(x)}\left|\int_{-\pi}^{\pi}\frac{\sin^2\dfrac{\varepsilon x\lambda}{2}}{\sin^2\dfrac{\lambda}{2}}f(\lambda)d\lambda+\int_{-\pi}^{\pi}\frac{\sin\varepsilon x\lambda\sin ax\lambda}{2\sin^2\dfrac{\lambda}{2}}f(\lambda)d\lambda\right|$$

$$\leqslant\frac{\psi(\varepsilon x)}{\psi(x)}+\left(\frac{\psi(\varepsilon x)}{\psi(x)}\right)^{\frac{1}{2}}.$$

只需证明 $\psi_1(a)$和 $\psi_2(a)$在 $a=0$ 连续. 利用关于缓变函数的性质 A4(见附录),对充分小的 $\varepsilon>0$,当 $x\to0$ 时,我们有

$$\frac{\psi(\varepsilon x)}{\psi(x)}=\frac{[\varepsilon x]h\left(\dfrac{[\varepsilon x]}{[x]}[x]\right)}{[x]h([x])}(1+o(1))$$

$$\leqslant\varepsilon^{1/2}(1+o(1)),$$

因此 $\psi_1(a)$和 $\psi_2(a)$在 $a=0$ 都是连续的.

定理 2.1.2 得证.

**注 2.1.1** 在定理 2.1.2 的证明中,$\varphi$-混合性仅被应用于给出

不等式

$$E\left|S_n\sum_{j=n+p}^{m+n+p}X_j\right|\leqslant 2\varphi(p)^{1/2}\|S_n\|_2\|S_m\|_2.$$

这样对 $\alpha$-混合情形，我们也有

**定理 2.1.3** 设$\{X_n,n\geqslant1\}$是强平稳 $\alpha$-混合序列，满足 $EX_1=0$，$EX_1^2<\infty$，$\sigma_n^2=ES_n^2\to\infty$ 且$\{S_n^2/\sigma_n^2,n\geqslant1\}$一致可积. 那么定理 2.1.2 的结论成立.

**证** 由定理 2.1.2 的证明和注 2.1.1 只需证明下列事实.

1. $\sigma_n^2\to\infty$;

2. 对任一 $\varepsilon>0$，存在 $p=p(\varepsilon)$，$N=N(\varepsilon)$，使得

$$\left|ES_n\sum_{j=n+p}^{m+n+p}X_j\right|\leqslant\varepsilon\sigma_n\sigma_m\qquad\text{当 } n,m\geqslant N(\varepsilon)\text{ 时}.$$

第一个事实是定理的一个假设. 考察后者. 从$\{S_n^2/\sigma_n^2\}$的一致可积性，对任给 $\varepsilon>0$，存在 $K>0$ 使对充分大的 $p$

$$\int_{S_n^2/\sigma_n^2\geqslant K}S_n^2/\sigma_n^2dP<\frac{\varepsilon}{4},K\alpha(p)<\varepsilon/16.$$

那么，由引理 1.2.1，Schwarz 不等式和强平稳性，我们得

$$\left|ES_n\sum_{j=n+p}^{m+n+p}X_j\right|/\sigma_n\sigma_m$$

$$\leqslant\int_{\left|\frac{S_n}{\sigma_n}\right|<\sqrt{K},\left|\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right|<\sqrt{K}}\left|\frac{S_n}{\sigma_n}\cdot\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right|dP$$

$$+\int_{\left|\frac{S_n}{\sigma_n}\right|\geqslant\sqrt{K}}\sqrt{K}\frac{S_n}{\sigma_n}dP+\int_{\left|\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right|\geqslant\sqrt{K}}\sqrt{K}$$

$$\cdot\left|\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right|dP$$

$$+\int_{\left|\frac{S_n}{\sigma_n}\right|\geqslant\sqrt{K},\left|\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right|\geqslant\sqrt{K}}\left|\frac{S_n}{\sigma_n}\cdot\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right|dP$$

$$\leqslant 4K\alpha(p)+\varepsilon/4+\varepsilon/4+\varepsilon/4\leqslant\varepsilon.$$

对 $\rho$-混合序列，Peligrad(1982)证明着下述一般结果. 记 $S_k(n)=S_{k+n}-S_k=\sum_{j=k+1}^{k+n}X_j$.

**定理 2.1.4** 设$\{X_n,n\geqslant1\}$是 $\rho$ 混合序列，$EX_n=0$，满足下述

条件：

（i） $\sup\limits_n EX_n^2=\sigma_0^2<\infty$,

（ii） 当 $n\to\infty$ 时 $ES_n^2\to\infty$,

（iii） $\lim\limits_{n\to\infty}\frac{ES_k^2(n)}{ES_n^2}=1$ 关于 $k$ 一致地成立.

那么

$$ES_n^2 = nh(n),$$

其中 $h(n)$ 是缓变函数且它的定义域可延拓于 $R$ 使得 $h(x)$ 也是 $R$ 上的缓变函数. 若附设

(iv) $\sum\limits_{n=1}^{\infty}\rho(2^n)<\infty$,

那么 $ES_n^2/n\to\sigma^2>0$.

为证明定理 2.1.4，我们需要下述引理.

**引理 2.1.1** 设 $\{X_n,n\geqslant 1\}$ 是 $\rho$-混合序列，$EX_1=0$. 若定理 2.1.4 的条件(i)被满足，那么对于自然数 $p,q,m$ 当 $p+q=m$ 时有

$$(2.1.12)\quad \begin{aligned}&(1-\rho(n))(ES_{km}^2(p)+ES_{km+p}^2(q))-C_1\leqslant ES_{km}^2(m)\\&\leqslant(1+\rho(n))(ES_{km}^2(p)+ES_{km+p}^2(q))+C_1,\end{aligned}$$

其中 $k$ 和 $n$ 是正整数且

$$\begin{aligned}C_1=C_1(m,p,n)\leqslant 20\sigma_0^2n^2+12\sigma_0 n(\|S_{km}(p)\|_2\\+\|S_{km+p}(q)\|_2),\end{aligned}$$

进一步有

$$(2.1.13)\quad (1-\rho(n))^{1/2}\|S_{km}(p)\|_2\leqslant\|S_{km}(m)\|_2+C_2,$$

其中 $C_2\leqslant 2\sigma_0 n$.

**证** 由 $\rho$ 混合的定义，我们有

$$(2.1.14)\quad \begin{aligned}&|E(S_{km}(p)+S_{km+p+i}(q))^2-(ES_{km}^2(p)+ES_{km+p+i}^2(q))|\\&\leqslant\rho(i)(ES_{km}^2(p)+ES_{km+p+i}^2(q)).\end{aligned}$$

注意到 $S_{km+p+n}(q)=S_{km+p}(q)-S_{km+p}(n)+S_{(k+1)m}(n)$，我们得

$$(2.1.15)\qquad ES_{km+p+n}^2(q)=ES_{km+p}^2(q)+\theta_1,$$

其中 $|\theta_1|\leqslant 2\sigma_0 n$. 因此从(2.1.14)和(2.1.15)即得

$$(2.1.16)\quad (1-\rho(n))(ES_{km}^2(p)+ES_{km+p}^2(q)+\theta_2)$$

$$\leqslant E(S_{km}(p)+S_{km+p+n}(q))^2$$
$$\leqslant(1+\rho(n))(ES^2_{km}(p)+ES^2_{km+p}(q)+\theta_2),$$
其中$|\theta_2|\leqslant 4\sigma_0^2n^2+4\sigma_0 n\|S_{km+p}(q)\|_2$. 我们写
$$S_{km}(m)=S_{km}(p)+S_{km+p}(n)+S_{km+p+n}(q)-S_{(k+1)m}(n).$$
那么
$$\|S_{km}(m)\|_2=\|S_{km}(p)+S_{km+p+n}(q)\|_2+\theta_3, \tag{2.1.17}$$
其中$|\theta_3|\leqslant 2\sigma_0 n$. 因此 $ES^2_{km}(m)=E(S_{km}(p)+S_{km+p+n}(q))^2+\theta_4$,
其中$|\theta_4|\leqslant 12\sigma_0^2n^2+4\sigma_0 n(\|S_{km}(p)\|_2+\|S_{km+p+n}(q)\|_2)$.
把它代入于(2.1.16)中就得证(2.1.12),此时
$$C_1=\max(|(1-\rho(n))\theta_2+\theta_4|,|(1+\rho(n))\theta_2+\theta_4|)$$
$$\leqslant 20\sigma_0^2n^2+12\sigma_0 n(\|S_{km}(p)\|_2+\|S_{km+p}(q)\|_2).$$
关于(2.1.13)的证明,由(2.1.14)蕴含着
$$(1-\rho(n))ES^2_{km}(p)\leqslant E(S_{km}(P)+S_{km+p+n}(q))^2,$$
那么从(2.1.17)即得(2.1.13)成立.

**定理 2.1.4 的证明** 首先我们证明对任意正整数 $h$
$$\lim_{n\to\infty} ES^2_{hn}/ES^2_n=h. \tag{2.1.18}$$
由(2.1.12)(令 $k=0,m=hn,p=(h-1)n,q=n,n=[(ES^2_n)^{1/3}]$),我们有
$$(1-\rho(n))(ES^2_{(h-1)n}+ES^2_{(h-1)n}(n))-C_0\leqslant ES^2_{hn}$$
$$\leqslant(1+\rho(n))(ES^2_{(h-1)n}+ES^2_{(h-1)n}(n))+C_0,$$
其中 $C_0=20\sigma_0^2n^2+12\sigma_0 n(\|S_{(h-1)n}\|_2+\|S_{(h-1)n}(n)\|_2)$. 利用条件(ii)和(iii),对 $h$ 用归纳法即得(2.1.18). 所以
$$h(n):=ES^2_n/n$$
是一缓变函数. 由下式来扩充它的定义域
$$h(t)=ES^2_{[t]}/t.$$
我们来证明
$$\lim_{n\to\infty} h((1-\epsilon_n)n)/h(n)=1, \tag{2.1.19}$$
其中 $\epsilon_n\downarrow 0$ 且使 $n\epsilon_n$ 是整数. 由附录性质 $A4$,
$$\lim_{n\to\infty}\frac{h(n\epsilon_n)}{h(n)}\epsilon_n^{\delta}=0. \tag{2.1.20}$$

令 $hn=\max(h;hn\varepsilon_n<(1-\varepsilon_n)n)$. 注意到

$$S_{n-n\varepsilon_n}=S_n-S_{h_n n\varepsilon_n}(n\varepsilon_n)+S_{h_n n\varepsilon_n}(p)-S_{h_n n\varepsilon_n+n\varepsilon_n}(p),$$

其中 $p=n-(h_n+1)n\varepsilon_n\leqslant n\varepsilon_n$，由(2.1.13)($m=n\varepsilon_n,k=h_n$ 和 $h_n+1$)，我们有

$$\left|\|S_{n-n\varepsilon_n}\|_2-\|S_n\|_2\right|\leqslant\|S_{h_n n\varepsilon_n}(n\varepsilon_n)\|_2+\frac{1}{(1-\rho(i))^{1/2}}$$

$$\cdot(\|S_{h_n n\varepsilon_n}(n\varepsilon_n)\|_2+\|S(h_n+1)_{n\varepsilon_n}(n\varepsilon_n)\|_2+4\sigma_0 i),$$

其中取 $i$ 使得 $\rho(i)<1$. 用 $\|S_n\|_2$ 除上述不等式两边，并利用(iii)和(2.1.20)就得(2.1.19)。对整数 $k>0$、存在 $n_k$ 使对每一 $n\geqslant n_k$，由于 $h(n)$ 是缓变的，我们有

$$\left|\log\frac{h(nk)}{h(n)}\right|<\frac{1}{k}. \tag{2.1.21}$$

不失一般性，可设 $n_k$ 关于 $k$ 是严格增加的. 令 $t>1$，对整数 $n>0$，定义 $k=q_n$ 使得 $n_k\leqslant nt<n_{k+1}$. 那么(2.1.21)蕴含着

$$\lim_{n\to\infty}\log(h([nt]q_n)/h(nt))=0. \tag{2.1.22}$$

令 $p_n=[q_nt]$. 那么 $p_n=[kt]\geqslant k$. 因此由(2.1.21)也可推出

$$\lim_{n\to\infty}\log(h(np_n)/h(n))=0. \tag{2.1.23}$$

另外，从(2.1.19)有

$$\lim_{n\to\infty}h([nt]q_n)/h(np_n)=1.$$

它与(2.1.22)和(2.1.23)相结合就得

$$\lim_{n\to\infty}h(nt)/h(n)=1.$$

所以由附录性质 A1 就得证

$$\lim_{x\to\infty}\frac{h(xt)}{h(x)}=\lim_{x\to\infty}\frac{h([x]t)}{h([x])}=1.$$

现在我们来考察定理的第二部分．由(2.1.12)让 $m=2N,p=q=N,n=[N^{1/3}]$，我们有

(2.1.24)

$$(1-\rho([N^{1/3}]))(ES^2_{2kN}(N)+ES^2_{(2k+1)N}(N))(1-\alpha_N)$$
$$\leqslant ES^2_{2kN}(2N)$$

$$\leqslant(1+\rho([N^{1/3}]))(ES^2_{2kN}(N)+ES^2_{(2k+1)N}(N))(1+\alpha N),$$

其中

$\alpha_N=$

$$\sup_k \frac{20\sigma_0^2N^{2/3}+12\sigma_0^2N^{1/3}(\|S^2_{2kN}(N)\|_2+\|S^2_{(2k+1)N}(N)\|_2)}{(1-\rho([N^{1/3}]))(ES^2_{2kN}(N)+ES^2_{(2k+1)N}(N))},$$

且 $N_0$ 充分大使当 $N\geqslant N_0$ 时 $\rho([N^{1/3}])<1$. 条件(i)和(iii)蕴含着

$$\alpha_N=O\left(\frac{N^{2/3}+N^{1/3}\|S_N\|_2}{\|S_N\|_2^2}\right).$$

由附录的性质 $A2$,对任给 $0<\varepsilon<1/6$

$$\lim_{N\to\infty}N^{\varepsilon}ES_N^2/N=\infty.$$

因此$(ES_N^2)^{-1}=O(N^{-1+\varepsilon})$,且进一步有

$$\alpha_N=O(N^{-\frac{1}{6}+\varepsilon}). \tag{2.1.25}$$

那么从(2.1.24),对整数 $r>p\geqslant N_0$, $\rho([2^{N_0/3}])<1$,有

$$\prod_{i=p}^{r-1}(1-\rho([2^{i/3}]))(1-\alpha_{2^i})\sum_{i=0}^{2^{r-p}-1}ES^2_{i2^p}(2^p)\leqslant ES^2_{2^r} \tag{2.1.26}$$

$$\leqslant\prod_{i=p}^{r-1}(1+\rho([2^{i/3}]))(1+\alpha_{2^i})\sum_{i=0}^{2^{r-p}-1}ES^2_{i2^p}(2^p).$$

由条件(iv), $\sum_i\rho([2^{i/3}])<\infty$. 另外,从(2.1.25)也有 $\sum_i\alpha(2^i)<\infty$. 所以从(2.1.26)我们得

$$\lim_{r>p\to\infty}ES^2_{2^r}\Big/\sum_{i=0}^{2^{r-p}-1}ES^2_{i2^p}(2^p)=1.$$

这样从条件(iii)即得

$$\lim_{r>p\to\infty}h(2^r)/h(2^p)=1,$$

且进一步,$h(2^r)$收敛于一个正的常数. 应用性质 $A3$ 于 $h(t)$和 $1/h(t)$,就得 $h(n)$与 $h(2^r)$收敛于同一个极限. 定理 2.1.4 证毕.

对强平稳 $\rho$ 混合序列,我们也有下列结果.

**定理 2.1.5**(Ibragimov 1975) 设$\{X_n,n\geqslant1\}$是强平稳 $\rho$ 混合序列,$EX_1=0$, $EX_1^2<\infty$ 且 $\sum_{n=1}^{\infty}\rho(2^n)<\infty$. 那么$\{X_n\}$具有连续谱密度且当 $f(0)\neq0$ 时

(2.1.27) $\quad (\operatorname{Var} S_n) = 2\pi f(0)n + o(n), \quad n \to \infty.$

定理 2.1.5 的证明参见 Ibragimov 和 Rozanov(1978) 的引理 17. 其证明概略是这样的：首先在条件 $\sum_{n=1}^{\infty}\rho(2^n) < \infty$ 下证明 $\{X_n\}$ 具有一有界谱密度 $f(\lambda)$. 然后证明

$$E_n(f) \leqslant 128 \max_{\lambda} f(\lambda) \sum_{k=0}^{\infty} \rho(2^{k-1}n),$$

其中 $E_n(f)$ 是指在 $[-\pi,\pi]$ 上函数 $f(\lambda)$ 用次数不超过 $n$ 的三角多项式最佳逼近的误差．借助这一结果可知谱密度 $f(\lambda)$ 是 $[-\pi,\pi]$ 上的连续函数．最后利用定理 2.1.1 即得(2.1.27)成立．

## §2.2 进一步的不等式

为证明关于混合序列的极限定理，除去 §1.2 中的基本不等式以外，常常需要某些进一步的不等式。

下述关于 $\alpha$ 混合序列的拓广 Ottaviani 不等式是由林正炎(1982)给出的．对随机变量序列 $\{X_n\}$，记 $\mathscr{F}_a^b = \sigma(x_i, a \leqslant i \leqslant b)$.

**引理 2.2.1** 设 $\{X_n, n \geqslant 1\}$ 是 $\alpha$ 混合序列．对任给正整数 $p, q$ 和 $k$，设 $\xi_j$ 是 $\mathscr{F}_{(j-1)(p+q)+1}^{jp+(j-1)q}$ 可测的，$j = 1, 2, \cdots, k$. 若

$$P\{|\xi_{l+1} + \cdots + \xi_k| \leqslant C\} \geqslant \frac{1}{2}, \quad l = 1, \cdots, k-1,$$

那么

$$\begin{aligned} &P\{\max_{1 \leqslant l \leqslant k} |\xi_1 + \cdots + \xi_l| > 2C\} \\ &\leqslant 2P\{|\xi_1 + \cdots + \xi_k| > C\} + 2k\alpha(q). \end{aligned}$$

**证** 设事件

$$A = P\{\max_{1 \leqslant l \leqslant k} |\xi_1 + \cdots + \xi_l| > 2C\},$$

$$B = \{|\xi_1 + \cdots + \xi_k| > C\},$$

$$A_1 = \{|\xi_1| > 2C\},$$

$$A_l = \{\max_{1 \leqslant r \leqslant l-1} |\xi_1 + \cdots + \xi_r| \leqslant 2C, |\xi_1 + \cdots + \xi_l| > 2C\},$$

$$l = 2, \cdots, k,$$

$$B_l = \{|\xi_{l+1} + \cdots + \xi_k| \leqslant C,\}, l = 1,\cdots,k-1, B_k = \Omega.$$

那么

$$A_iA_j = \phi(i \neq j), A = \bigcup_{l=1}^{k} A_l, \bigcup_{l=1}^{k} A_lB_l \subseteq B.$$

由引理的条件

$$P(A_lB_l) \geqslant P(A_l)P(B_l) - \alpha(q) \geqslant \frac{1}{2}P(A_l) - \alpha(q),$$

因此

$$P(B) \geqslant \sum_{l=1}^{k} P(A_lB_l) \geqslant \frac{1}{2}\sum_{l=1}^{k} P(A_l) - k\alpha(q)$$

$$= \frac{1}{2}P(A) - K\alpha(q).$$

引理证毕.

下述引理大部分是关于部分和的矩的阶. 对 $\rho$ 混合序列,较早的工作是属于 Peligrad(1982,1987)的. 邵启满(1988b,1989a,b)改进并拓广了她的结果.

**引理 2.2.2** 设$\{X_n, n\geqslant 1\}$是 $\rho$ 混合序列,对每一 $n\geqslant 1$, $EX_n=0$, $EX_n^2<\infty$. 那么对任给 $\epsilon>0$ 存在 $C=C(\epsilon)>0$ 使得对每一 $k\geqslant 1$ 和 $n\geqslant 1$

$$ES_k^2(n) \leqslant Cn\exp\left\{(1+\epsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}\max_{k<i\leqslant k+n} EX_i^2,$$

其中 $S_k(n) = \sum_{i=k+1}^{k+n} X_i$.

**证** 不失一般性可设 $0<\epsilon<1/4$. 设 $C_n$ 是不减的数列使得

(2.2.1) $\quad ES_k^2(n) \leqslant C_n n \max_{k<i\leqslant k+n} EX_i^2.$

对 $n\leqslant 2^{1/\epsilon}$,由 Minkowski 不等式我们仅需取 $C_n\geqslant 2^{1/\epsilon}$. 设 $C_1=2^{1/\epsilon}$. 假设 $C_m, m=1,\cdots,n-1$ 已如引理的要求被定义了. 令 $n_1=[n/2]$, $n_2=n-n_1$, $n_3=[n^{\frac{1}{1+\epsilon}}]+1$. 显然地

(2.2.2) $\quad ES_k^2(n)=ES_k^2(n_1)+ES_{k+n_1}^2(n_2)+2ES_k(n_1)S_{k+n_1}(n_2),$

且进一步有

$$|ES_k(n_1)S_{k+n_1}(n_2)| \leqslant |ES_k(n_1)S_{k+n_1}(n_3)|$$

$$
\begin{aligned}
&+ |ES_k(n_1)S_{k+n_1+n_3}(n_2-n_3)| \\
&\leqslant \|S_k(n_1)\|_2 \|S_{k+n_1}(n_3)\|_2 + \rho(n_3)\|S_k(n_1)\|_2 \\
&\quad \cdot \|S_{k+n_1+n_3}(n_2-n_3)\|_2 \\
&\leqslant 2\|S_k(n_1)\|_2\|S_{k+n_1}(n_3)\|_2 + \rho(n_3)\|S_k(n_1)\|_2 \\
&\quad \cdot \|S_{k+n_1}(n_2)\|_2.
\end{aligned}
$$

把上述不等式代入(2.2.2)中，并注意到(2.2.1)我们得

$$
\begin{aligned}
ES_k^2(n) &\leqslant \{ES_k^2(n_1)+ES_{k+n_1}^2(n_2)\}(1+\rho(n_3)) \\
&\quad +4\|S_k(n_1)\|_2\|S_{k+n_1}(n_3)\|_2 \\
&\leqslant C_{n_2}\cdot n(1+\rho(n_3))\max_{k<i\leqslant k+n} EX_i^2+4C_{n_1}n_1^{\frac{1}{2}}n_3^{\frac{1}{2}}\max_{k<i\leqslant k+n} EX_i^2 \\
&\leqslant C_{n_2}(1+\rho(n_2^{\frac{1}{1+\varepsilon}})+4n_2^{-\frac{\varepsilon}{2(1+\varepsilon)}})n\max_{k<i\leqslant k+n} EX_i^2,
\end{aligned}
$$

其中

$$\rho(x)=(\rho(i+1)-\rho(i))(x-i)+\rho(i) \text{ 若 } i<x<i+1.$$

因此对 $n\geqslant 2$，我们定义

$$C_n=C_{n_2}\left\{1+\rho\left(n_2^{\frac{1}{1+\varepsilon}}\right)+4n_2^{-\frac{\varepsilon}{2(1+\varepsilon)}}\right\}.$$

显然 $C_n$ 是不减的，且

$$
\begin{aligned}
(2.2.3)\quad C_{2^n} &= C_{2^{n-1}}\left\{1+\rho\left(2^{\frac{n-1}{1+\varepsilon}}\right)+4\cdot 2^{-\frac{\varepsilon(n-1)}{2(1+\varepsilon)}}\right\} \\
&= C_1\prod_{i=0}^{n-1}\left\{1+\rho\left(2^{\frac{i}{1+\varepsilon}}\right)+4\cdot 2^{-\frac{\varepsilon i}{2(1+\varepsilon)}}\right\} \\
&\leqslant C_1\exp\left\{\sum_{i=0}^{n-1}\left(\rho\left(2^{\frac{i}{1+\varepsilon}}\right)+4\cdot 2^{-\frac{\varepsilon i}{2(1+\varepsilon)}}\right)\right\} \\
&\leqslant C_1\exp\left\{3+\int_2^{n-1}\rho\left(2^{\frac{x}{1+\varepsilon}}\right)dx+c_\varepsilon\right\} \\
&\leqslant C_1\exp\left\{3+(1+\varepsilon)\int_{2/(1+\varepsilon)}^{n-1}\rho(2^x)dx+C_\varepsilon\right\} \\
&\leqslant C_1\exp\left\{3+(1+\varepsilon)\sum_{i=1}^{n}\rho(2^i)+C_\varepsilon\right\},
\end{aligned}
$$

其中 $C_\varepsilon=4/(1-2^{-\frac{\varepsilon}{2(1+\varepsilon)}})$. 令 $d_\varepsilon=2^{1/\varepsilon}\exp(3+C_\varepsilon)$. 我们得

$$C_{2^n}\leqslant d_\varepsilon\exp\left\{(1+\varepsilon)\sum_{i=1}^{n}\rho(2^i)\right\}.$$

对任一 $n$，存在 $m$ 使 $2^m \leqslant n < 2^{m+1}$. 利用 $C_n$ 的单调性即得

$$C_n \leqslant C_{2^{m+1}} \leqslant d_\varepsilon \exp\left\{(1+\varepsilon)\sum_{i=1}^{m+1}\rho(2^i)\right\}$$

$$\leqslant d_\varepsilon \exp\left\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}.$$

引理证毕.

**引理 2.2.3** 设 $\{X_n, n\geqslant 1\}$ 是 $\rho$ 混合序列，对每一 $n\geqslant 1$，$EX_n = 0$，$EX_n^2 < \infty$. 假设对 $k$ 一致地有

(2.2.4) $$ES_k^2(n) / \min_{k<i\leqslant k+n} EX_i^2 \to \infty, \quad n\to\infty,$$

且对某 $a\geqslant 1$ 有

(2.2.5) $$\max_{k<i\leqslant k+n} EX_i^2 \leqslant a \min_{k<i\leqslant k+n} EX_i^2.$$

那么对任一 $\varepsilon>0$，存在 $C'=C'(\varepsilon,\rho(\cdot),a)>0$ 和整数 $N$ 使对每一 $k\geqslant 0$ 和 $n\geqslant N$

$$ES_k^2(n) \geqslant C' n \exp\left\{-(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}\min_{k<i\leqslant k+n} EX_i^2.$$

**证** 不失一般性可设 $0<\varepsilon<1/400$. 由此可得

$$1 - 5\varepsilon^2 > (3/2)^{-\varepsilon/6}.$$

因此，注意到 $\rho(n)\to 0\ (n\to\infty)$，对某一充分大 $m_0$ 我们有

$$1 - 5\varepsilon^2 - \rho(m_0) > (3/2)^{-\varepsilon/6}.$$

不难验证 $\exp\left\{2\sum_{i=0}^{[\log n]}\rho(2^i)\right\}$ 是一个缓变函数. 由引理 2.2.2 和条件 (2.2.4)，存在整数 $n_0$ 使当 $n\geqslant n_0$ 时

(2.2.6) $$ES_k^2(n) \leqslant n^{1+\varepsilon^4}\max_{k<i\leqslant k+n} EX_i^2,$$

(2.2.7) $$ES_k^2(n) \geqslant \frac{4am_0^2}{\varepsilon^4}\min_{k<i\leqslant k+n} EX_i^2.$$

当 $n\geqslant 2n_0$ 时，令 $n_1=[n/2]$，$n_2=n-n_1$. 那么

(2.2.8) $$\begin{aligned} ES_k^2(n) &= ES_k^2(n_1) + ES_{k+n_1}^2(n_2) \\ &\quad + 2ES_k(n_1)S_{k+n_1}(m_0) + 2ES_k(n_1)S_{k+n_1+m_0}(n_2-m_0) \\ &\geqslant ES_k^2(n_1) + ES_{k+n_1}^2(n_2) - 2\|S_k(n_1)\|_2 \\ &\quad \cdot \|S_{k+n_1}(m_0)\|_2 - 2\rho(m_0)\|S_k(n_1)\|_2 \end{aligned}$$

$$\cdot \| S_{k+n_1+m_0}(n_2-m_0) \|_2$$

$$\geqslant (1-\rho(m_0))(ES_k^2(n_1)+ES_{k+n_1}^2(n_2))-4\| S_k(n_1) \|_2$$

$$\cdot \| S_{k+n_1}(m_0) \|_2$$

$$\geqslant (1-\rho(m_0))(ES_k^2(n_1)+ES_{k+n_1}^2(n_2))-4\varepsilon^2 ES_k^2(n_1)$$

$$-\frac{4}{\varepsilon^2}m_0^2 \max_{k<i\leqslant k+n} EX_i^2$$

$$\geqslant (1-4\varepsilon^2-\rho(m_0))(ES_k^2(n_1)+ES_{k+n_1}^2(n_2))$$

$$-\frac{4}{\varepsilon^2}m_0^2 a \min_{k<i\leqslant k+n} EX_i^2$$

$$\geqslant (1-5\varepsilon^2-\rho(m_0))(ES_k^2(n_1)+ES_{k+n_1}^2(n_2))$$

$$\geqslant (3/2)^{-\varepsilon/6}(ES_k^2(n_1)+ES_{k+n_1}^2(n_2)).$$

首先我们来证明对每一 $n\geqslant n_0$

(2.2.9) $$ES_k^2(n)\geqslant C_2 n^{1-\varepsilon/6}\min_{k<i\leqslant k+n} EX_i^2,$$

其中 $C_2=2am_0^2n_0^{-1}\varepsilon^{-4}$. 由(2.2.7)，对于 $n_0\leqslant n<2n_0$ (2.2.9)成立. 当 $n\geqslant 2n_0$ 时，我们假设对小于 $n$ 的每一正整数(2.2.9)成立. 那么它对 $n$ 也成立. 事实上，利用(2.2.8)我们得

$$ES_k^2(n)\geqslant (3/2)^{-\varepsilon/6}C_2(n_1^{1-\varepsilon/6}+n_2^{1-\varepsilon/6})\min_{k<i\leqslant k+n} EX_i^2$$

$$\geqslant \left(\frac{3}{2}\right)^{-\varepsilon/6}C_2\left(\left(\frac{1}{3}\right)^{1-\varepsilon/6}+\left(\frac{2}{3}\right)^{1-\varepsilon/6}\right)$$

$$\cdot n^{1-\varepsilon/6}\min_{k<i\leqslant k+n} EX_i^2$$

$$\geqslant C_2 n^{1-\varepsilon/6}\min_{k<i\leqslant k+n} EX_i^2.$$

下面我们来证明引理的断言. 对 $n\geqslant n_0^{1+\varepsilon}$，令 $n_1=\left[\frac{n}{2}\right]$，$n_2=n-n_1$，$n_3=[n^{1/(1+\varepsilon)}]+1$. 从(2.2.6)和(2.2.9)即得

(2.2.10) $$ES_k^2(n)=ES_k^2(n_1)+ES_{k+n_1}^2(n_2)$$

$$+2ES_k(n_1)ES_{k+n_1}(n_3)$$

$$+2ES_k(n_1)S_{k+n_1+n_3}(n_2-n_3)$$

$$\geqslant ES_k^2(n_1)+ES_{k+n_1}^2(n_2)$$

$$-4\| S_k(n_1) \|_2 \| S_{k+n_1}(n_3) \|_2-2\rho(n_3)\| S_k(n_1) \|_2$$

$$\cdot \| S_{k+n_1}(n_2) \|_2$$

$$\geqslant (1-\rho(n_3))(ES_k^2(n_1)+ES_{k+n_1}^2(n_2))$$
$$-4(n_1 n_3)^{(1+\varepsilon^4)/2}\max_{k<i\leqslant k+n} EX_i^2$$
$$\geqslant (1-\rho(n_3))(ES_k^2(n_1)+ES_{k+n_1}^2(n_2))$$
$$-4n^{1-(\varepsilon-3\varepsilon^4)/2(1+\varepsilon)}\max_{k<i\leqslant k+n} EX_i^2$$
$$\geqslant (1-\rho(n_3))(ES_k^2(n_1)+ES_{k+n_1}^2(n_2))$$
$$-4an^{1-\varepsilon/5}\min_{k<i\leqslant k+n} EX_i^2$$
$$\geqslant (1-\rho(n_3))(ES_k^2(n_1)+ES_{k+n_1}^2(n_2))$$
$$-8aC_2^{-1}n^{-\varepsilon/30}ES_k^2(n_1)$$
$$\geqslant (1-\rho(n_3)-n^{-\varepsilon/40})(ES_k^2(n_1)+ES_{k+n_1}^2(n_2)).$$

最后的不等号对 $n\geqslant(8ac_2^{-1})^{120/\varepsilon}$成立. 令

$$n'_0=\max(n_0^{1+\varepsilon},(8aC_2^{-1})^{120/\varepsilon}).$$

设 $C_n$ 是不增的，使得对 $n\geqslant n_0$

$$ES_k^2(n)\geqslant C_n n\min_{k<i\leqslant k+n} EX_i^2.$$

那么由(2.2.10)对 $n\geqslant n_0'$ 有

$$ES_k^2(n)\geqslant\left(1-\rho(n_3)-n_2^{-\varepsilon/40}\right)C_{n_2}n\min_{k<i\leqslant k+n} EX_i^2$$
$$\geqslant\left(1-\rho\left(n_2^{\frac{1}{1+\varepsilon}}\right)-n_2^{-\varepsilon/40}\right)C_{n_2}n\min_{k<i\leqslant k+n} EX_i^2.$$

因此对 $n\geqslant n'_0$ 我们可选

$$C_n=C_{n_2}\left(1-\rho\left(n_2^{\frac{1}{1+\varepsilon}}\right)-n_2^{-\varepsilon/40}\right). \tag{2.2.11}$$

容易验证存在 $n''_0$ 使对 $n\geqslant n''_0$ 有

$$1-\rho\left(n_2^{\frac{1}{1+\varepsilon}}\right)-n_2^{-\varepsilon/40} \tag{2.2.12}$$
$$\geqslant\exp\left\{-(1+\varepsilon)\left(\rho\left(n_2^{\frac{1}{1+\varepsilon}}\right)+n_2^{-\varepsilon/40}\right)\right\}.$$

令 $n_0^*=n'_0\vee n''_0$. 由(2.2.7)，我们取 $Cn_0^*=4am_0^2/(\varepsilon^4 n_0^*)$. 显然，由(2.2.11)定义的$\{C_n,n\geqslant 2n_0^*\}$是不增的，从(2.2.11)和(2.2.12)，我们得对 $2^m>n_0^*$ 有

$$C_{2^m}=C_{2^{m-1}}\left(1-\rho\left(2^{\frac{m-1}{1+\varepsilon}}\right)-2^{-(m-1)\varepsilon/40}\right)$$
$$\geqslant C_{2^{m-1}}\exp\{-(1+\varepsilon)(\rho(2^{\frac{m-1}{1+\varepsilon}})+2^{-(m-1)\varepsilon/40})\}.$$

由此可得

$$C_{2^m} \geqslant C_{n_0'} \prod_{i=0}^{m-1} \exp\{-(1+\varepsilon)(\rho(2^{i/(1+\varepsilon)}) + 2^{-i\varepsilon/40})\}$$

$$= C_{n_0'} \exp\Big\{-(1+\varepsilon)\sum_{i=0}^{m-1}(\rho(2^{i/(1+\varepsilon)}) + 2^{-i\varepsilon/40})\Big\}.$$

类似于(2.2.3)，存在 $d_\varepsilon>0$ 使得

$$\exp\Big\{-(1+\varepsilon)\sum_{i=0}^{m-1}(\rho(2^{i/(1+\varepsilon)}) + 2^{-i\varepsilon/40})\Big\}$$

$$\geqslant d_\varepsilon \exp\Big\{-(1+\varepsilon)^2\sum_{i=0}^{m}\rho(2^i)\Big\}.$$

所以

$$C_{2^m} \geqslant d_\varepsilon C_{n_0'} \exp\Big\{-(1+\varepsilon)^2\sum_{i=0}^{m}\rho(2^i)\Big\}.$$

对任一 $n>n_0'$ 存在 $m$ 使 $2^m\leqslant n<2^{m+1}$. 由 $C_n$ 的单调性，我们得

$$C_n \geqslant C_{2^{m+1}} \geqslant d_\varepsilon C_{n_0'} \exp\Big\{-(1+\varepsilon)^2\sum_{i=0}^{m+1}\rho(2^i)\Big\}$$

$$\geqslant \frac{1}{8} d_\varepsilon C_{n_0'} \exp\Big\{-(1+\varepsilon)^2\sum_{i=0}^{[\log n]}\rho(2^i)\Big\},$$

因此就得引理的断言.

有时我们需要高于 2 阶矩的界.

**引理 2.2.4** 设 $\{X_n, n\geqslant 1\}$ 是 $\rho$ 混合序列，$EX_n=0$，对某 $0<\delta<1$，$\sup_n E|X_n|^{2+\delta}<\infty$ 且

$$\sum_{n=1}^{\infty}\rho(2^n) < \infty. \tag{2.2.13}$$

那么存在 $C=C(\delta,\rho(\cdot))>0$ 使对每一 $n\geqslant 1$ 有

$$\sup_{k\geqslant 1} E|S_k(n)|^{2+\delta} \leqslant C\{n^{1+\delta/2}(\sup_{k\geqslant 1} EX_k^2)^{1+\delta/2}$$

$$+ n\exp\{(C\log n)^{\delta/(2+\delta)}\}\sup_{k\geqslant 1} E|X_k|^{2+\delta}\}.$$

**证** 不难验证对 $x\geqslant 0$

$$(1+x)^{2+\delta}\leqslant 1+(2+\delta)^2(x+x^{1+\delta})+x^{2+\delta} \tag{2.2.14}$$

$$\leqslant 1+9(x+x^{1+\delta})+x^{2+\delta}.$$

令

$$a_m = \sup_{k\geq 1} \| S_k(m) \|_{2+\delta}, \sigma_m = \sup_{k\geq 1} \| S_k(m) \|_2.$$

显然地

$$\| S_k(2m) \|_{2+\delta} \leqslant \| S_k(m) + S_{k+m+[m^{1/5}]}(m) \|_{2+\delta} + 2m^{1/5}a_1.$$

由(2.2.14)，令 $m_1 = m + [m^{1/5}]$，我们有

$$E|S_k(m) + S_{k+m_1}(m)|^{2+\delta} \leqslant 2a_m^{2+\delta} + 9E|S_k(m)|^{1+\delta}|S_{k+m_1}(m)|$$
$$+ 9E|S_k(m)| |S_{k+m_1}(m)|^{1+\delta}.$$

由 Schwarz 不等式和引理 1.2.7 我们有

$$E|S_k(m)|^{1+\delta}|S_{k+m_1}(m)| \leqslant \| S_k(m) \|_{2+\delta}^{\delta}$$
$$\cdot \| S_k(m) S_{k+m_1}(m) \|_{(2+\delta)/2}$$
$$\leqslant a_m^{\delta} \{ \sigma_m^{2-\delta} + 4\rho([m^{1/5}]) a_m^{2+\delta} \}^{2/(2+\delta)}$$
$$\leqslant a_m^{\delta}\sigma_m^2 + 4\rho^{2/(2+\delta)}([m^{1/5}]) a_m^{2+\delta}.$$

类似地

$$E|S_k(m)| |S_{k+m_1}(m)|^{1+\delta} \leqslant a_m^{\delta}\sigma_m^2 + 4\rho^{2/(2+\delta)}([m^{1/5}])a_m^{2+\delta}.$$

结合这些不等式得

$$E|S_k(m) + S_{k+m_1}(m)|^{2+\delta}$$
$$\leqslant 2a_m^{2+\delta} + 18(a_m^{\delta}\sigma_m^2 + 4\rho^{2/(2+\delta)}([m^{1/5}])a_m^{2+\delta})$$
$$\leqslant \{[2(1+36\rho^{2/(2+\delta)}([m^{1/5}]))]^{1/(2+\delta)}a_m + 18\sigma_m\}^{2+\delta},$$

由此即得

$$(2.2.15) \quad a_{2m} \leqslant \{2(1+36\rho^{2/(2+\delta)}([m^{1/5}]))\}^{1/(2+\delta)}a_m$$
$$+ 18\sigma_m - 2m^{1/5}a_1.$$

注意到 $\rho(n)$ 的单调性和条件(2.2.13)，我们有

$$\rho(n) \leqslant c/\log n,$$

这里及以后 $C$ 均表示一个正的常数，在不同地方可取不同的值．因此，应用引理 2.2.2，我们得

$$(2.2.16) \quad a_{2^r} \leqslant \{2(1 + 36\rho^{2/(2+\delta)}([2^{(r-1)/5}]))\}^{1/(2+\delta)}a_{2^{r-1}}$$
$$+ 18\sigma_{2^{r-1}} + 2 \cdot 2^{(r-1)/5}a_1$$
$$\leqslant 2^{(r-1)/(2+\delta)} \prod_{i=0}^{r-1} (1 + 36\rho^{2/(2+\delta)}([2^{j/5}]))^{1/(2+\delta)}a_1$$

$$+c\sigma_1\sum_{i=0}^{r-1}2^{i/2}\prod_{j=i+1}^{r-1}\{2(1+9\rho^{2/(2+\delta)}([2^{j/5}]))\}^{1/(2+\delta)}$$

$$+2a_1\sum_{i=0}^{r-1}2^{i/5}\prod_{j=i+1}^{r-1}\{2(1+9\rho^{2/(2+\delta)}([2^{j/5}]))\}^{1/(2-\delta)}$$

$$\leqslant C2^{r/2}\sigma_1+2^{r/(2+\delta)}\exp(Cr)^{\delta/(2+\delta)}a_1.$$

这就得到引理的结论．

类似地，通过细致的估计，邵启满(1989a)证明了下列结果，其证明不在此详述．

**引理 2.2.5** 设$\{X_n,n\geqslant1\}$是$\rho$混合序列，$EX_n=0$，对某$0\leqslant\delta<1$，$\sup\limits_n E|X_n|^{2+\delta}<\infty$．那么对任给$\varepsilon>0$，存在$C=C(\delta,\rho(\cdot),\varepsilon)>0$使对每一$n\geqslant2$有

$$E|S_k(n)|^{2+\delta}\leqslant C\Big\{\Big(n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\max_{k<i\leqslant k+n}EX_i^2\Big)^{1+\delta/2}$$

$$+n\exp\Big\{C\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big\}\max_{k<i\leqslant k+n}E|X_i|^{2+\delta}\Big\}.$$

**引理 2.2.6** 设$\{X_n,n\geqslant1\}$是$\rho$混合序列，$EX_n=0$，对某$q>2$，$E|X_n|^q<\infty$．假设存在函数$h(n)$使对每一$k\geqslant0,n\geqslant1$有

$$ES_k^2(n)\leqslant nh(n)\max_{k<i\leqslant k+n}EX_i^2,$$

且有正整数$n_0$和常数$0<\theta<2^{1-2/(q\wedge3)}$使当$n\geqslant n_0$时有

$$\max(h([n/2]),h(n-[n/2]))\leqslant\theta h(n).$$

进一步，当$q>3$时假设存在$C>0$使得

$$h(n)\geqslant\frac{1}{C}\exp\Big\{-C\sum_{i=0}^{[\log n]}\rho^{2/q}(2^i)\Big\}.$$

那么存在常数$K=K(q,n_0,\theta,C,\rho(\cdot))$使对每一$k\geqslant0,n\geqslant1$有

$$E|S_k(n)|^q\leqslant K\Big\{\Big(nh(n)\max_{k<i\leqslant k+n}EX_i^2\Big)^{q/2}$$

$$+n\exp\Big\{K\sum_{i=0}^{[\log n]}\rho^{2/q}(2^i)\Big\}\max_{k<i\leqslant k+n}E|X_i|^q\Big\}.$$

现在我们来讨论$\varphi$混合序列的有关不等式．Peligrad(1985)给出了如下的尾概率不等式(参见邵启满1988a)．

**引理 2.2.7** 设$\{X_n,n\geqslant1\}$是$\varphi$混合序列，$0<\eta<1$．假设存在

整数 $p,1\leqslant p\leqslant n$，和数 $A>0$ 使得

(2.2.17) $$\varphi(p)+\max_{p\leqslant i\leqslant n}P\{|S_n-S_i|\geqslant A\}\leqslant\eta.$$

那么对任给的 $a\geqslant0,b\geqslant0$，我们有

(2.2.18) $$P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant a+A+b\}$$
$$\leqslant\frac{1}{1-\eta}P\{|S_n|\geqslant a\}+\frac{1}{1-\eta}P\left\{\max_{1\leqslant i\leqslant n}|X_i|\geqslant\frac{b}{p-1}\right\}.$$

(2.2.19) $$P\{|S_n|\geqslant a+A+b\}\leqslant\eta P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant a\}$$
$$+P\left\{\max_{1\leqslant i\leqslant n}|X_i|\geqslant\frac{b}{p}\right\}.$$

**证** 令 $E_i=\{\max_{1\leqslant j<i}|S_j|<a+A+b\leqslant|S_i|\}$. 那么

$$P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant a+A+b\}\leqslant P(|S_n|\geqslant a)$$
$$+\sum_{i=1}^{n-1}P(E_i\cap\{|S_n-S_i|\geqslant A+b\})$$

且

$$\sum_{i=1}^{n-1}P(E_i\cap\{|S_n-S_i|\geqslant A+b\})$$
$$\leqslant\sum_{i=1}^{n-p-1}P(E_i\cap\{|S_{i+p-1}-S_i|\geqslant b\})$$
$$+\sum_{i=1}^{n-p-1}P(E_i\cap\{|S_n-S_{i+p-1}|\geqslant A\})$$
$$+\sum_{i=n-p}^{n-1}P(E_i\cap\{|S_n-S_i|\geqslant A+b\})$$
$$\leqslant\sum_{i=1}^{n-1}P\left\{E_i\cap\left\{\max_{1\leqslant j\leqslant n}|X_j|\geqslant\frac{b}{p-1}\right\}\right\}$$
$$+\sum_{i=1}^{n-p-1}P(E_i)(P\{|S_n-S_{i+p-1}|\geqslant A\}+\varphi(p))$$
$$\leqslant P\left\{\max_{1\leqslant j\leqslant n}|X_j|\geqslant\frac{b}{p-1}\right\}+\eta P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant a+A+b\right\},$$

其中在最后一个不等式中应用了条件(2.2.17). 由此得证(2.2.18)成立.

对(2.2.19)，令 $E'_i=\{\max_{1\leqslant j<i}|S_j|<a\leqslant|S_i|\}$ 并注意到对 $1\leqslant j$

$\leqslant n-p,|S_n-S_{j+p-1}|\geqslant||S_n|-|S_{j-1}|-p\max\limits_{1\leqslant i\leqslant n}|X_i||$，我们有

$$P\{|S_n|\geqslant a+A+b\}$$

$$\leqslant P\left\{|S_n|\geqslant a+A+b,\max_{1\leqslant i\leqslant n-p}|S_i|\geqslant a,\max_{1\leqslant i\leqslant n}|X_i|\leqslant\frac{b}{p}\right\}$$

$$+P\left\{\max_{1\leqslant i\leqslant n}|X_i|>\frac{b}{p}\right\}$$

$$\leqslant\sum_{i=1}^{n-p}P(E_i\cap\{|S_n-S_{i+p-1}|>A\})+P\left\{\max_{1\leqslant i\leqslant n}|X_i|\geqslant\frac{b}{p}\right\}$$

$$\leqslant\eta P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant a\}+P\left\{\max_{1\leqslant i\leqslant n}|X_i|\geqslant\frac{b}{p}\right\}.$$

引理证毕．

引理 2.2.8 是属于邵启满，陆传荣(1986)的．

**引理 2.2.8** 设$\{X_n,n\geqslant1\}$是$\varphi$混合序列，$EX_n=0$ 且对某$\delta>0$，$\sup\limits_n E|X_n|^{2+\delta}<\infty$．假设对某 $M>0$．

$$\sup_k ES_k^2(n)\leqslant Mn\sup_k EX_k^2 \tag{2.2.20}$$

那么存在$C=C(\delta,M,\varphi(\cdot))>0$ 使对每一 $n\geqslant1$ 有

$$\sup_k E|S_k(n)|^{2+\delta}\leqslant Cn^{1+\delta/2}\sup_k E|X_k|^{2+\delta}.$$

**证** 容易看到对 $r\geqslant1$ 和 $x\geqslant0$ 有

$$(1+x)^r\leqslant\sum_{k=0}^{[r]}\binom{r}{k}x^k+\delta_r x^r, \tag{2.2.21}$$

其中当 $r$ 不是整数时 $\delta_r=1$，不然的话 $\delta_r=0$，现在对 $r=2+\delta$ 用归纳法来证明引理．假设引理对 $l\leqslant[r]$成立，$r$ 不是整数．记 $a_m=\sup\limits_k\|S_k(m)\|_r$，从(2.2.21)我们得

$$\begin{aligned}(2.2.22)\quad & E|S_k(m)+S_{k+m+k_0}(m)|^r\\ &\leqslant E|S_k(m)|^r+E|S_{k+m+k_0}(m)|^r\\ &\quad+\sum_{j=1}^{[r]}\binom{r}{j}E|S_k(m)|^j|S_{k+m+k_0}(m)|^{r-j}\\ &\leqslant\left(2+2\sum_{j=1}^{[r]}\binom{r}{j}\varphi^{j/r}(k_0)\right)a_m^r\\ &\quad+\sum_{j=1}^{[r]}\binom{r}{j}E|S_k(m)|^jE|S_{k+m+k_0}(m)|^{r-j}\end{aligned}$$

$$=:I_1+I_2.$$

由归纳假设，我们有

$$I_2\leqslant\sum_{j=1}^{[r]}\binom{r}{j}(E|S_k(m)|^{[r]})^{j/[r]}(E|S_{k+m+k_0}(m)|^{[r]})^{(r-j)/[r]}$$

$$\leqslant c\sum_{j=1}^{[r]}(m^{[r]/2}\sup_k E|X_k|^{[r]})^{r/[r]}\leqslant cm^{r/2}a_1^r.$$

把上述不等式代入(2.2.22)，得

$$a_{2m}\leqslant\left(2+2\sum_{j=1}^{[r]}\binom{r}{j}\varphi^{j/r}(k_0)\right)^{1/r}a_m+cm^{1/2}a_1.$$

选取充分大的 $k_0$ 并如引理 2.2.4 的证明中同样讨论，就可推得此时引理成立. 类似地对$[r]+1$ 引理成立. 引理证毕.

利用引理 2.2.7，邵启满(1988a)证明着下述引理.

**引理 2.2.9** 设$\{X_n,n\geqslant1\}$是满足(2.2.17)的 $\varphi$ 混合序列，$q>0$ 满足 $\eta4^q<1-\eta$. 那么

$$E\max_{1\leqslant i\leqslant n}|S_i|^q\leqslant(1-\eta-\eta4^q)^{-1}\{(8A)^q+2(4p)^qE\max_{1\leqslant i\leqslant n}|X_i|^q\},$$

其中 $\eta,p,A$ 如引理 2.2.7 中所定义.

**证** 由引理 2.2.7，对 $x\geqslant8A$ 我们有

$$P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant x\}$$

$$\leqslant\frac{1}{1-\eta}\left(P\left\{|S_n|\geqslant\frac{5}{8}x\right\}+P\left\{\max_{1\leqslant i\leqslant n}|X_i|\geqslant\frac{x}{4p}\right\}\right)$$

$$\leqslant\frac{1}{1-\eta}\left(\eta P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant\frac{x}{4}\right\}+2P\left\{\max_{1\leqslant i\leqslant n}|X_i|\geqslant\frac{x}{4p}\right\}\right).$$

因此对任给的 $B>8A$ 有

$$\int_0^B qy^{q-1}P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant y\}dy\leqslant\int_0^{8A}qy^{q-1}P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant y\}dy$$

$$+\frac{\eta}{1-\eta}\int_{8A}^B qy^{q-1}P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant\frac{y}{4}\right\}dy$$

$$+\frac{2}{1-\eta}\int_{8A}^B qy^{q-1}P\left\{\max_{1\leqslant i\leqslant n}|X_i|\geqslant\frac{y}{4p}\right\}dy$$

$$\leqslant(8A)^q+\frac{\eta}{1-\eta}4^q\int_0^B qy^{q-1}P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant y\}dy$$

$$+\frac{2(4p)^q}{1-\eta}\int_0^\infty qy^{q-1}P\{\max_{1\le i\le n}|X_i|\ge y\}dy.$$

由此可得

$$\int_0^B qy^{q-1}P\{\max_{1\le i\le n}|S_i|\ge y\}dy$$

$$\le\left(1-\frac{\eta 4^q}{1-\eta}\right)^{-1}\left((8A)^q+\frac{2(4p)^q}{1-\eta}\int_0^\infty qy^{q-1}P\left\{\max_{1\le i\le n}|X_i|\ge y\right\}dy\right)$$

$$\le(1-\eta-\eta 4^q)^{-1}((8A)^q+2(4p)^q E\max_{1\le i\le n}|X_i|^q).$$

让 $B\to\infty$ 就得引理的断言.

一个类似的结果是下述引理.

**引理 2.2.10** 设 $\{X_n,n\ge1\}$ 是 $\varphi$ 混合序列,假设存在正数组列 $\{C_{kn}\}$ 使得

$$\max_{1\le i\le n}ES_k^2(i)\le c_{kn}. \tag{2.2.23}$$

那么对任一 $q\ge2$ 存在 $C=C(q,\varphi(\cdot))$ 使得

$$E\max_{1\le i\le n}|S_k(i)|^q\le C(c_{kn}^{q/2}+E\max_{k<i\le k+n}|X_i|^q). \tag{2.2.24}$$

**证** 取 $\eta=4^{-2q}$, $A^2=2c_{kn}/\eta$. 由于 $\varphi(p)\to0(p\to\infty)$,所以存在 $p_0$ 使得 $\varphi(p_0)\le\eta/2$. 利用(2.2.23)即可验证(2.2.17)被满足. 因此从引理 2.2.9 即得(2.2.24).

# 第Ⅱ部分　弱收敛

在这一部分中，我们研究形如

$$\text{(Ⅱ 1)}\qquad \frac{1}{B_n}\sum_{j=1}^{n}X_j - A_n$$

的正则化和的概率测度(或分布)的弱收敛性. 对于独立情形我们已有一系列结果(参见 Petrov 1971 和 Billingsley 1968). 一个自然的问题就是对相依情形如何呢? 对于弱收敛性仅仅假设弱相依是不够的. 例如，设$\{\xi_n, n\geq 1\}$是 $i.i.d.$ 随机变量序列具有特征函数 $f(t)$，又设

$$X_n = \xi_{n+1} - \xi_n.$$

那么$\{X_n, n\geq 1\}$是强平稳序列且满足 § 1.1 中所述任何混合条件. 和式

$$\text{(Ⅱ 2)}\qquad \sum_{k=1}^{n}X_k = \xi_{n+1} - \xi_1$$

对所有 $n$ 具有特征函数 $|f(t)|^2$. 引入某些限制使得和式 $\sum_{k=1}^{n}X_k$ 的方差当 $n\to\infty$ 时趋向无穷是合理的. 因此我们总设当 $n\to\infty$ 时 (Ⅱ 1) 中的 $B_n$ 趋向无穷.

首先，我们叙述对混合随机变量极限定理证明中十分有用的 Bernstein 分段法. 设正整数 $p=p(n)$, $q=q(n)$, $k=k(n)$ 满足 $1\leq p\leq n$, $q=o(p)$, $k=[n/(p+q)]$[1]，且记

---

1) 在本书中，方括号[ · ]有时表示最大整数部分，有时作为括号，从行文中能明显地看出.

$$\xi_j = \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} X_i, \qquad \eta_j = \sum_{i=jp+(j-1)q+1}^{j(p+q)} X_i, j=1,\cdots,k,$$

(Ⅱ3)

$$\eta_{k+1} = \sum_{i=k(p+q)+1}^{n} X_i.$$

那么

(Ⅱ4)
$$S_n = \sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k+1}\eta j.$$

由弱相依性知，当 $q=q(n)$ 充分大时，$\xi_1,\xi_2,\cdots,\xi_n$ 是渐近独立的. 另一方面，注意到 $q=o(p)$，与 $S_n$ 相比较，和 $\sum_{j=1}^{k}\eta_j$ 是可略的. 由此可见，Bernstein 方法允许我们把混合相依随机变量和作为独立和来考察.

利用这一方法，对 $\alpha$ 混合序列，可通过对独立序列类似的步骤，来证明下述关于和的可能极限分布族的定理

**定理Ⅱ1** 设 $\{X_n, n\geqslant 1\}$ 是强平稳 $\alpha$ 混合序列，$\{A_n\}$ 和 $\{B_n\}$ 是两个实数列，$B_n\to\infty\,(n\to\infty)$. 假设和

$$\frac{1}{B_n}\sum_{k=1}^{n} X_k - A_n$$

的分布函数 $F_n(x)$ 依分布收敛于分布函数 $F(x)$. 那么 $F(x)$ 是指数为 $a$ 的稳定分布，而且

$$B_n = n^{1/a}h(n),$$

其中 $h(n)$ 是正整变量的缓变函数.

# 第三章　α混合序列的弱收敛

## §3.1　中心极限定理的充分必要条件

对α混合序列$\{X_n,n\geqslant 1\}$，Ibragimov(1959,1962)首先给出了中心极限定理成立的充分必要条件.在本章中除非特别声明总设$\{X_n,n\geqslant 1\}$是强平稳α混合序列，记$S_n=\sum_{j=1}^{n}X_j$，$\sigma_n^2=\mathrm{Var}S_n$.

**定理3.1.1**　设$EX_1=0$，$EX_1^2<\infty$.那么当$\{X_n\}$服从中心极限定理且$\lim\limits_{n\to\infty}\sigma_n^2=\infty$时，下述条件是必要的：

(1) $\sigma_n^2=nh(n)$，其中$h(x)$是连续变量$x>0$的缓变函数，

(2) 当$n\to\infty$时对任一对满足如下条件的序列$p=p(n)$，$q=q(n)$

(a) $p\to\infty$，$q\to\infty$，$q=o(p)$，$p=o(n)$，

(b) 对一切$\beta>0$，$n^{1-\beta}q^{1+\beta}p^{-2}\to 0$，

(c) $np^{-1}\alpha(q)\to 0$，

且对任给$\varepsilon>0$

$$\lim_{n\to\infty}\frac{n}{p\sigma_n^2}\int_{|x|>\varepsilon\sigma_n}x^2dF_p(x)=0 \tag{3.1.1}$$

其中$F_p(x)=p(S_p<x)$.

反之，若条件(1)成立且若对某一对满足(2)中(a)—(c)的函数$p$和$q$，(3.1.1)被满足，那么中心极限定理成立.

**证**　先证(1)是必要的，由定理Ⅰ1知$h(n)$是$n$的缓变函数.设分布函数$G_n(x)=P\{Sn/\sigma_n<x\}$依分布收敛于标准正态分布$\Phi(x)$，那么对给定的$N>0$

$$\lim_{n\to\infty}\int_{|x|\leqslant N}x^2dG_n(x)=\int_{|x|\leqslant N}x^2d\Phi(x).$$

因此

$$(3.1.2)\quad \lim_{n\to\infty}\int_{|x|>N} x^2 dG_n(x) = 1 - \lim_{n\to\infty}\int_{|x|\leqslant N} x^2 dG_n(x)$$

$$= 1 - \int_{|x|\leqslant N} x^2 d\Phi(x) = \int_{|x|>N} x^2 d\Phi(x),$$

$$(3.1.3)\quad \lim_{N\to\infty}\lim_{n\to\infty}\int_{|x|>N} x^2 dG_n(x) = 0.$$

记

$$\xi = \sum_{j=1}^{n} x_j, \eta = \sum_{j=n+p}^{2n+p} x_j.$$

显然 $E\xi^2 = E\eta^2 = \sigma_n^2$. 由定理 2.1.3 的证明，我们只需验证对任给 $\varepsilon > 0$，有充分大 $p = p(\varepsilon)$ 使得

$$|E\xi\eta| \leqslant \varepsilon\sigma_n^2.$$

应用引理 1.2.1 不难导出不等式

$$|E\xi\eta| \leqslant 4N_1^2\alpha(p) + 6\sigma_n\left(\int_{|x|>N_1} x^2 dG_n(x/\sigma_n)\right)^{1/2},$$

其中 $N_1$ 是任给正数. 特别地，令 $2N_1 = \sigma_n/\alpha(p)^{1/4}$，有

$$(3.1.4)\quad |E\xi\eta| \leqslant \sigma_n^2\alpha(p)^{1/2} + 6\sigma_n^2\left\{\int_{|x|>(2\alpha(p)^{1/4})^{-1}} x^2 dG_n(x)\right\}^{1/2},$$

由 $\alpha$ 混合性及(3.1.3)，可取 $p = p(\varepsilon)$ 充分大使(3.1.4)式右边小于 $\varepsilon\sigma_n^2$. 得证(1)是必要的.

在余下的证明中总设(1)被满足. 写

$$(3.1.5)\quad S_n = \sum_{i=0}^{k-1}\xi_i + \sum_{i=0}^{k}\eta_i =: S'_n + S''_n,$$

其中

$$\xi_i = \sum_{j=i(p+q)+1}^{(i+1)p+iq} X_j, \qquad i = 0,1,\cdots,k-1,$$

$$\eta_i = \sum_{j=(i+1)p+iq+1}^{(i+1)(p+q)} X_j, \qquad i = 0,1,\cdots,k-1,$$

$$\eta_k = \sum_{j=k(p+q)+1}^{n} X_j, \qquad k = \left[\frac{n}{p+q}\right],$$

其中数 $p$、$q$ 满足(2). 记

$$Z_n = S_n/\sigma_n = S'_n/\sigma_n + S''_n/\sigma_n =: Z'_n + Z''_n.$$

我们先来说明满足(2)中(a)－(c)的数组 $p,q$ 是存在的. 为此令 $a(n)=\max\{\alpha([n^{1/4}])^{1/3},(\log n)^{-1}\}$,

$$p = \max\left\{\left[\frac{n\alpha([n^{1/4}])}{a(n)}\right],\left[\frac{n^{3/4}}{a(n)}\right]\right\}, q = [n^{1/4}],$$

那么当 $n\to\infty$ 时就有

(a) $p\to\infty, q\to\infty, p=o(n), q=o(p)$,

(b) 对任一 $\beta>0, n^{1+\beta}q^{1-\beta}p^{-2}=O(n^{-\frac{1-3\beta}{4}})=o(1)$,

(c) $\frac{n}{p}\alpha(q)\leqslant\frac{a(n)}{\alpha([n^{1/4}])}\alpha([n^{1/4}])\to 0.$

现在来证 $E(Z''_n)^2\to 0$. 事实上

$$(3.1.6)\qquad E(Z''_n)^2 = \frac{1}{\sigma_n^2}\sum_{0\leqslant i,j\leqslant k-1}E\eta_i\eta_j + \frac{2}{\sigma_n^2}\sum_{i=0}^{k-1}E\eta_i\eta_k + \frac{1}{\sigma_n^2}E\eta_k^2$$
$$\leqslant (k^2\sigma_q^2 + 2k\sigma_q\sigma_{q'} + \sigma_{q'}^2)/\sigma_n^2,$$

其中 $q'$ 是 $\eta_k$ 中 $x_j$ 的项数, 易知 $q'\leqslant p+q$, 对(3.1.6)式右边各项, 由缓变函数性质及(a),(b),我们有

$$\frac{k^2\sigma_q^2}{\sigma_n^2}\leqslant\frac{n^2}{p^2}\frac{qh(q)}{nh(n)} = \left(\frac{q}{n}\right)^{\beta}\frac{h(n\cdot q/n)}{h(n)}\cdot\frac{n^{1+\beta}q^{1-\beta}}{p^2}\to 0,$$

$$\frac{\sigma_{q'}^2}{\sigma_n^2} = \frac{q'}{n}\cdot\frac{h(n\cdot q'/n)}{h(n)}\to 0, \frac{k\sigma_q\sigma_{q'}}{\sigma_n^2} = \frac{k\sigma_q}{\sigma_n}\cdot\frac{\sigma_{q'}}{\sigma_n}\to 0.$$

即得证 $E(Z''_n)^2\to 0$, 所以 $Z''_n\to O(P.)$ 从而 $Z_n$ 与 $Z'_n$ 有相同的极限分布.

记 $\xi_0/\sigma_n$ 的特征函数为 $\varphi_n(t)$, 由条件(c)我们有

$$\left|Ee^{itZ'_n} - \varphi_n^k(t)\right| = \left|Ee^{it\sum_{j=0}^{k-1}\xi_j/\sigma_n} - \prod_{j=0}^{k-1}Ee^{it\xi_j/\sigma_n}\right|$$
$$\leqslant\sum_{l=1}^{k-1}\left|Ee^{it\sum_{j=0}^{k-l}\xi_j/\sigma_n} - Ee^{it\sum_{j=0}^{k-l-1}\xi_j/\sigma_n}\cdot Ee^{it\xi_{k-l}/\sigma_n}\right|$$
$$\leqslant 16k\alpha(q)\to 0.$$

设 $\{\xi'_k\}$ 是相互独立同分布随机变量, 且 $\xi'_k$ 与 $\xi_k$ 同分布. 上式指出 $Z'n$ 与 $\{\xi'_k\}$ 的和有相同的极限分布, 而 $\{\xi'_k\}$ 服从中心极限定理

的充分必要条件是 Lindeberg 条件成立，即

$$g_k(\epsilon)=\frac{k}{\sigma_n^2}\int_{|x|>\epsilon\sigma_n}x^2dF_p(x)$$

$$\sim\frac{n}{p\sigma_n^2}\int_{|x|>\epsilon\sigma_n}x^2dF_p(x)\to 0.$$

定理 3.1.1 证毕．

关于中心极限定理的较简单的充分必要条件由 Denker (1985)给出．

**定理 3.1.2** 设 $EX_1=0, EX_1^2<\infty$ 且 $\sigma_n^2\to\infty(n\to\infty)$．那么 $\{X_n\}$ 服从中心极限定理的充要条件是 $\{S_n^2/\sigma_n^2, n\geqslant 1\}$ 是一致可积的．

**证** 条件必要 假设

$$S_n/\sigma_n\to N(0,1)$$

其中 $N(0,1)$ 是服从标准正态分布的随机变量．所以对任给 $\epsilon>0$，存在 $k>0$ 使得

$$\lim_{n\to\infty}\int_{|S_n/\sigma_n|>k}S_n^2/\sigma_n^2dp=\int_{|N|>k}N^2dp<\epsilon,$$

上式蕴含了 $\{S_n^2/\sigma_n^2, n\geqslant 1\}$ 的一致可积性．

条件充分 假设 $\{S_n^2/\sigma_n^2, n\geqslant 1\}$ 一致可积．由定理 2.1.3，我们有

$$\sigma_n^2=nh(n),$$

且 $h(x)$ 是 $(1,\infty)$ 上的缓变函数．假设 $p$ 和 $q$ 是满足定理 3.1.1 的条件(a)，(b)的函数．进一步可选 $p$、$q$ 满足定理 3.1.1 的(c)且使

(3.1.7) $\quad n^2p^{-2}\sigma_q^2\sigma_p^{-2}=n^2p^{-3}qh(q)/h(p)\to 0, n\to\infty.$

由 $\{S_n^2/\sigma_n^2, n\geqslant 1\}$ 一致可积，对任给 $\delta>0$ 存在 $K>0$ 使对每一 $n\geqslant 1$ 有

$$\int_{|S_n|>k\sigma_n}S_n^2dP>\delta\sigma_n^2. \tag{3.1.8}$$

由缓变函数性质 $A4$，当 $n\to\infty$ 时

$$\frac{\sigma_p^2}{\sigma_n^2}=\frac{ph(p)}{nh(n)}=o(1).$$

由此即得对所有充分大的 $n$ 有 $K\sigma_p<\varepsilon\sigma_n$. 因此

$$\frac{n}{p}\int_{|S_p|>\varepsilon\sigma_n}S_p^2dP\leqslant\frac{n}{p}\int_{|S_p|>K\sigma_p}S_p^2dP<\frac{n}{p}\delta\sigma_p^2<2\delta\sigma_n^2,$$

其中最后的不等式是由于(Ⅱ4),(3.1.7)及 $2(k+\varepsilon)>n/p$. 故得(3.1.1)成立. 由定理 3.1.1 得证条件充分.

如所周知 $\{E|S_n/\sigma_n|^{2+\delta},n\geqslant1\}$ 的有界性蕴含着 $\{S_n^2/\sigma_n^2,n\geqslant1\}$ 的一致可积性. 前者要求较高阶矩的存在性. 这说明附加在 $\{S_n\}$ 上的矩条件对于中心极限定理是很重要的. 下述有关结果是由 Dehling, Denker 和 Philipp(1986)给出的

**定理 3.1.3** 设 $EX_1=0,EX_1^2=1,\sigma_n^2=nh(n)$, 其中 $h(n)$ 是缓变函数, 那么 $\{S_n/\sigma_n,n\geqslant1\}$ 依分布收敛于 $\Phi(x)$ 的充分必要条件是

(3.1.9) $$\limsup_{n\to\infty}\sigma_n/E|S_n|\leqslant\sqrt{\pi/2}.$$

为证这一定理, 我们需要下述引理.

首先, 我们引入一些记号, 设整数 $p$ 和实数 $g$ 满足:

$$2\leqslant g\leqslant\alpha^{-1/4}(\sigma_p^{1/4})\wedge\sigma_p^{1/4},$$

其中 $\alpha(x)=\alpha([x])$ 且设

(3.1.10) $$v^2=\sigma_p^{-2}\int_{g^{1/2}<|S_p|/\sigma_p\leqslant g}S_p^2dP,$$

(3.1.11) $$u^2=\int_{|S_p|\leqslant g\sigma_p}S_p^2dP,$$

(3.1.12) $$r=[g^2d],$$

其中 $d$ 满足 $2g^{-1/2}\vee v^2\leqslant d<1$,

(3.1.13) $$n=r(p+[\sigma_p^{1/4}]),\tau^2=ru^2.$$

**引理 3.1.1** 假设 $EX_1=0,EX_1^2=1$, 我们有

$$\begin{aligned}&|E\exp(itS_n/\tau)-\exp(-t^2/2)|\\&\leqslant2d+|t|\sigma_pd^{1/2}/u+|t|^3\sigma_p/(ug^{1/4})+|t|^3\sigma_p^3v/u^3\\&\quad+4\alpha^{1/2}(\sigma_p^{1/4})+t^4/g+t^2\sigma_p^2/(u^2g).\end{aligned}$$

**证** 注意到 $u\leqslant\sigma_p$. 因此对 $|t|>r^{1/2}$, 我们有

$$|t|^3\sigma_p/(ug^{1/4})\geqslant r^{3/2}/g^{1/4}\geqslant((g^2d-1)g^{-1/6})^{3/2}\geqslant g^2.$$

那么引理的结论显然成立.

现在我们假设$|t|\leqslant r^{1/2}$. 设$q=[\sigma_p^{1/4}]$且

$$\xi_j=\sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q}X_i,\ \eta_j=\sum_{i=jp+(j-1)q+1}^{j(p+q)}X_i,\qquad j=1,\cdots,r.$$

回顾$n$的定义，我们可写

$$S_n=\sum_{j=1}^{r}\xi_j+\sum_{j=1}^{r}\eta_j=:S_n'+S_n''.$$

由 Minkowski 不等式

$$ES_n''^2\leqslant r^2\sigma_p^{1/2}\leqslant g^4\sigma_p^{1/2}\leqslant\sigma_p^{3/2}.$$

因此

(3.1.14) $$|E\exp(itS_n/\tau)-E\exp(itS_n'/\tau)|$$
$$\leqslant|E\exp(itS_n''/\tau)-1|$$
$$\leqslant t^2ES''^2_n/\tau^2\leqslant t^2\sigma_p^{3/2}/(u^2r)$$
$$\leqslant t^2\sigma_p^2/(u^2g).$$

此外，由引理 1.2.1 我们有

(3.1.15) $$|E\exp(itS'_n/\tau)-(E\exp(itS_p/\tau))^r|$$
$$\leqslant 4r\alpha(\sigma_p^{1/4})\leqslant 4\alpha^{1/2}(\sigma_p^{1/4}).$$

现在我们来估计$|E\exp(itS_p/\tau)-(1-t^2/(2r))|$. 由 Chebyshev 不等式我们有

(3.1.16) $$\left|\int_{|S_p|>g\sigma_p}\exp(itS_p/\tau)dP\right|\leqslant g^{-2}\leqslant d/r.$$

由 Taylor 定理

(3.1.17) $$\left|\int_{|S_p|\leqslant g\sigma_p}\exp(itS_p/\tau)dP-(1-t^2/(2r))\right|$$
$$\leqslant\left|P(|S_p|\leqslant g\sigma_p)+\frac{it}{\tau}\int_{|S_p|\leqslant g\sigma_p}S_pdP\right.$$
$$\left.-\frac{t^2}{2\tau^2}\int_{|S_p|\leqslant g\sigma_p}S_p^2dP-\left(1-\frac{t^2}{2r}\right)\right|$$
$$+\frac{|t|^3}{\tau^3}\int_{|S_p|\leqslant g\sigma_p}|S_p|^3dP.$$

类似于(3.1.16)，

(3.1.18) $$|1-P(|S_p|\leqslant g\sigma_p)|\leqslant g^{-2}\leqslant d/r.$$

注意到 $ES_p=0$，我们得

$$(3.1.19)\qquad \tau^{-1}\left|\int_{|S_p|\leqslant g\sigma_p} S_p dP\right| = \tau^{-1}\left|\int_{|S_p|>g\sigma_p} S_p dP\right| \leqslant \sigma_p/\tau g \leqslant \sigma_p d^{1/2}/(ru).$$

显然地

$$(3.1.20)\qquad \frac{t^2}{2\tau^2}\int_{|S_p|\leqslant g\sigma_p} S_p^2 dP = \frac{t^2}{2r}.$$

(3.1.17)中的三次项估计如下：

$$(3.1.21)\qquad \tau^{-3}\int_{g^{1/2}\sigma_p<|S_p|\leqslant g\sigma_p} |S_p|^3 dP \leqslant \tau^{-3} g\sigma_p^3 v^2 \leqslant \sigma_p^3 v/(ru^3)$$

且

$$(3.1.22)\qquad \tau^{-3}\int_{|S_p|\leqslant g^{1/2}\sigma_p} |S_p|^3 dP \leqslant \tau^{-3} g^{1/2}\sigma_p u^2 \leqslant \sigma_p g^{1/2}/(ur^{3/2}) \leqslant \sigma_p/(rug^{1/4}).$$

因此，把(3.1.18)－(3.1.22)代入于(3.1.17)中，由(3.1.16)得

$$|E\exp(itS_p/\tau) - (1 - t^2/(2r))| \leqslant \eta/r,$$

其中

$$\eta = 2d + |t|\sigma_p d^{1/2}/u + |t|^3\sigma_p/(ug^{1/4}) + |t|^3\sigma_p^3 v/u^3.$$

注意到当 $|a|\leqslant 1$，$|b|\leqslant 1$ 时，$|a^r-b^r|\leqslant r|a-b|$. 对 $|t|<r^{1/2}$ 从(3.1.15)我们有

$$(3.1.23)\qquad |E\exp(itS_n'/\tau)-(1-t^2/(2r))^r|\leqslant \eta+4\alpha^{1/2}(\sigma_p^{1/4}).$$

此外，当 $|x|<1/2$ 时 $|e^x-1-x|\leqslant x^2$，故对 $|t|<r^{1/2}$，

$$|\exp(-t^2/2) - (1 - t^2/(2r))^r| \leqslant \frac{1}{4}t^4 r^{-1}.$$

因此从(3.1.23)和(3.1.14)得证引理成立．

**定理 3.1.3 的证明**　条件必要．若 $S_n/\sigma_n$ 依分布收敛于 $\Phi(x)$，那么对任一 $a>0$

$$\liminf_{n\to\infty} E|S_n|/\sigma_n \geqslant \int_{-a}^{a} \frac{|x|}{\sqrt{2\pi}} e^{-x^2/2} dx,$$

由此即得(3.1.9).

条件充分　设 $\rho_n = \sqrt{\pi/2}E|S_n|$. 若我们能证明

$$(3.1.24) \qquad S_n/\rho_n \xrightarrow{d} N(0,1) \quad n\to\infty,$$

那么由(3.1.9)对任一 $a>0$

$$\int_{|N|\leqslant a} N^2 dP = \lim_{n\to\infty} \rho_n^{-2} \int_{|S_n|/\rho_n \leqslant a} S_n^2 dP \leqslant \limsup_{n\to\infty} \sigma_n^2/\rho_n^2 \leqslant 1.$$

令 $a\to\infty$ 即得当 $n\to\infty$ 时 $\sigma_n/\rho_n \to 1$. 因此由(3.1.24)就得 $S_n/\sigma_n \xrightarrow{d} N(0,1)$.

现在我们来证(3.1.24). 首先,我们来证明存在一无穷序列 $Q\subset N$ 和实数 $\tau_n, n\in Q$ 使得

$$(3.1.25) \qquad S_n/\tau_n \xrightarrow{d} N(0,1) \quad n\to\infty, n\in Q.$$

为此,我们来证存在序列 $\{g(p),p\geqslant 1\}$ 和单调数列 $\{c(p),p\geqslant 1\}$,具有下列性质:

$$(3.1.26) \qquad g(p)\to\infty, c(p)\to 0, p\to\infty,$$

$$(3.1.27) \qquad g(p)\leqslant a^{-1/4}(\sigma_p^{1/4})\wedge \sigma_p^{1/4};$$

$$(3.1.28) \qquad v^2(p) := \sigma_p^{-2} \int_{g(p)^{1/2} < |S_p|/\sigma_p \leqslant g(p)} S_p^2 dP \to 0 \quad p\to\infty$$

和

$$(3.1.29) \qquad 2g(p)^{-1/2} \vee v^2(p) \leqslant c(p) < 1.$$

首先我们选数列 $\{z(p),p\geqslant 1\}$ 满足

$$(3.1.30) \qquad \lim_{p\to\infty} z(p) = \infty, \ z(p) \leqslant a^{-1/4}(\sigma_p^{1/4}) \wedge \sigma_p^{1/4}.$$

其次,我们选一数列 $\{i(p),p\geqslant 1\}$ 使得

$$(3.1.31) \qquad i(p)\to\infty, \ 2^{-i(p)}\log z(p)\to\infty, \ p\to\infty.$$

对固定的 $p\in N$. 因区间 $I_i(p) := (z(p)^{2^{i-1}}, z(p)^{2^i}], 0\leqslant i<i(p)$ 是互不相交的,故存在整数 $k=k(p), 0\leqslant k\leqslant i(p)$ 使得

(3.1.32) $$\sigma_p^{-2}\int_{|S_p|/\sigma_p\in I_k(p)} S_p^2 dP \leqslant 1/i(p).$$

设

(3.1.33) $$g(p) = z(p)^{2-k(p)}.$$

那么由(3.1.31)，当 $p\to\infty$ 时 $g(p)\to\infty$. 因为(3.1.30)—(3.1.32)，(3.1.27)和(3.1.28)被满足. 由于当 $p\to\infty$ 时 $2g(p)^{-1/2}\vee v^2(p)\to 0$，我们可选 $\{c(p)\}$ 使 $c(p)\downarrow 0$ 且满足(3.1.29).

对这样选取的 $\{g(p)\}$ 和 $\{c(p)\}$，我们分别地由(3.1.11)，(3.1.12)和(3.1.13)来定义 $u(p)$，$r(p)$ 和 $n(p)$. 令 $Q=\{n(p), p\geqslant 1\}$ 并由(3.1.13)定义 $\tau_n^2, n\in Q$. 由于

(3.1.34) $$\begin{aligned}\sigma_p^{-1}E|S_p| &= \sigma_p^{-1}\int_{|S_p|/\sigma_p\leqslant g(p)} |S_p| dP \\ &\quad + \sigma_p^{-1}\int_{|S_p|/\sigma_p> g(p)} |S_p| dP \\ &\leqslant \sigma_p^{-1}u(p) + g(p)^{-1},\end{aligned}$$

由(3.1.9)，对充分大的 $p$ 我们有

(3.1.35) $$u(p)/\sigma_p \geqslant \gamma/2,$$

其中 $\gamma = \inf\{E|S_p|/\sigma_p, p\geqslant 1\} > 0$. 这样从引理3.1.1可得(3.1.24)成立.

其次我们来证

(3.1.36) $$\lim_{\substack{n\to\infty\\ n\in Q}} \tau_n/\rho_n = 1.$$

为此我们取一数列 $\{b(m), m\geqslant 1\}$ 满足：

(3.1.37) $$\lim_{m\to\infty} b(m) = \infty,$$

$$\lim_{m\to\infty}\sup\{|h(tm)/h(m) - 1|, 1\leqslant t\leqslant b(m)\} = 0.$$

这是可能的. 事实上，由 Karamata 定理(见附录定理 A1)，存在一个增加的数列 $\{m_k, k\geqslant 2\}$ 使得

$$\left|\sup_{1\leqslant t\leqslant k} h(tm)/h(m) - 1\right| \leqslant \frac{1}{k}, \quad m\geqslant m_k.$$

那么由 $b(m)=k$，当 $m_k<m\leqslant m_{k+1}$，定义 $b(\cdot)$，它具有所要求的性质．当然，我们可设所选的 $\{z(p),p\geqslant 1\}$ 具有性质(3.1.30)且 $z(p)\leqslant b(p)^{1/2}/2$. 那么由(3.1.13)，$\sigma_p^2\leqslant p^2$ 和(3.1.37)，对充分大的 $p$ 有

$$\frac{\sigma^2(n(p))}{r(p)\sigma_p^2}=\frac{r(p)(p+[\sigma_p^{1/4}])h(r(p)(p+[\sigma_p^{1/4}]))}{r(p)ph(p)}$$

$$=(1+O(p^{-1/2}))\frac{h(r(p)(p+[\sigma_p^{1/4}]))}{h(p)}=1+o(1).$$

这样，由(3.1.13)和(3.1.34)对充分大的 $p$ 有

(3.1.38) $\quad E(S_{n(p)}/\tau_{n(p)})^2=\sigma_{n(p)}^2/\tau_{n(p)}^2\leqslant 2\sigma_p^2/u(p)^2\leqslant 8/\gamma^2.$

因此 $\{S_n/\tau_n,n\geqslant 1\}$ 是一致可积的，这样由(3.1.25)得

$$\lim_{\substack{n\to\infty\\ n\in Q}}E|S_n|/\tau_n=E|N|=\sqrt{2/\pi}.$$

这就证明了(3.1.36).

若 $Q=\mathbb{N}$，(3.1.25)和(3.1.36)蕴含着

(3.1.39) $\qquad S_n/\rho_n\xrightarrow{d}N(0,1),\qquad n\to\infty.$

观察 $Q\subset\mathbb{N}$ 情形. 假设 $\{z(p),p\geqslant 1\}$ 如(3.1.25)的证明中，满足

(3.1.40) $\quad z(p)\leqslant\min(\alpha(\sigma_p^{1/4})^{1/16},\sigma_p^{1/16},p^{1/4},b(p)^{1/8}/2)$

和

(3.1.41) $\qquad z(p)\leqslant z(q)\leqslant z(p)^{3/2},p\leqslant q\leqslant p^2.$

这样的数列可如下构造：先取一满足(3.1.40)的增加数列 $y(p)$. 对 $p=p_k=2^{2^k},p_k+1,\cdots,p_{k+1}-1$ 关于 $k$ 归纳地定义 $z(p)=y(p_k)\wedge z(p_{k-1})^{3/2}$. 设 $\{l(n),n\in Q\}$ 和 $\{j(n),n\in Q\}$ 是任两个趋于∞的实数列且 $l(n)<j(n),n\in Q$.

回顾条件(3.1.5)并注意到 $E|S_n|\leqslant\sigma_n$. 我们有

(3.1.42) $\qquad \sqrt{2/\pi}\leqslant\limsup_{n\to\infty}\sigma_n/\rho_n\leqslant 1.$

其次，我们已证 $S_n/\rho_n\xrightarrow{d}N(0,1),n\to\infty,n\in Q$. 由此即得对任给

$\alpha>0$

$$\int_{|N|\leqslant\alpha}N^2dP=\lim_{n\in Q}\int_{|S_n|/\rho_n\leqslant\alpha}S_n^2/\rho_n^2dP$$

$$\leqslant\liminf_{n\in Q}\rho_n^{-2}\int_{|S_n|/\sigma_n\leqslant l(n)}S_n^2dP,$$

让 $\alpha\to\infty$ 就得

(3.1.43)
$$1\leqslant\liminf_{n\in Q}\rho_n^{-2}\int_{|S_n|/\sigma_n\leqslant l(n)}S_n^2dP$$

$$\leqslant\limsup_{n\in Q}\rho_n^{-2}\int_{|S_n|/\sigma_n\leqslant j(n)}S_n^2dP\leqslant 1.$$

(3.1.42)和(3.1.43)蕴含着

(3.1.44)
$$\lim_{n\in Q}\sigma_n^{-2}\int_{l(n)<|S_n|/\sigma_n\leqslant j(n)}S_n^2dP=0.$$

为再次应用引理 3.1.1，对 $n=n(p),p\geqslant1$，令

(3.1.45)
$$l(n)=l(n(p))=z(n(p)),$$

(3.1.46)
$$j(n)=j(n(p))=\min(\alpha^{-1/4}(\sigma_n^{1/4}),\sigma_n^{1/4},b(n)^{1/2}/2).$$

由(3.1.26)，(3.1.33)和(3.1.40)，对充分大的 $p$ 我们有

(3.1.47)
$$n(p)\leqslant g^2(p)c(p)(p+p^{1/4})\leqslant z^2(p)p\leqslant p^{1/2}p<p^2.$$

由(3.1.40)，(3.1.46)和(3.1.45)有

(3.1.48)
$$j(n)\geqslant z^4(n)=l^4(n)>l(n),n\in Q.$$

由此，从(3.1.45)

(3.1.49)
$$w^2(n):=\sigma_n^{-2}\int_{l^{1/2}(n)<|S_n|/\sigma_n\leqslant j(n)}S_n^2dP\to0,n\in Q,$$

我们可选一非增数列$\{d(n),n\in Q\}$使得

(3.1.50)
$$\lim_{n\in Q}d(n)=0,\quad d(n)\geqslant2h(n)^{-1/2}\wedge w^2(n)\ ,n\in Q.$$

设 $Q=\{n_k,k\geqslant1\}$ 是按增加顺序的一个排列且设 $J_k$ 是区间

$$J_k=[n_kl^2(n_k)d(n_k),n_kj^2(n_k)d(n_k)].$$

我们来证存在 $k_0$ 使

(3.1.51)
$$J_k\cap J_{k+1}\neq\phi\ k\geqslant k_0.$$

显然 $n(n_{k+1})=r(n_k)(n_k+[\sigma(n_k)^{1/4}])\in Q$. $n_{k+1}$是 $Q$ 中大于 $n_k$ 的最

小数，对充分大 $k$ 由(3.1.47)有

$$n_{k+1}\leqslant n(n_k)\leqslant n_k z^2(n_k)<n_k^2.$$

因此，由(3.1.45)和(3.1.41)，对充分大 $k$，$J_{k+1}$的左端点不超过

$$\begin{aligned}n_{k+1}l^2(n_{k+1})d(n_{k+1})&\leqslant n_k z^2(n_k)z^2(n_{k+1})\\&\leqslant n_k z^2(n_k)z^2(n_k^2)\leqslant n_k z^5(n_k).\end{aligned}$$

另一方面，由(3.1.50)和(3.1.48)，对充分大 $k$，$J_k$ 的右端点大于

$$n_k j^2(n_k)d(n_k)\geqslant n_k j^2(n_k)l^{-1/2}(n_k)\geqslant n_k z(n_k)^{15/2}.$$

因 $z(n_k)\to\infty$，我们得(3.1.51)．令 $m\geqslant\min\{l,l\in J_{k_0}\}$．那么存在 $k\geqslant k_0$使 $m\in J_k$. 这样对某 $g\in[l(n_k),j(n_k)]$和某$|\theta|\leqslant 2$ 我们有

$$\begin{aligned}(3.1.52)\qquad m&=g^2d(n_k)n_k=[g^2d(n_k)](n_k+[\sigma^{1/4}(n_k)])+\theta n_k\\&=:M_k+\theta n_k.\end{aligned}$$

现在由(3.1.12)，$M_k$ 具有(3.1.13)形式，因此我们可应用引理3.1.1. 令 $p=n_k,d=d(n_k)$. 由(3.1.35)，$u(n_k)/\sigma(n_k)\geqslant\gamma/2>0$. 现在 $g\geqslant l(n_k)\to\infty$ 且 $\alpha(\sigma^{1/4}(n_k))\to 0$. 最后，由(3.1.49)并注意到$l(n_k)\leqslant g\leqslant j(n_k)$，我们有

$$v^2(n_k):=\sigma^{-2}(n_k)\int_{g^{1/2}<|S_{n_k}|/\sigma(n_k)\leqslant g}S_{n_k}^2dP\leqslant w^2(n_k)\to 0.$$

因此，由引理 3.1.1

$$(3.1.53)\qquad S_{M_k}/\tau(M_k)\xrightarrow{d}N(0,1).$$

由于$|\theta|\leqslant 2$，由(3.1.37)对充分大 $k$ 有

$$\frac{ES^2_{|\theta|n_k}}{\sigma^2(n_k)}\leqslant\frac{|\theta|n_kh(|\theta|n_k)}{n_kh(n_k)}\leqslant 4.$$

由此推得

$$(3.1.54)\qquad E(S_m-S_{M_k})^2=ES^2_{|\theta|n_k}\leqslant 4\sigma^2(n_k).$$

记 $r^*(n_k)=[g^2d(n_k)]$，由(3.1.37)对充分大 $k$ 我们得

$$\frac{\sigma^2(M_k)}{r^*(n_k)\sigma^2(n_k)}\geqslant\frac{r^*(n_k)n_kh(r^*(n_k)(n_k+[\sigma^{1/4}(n_k)]))}{r^*(n_k)n_kh(n_k)}\geqslant\frac{1}{2}.$$

由于由(3.1.45)有 $r^*(n_k)\leqslant g^2d(n_k)\leqslant j^2(n_k)d(n_k)\leqslant b(n_k)/2$，因此

从(3.1.54)，当 $r^*(n_k)\to\infty$ 时我们有

$$E(S_m - S_{M_k})^2/\sigma^2(M_k) \to 0, k \to \infty. \tag{3.1.55}$$

同样如(3.1.38)我们可证

$$\sigma^2(M_k)/\tau^2(M_k) \leqslant 8/r^2. \tag{3.1.56}$$

当 $m$ 和 $M_k$ 如(3.1.51)时，令 $\tau_m = \tau(M_k)$，那么由(3.1.53)，(3.1.55)和(3.1.56)有

$$S_m/\tau_m \xrightarrow{d} N(0,1). \tag{3.1.57}$$

由于(3.1.55)和(3.1.56)，序列 $\{S_m/\tau_m, m\geqslant 1\}$ 是一致可积的，从(3.1.57)和(3.1.36)得(3.1.39)成立．

借此类似于(3.1.43)我们有 $\sigma_n/\rho_n\to 1(n\to\infty)$，定理 3.1.3 证毕．

## §3.2 中心极限定理及弱不变原理的充分条件

在上节中，我们已给出了关于中心极限定理的若干充分必要条件，但它们都不易验证．在本节中，我们将给出关于中心极限定理和弱不变原理成立的若干充分条件．Rosenblatt(1956)首先给出了关于中心极限定理的充分条件．自此以后，许多作者(如 Ibragimov 等)讨论了这一课题且获得了若干进一步结果．最佳结果是属于 Herrndorf(1984,1985)的．下述定理是 Gordin(1969)的贡献，也被 Hall 和 Heyde(1980)(推论 5.3(ii))再次提出和证明，他们通过一自然相关的平稳遍历鞅差来逼近 $S_n$，这些将不在此详细陈述。

**定理 3.2.1** 设 $EX_1=0$，$Z|X_1|^{2+\delta}<\infty$(某 $\delta>0$)且

$$\sum_{n=1}^{\infty}\alpha(n)^{\delta/(2+\delta)} < \infty. \tag{3.2.1}$$

那么

$$\sigma^2 := EX_1 + 2\sum_{j=2}^{\infty}EX_1X_j < \infty. \tag{3.2.2}$$

且当 $\sigma\neq 0$ 时

$$S_n/\sigma\sqrt{n}\xrightarrow{d}N(0,1). \tag{3.2.3}$$

**注 3.2.1** 对有界变量情形，即 $\delta=\infty$ 时，条件(3.2.1)归结为

$$\sum_{n=1}^{\infty}\alpha(n)<\infty. \tag{3.2.4}$$

**注 3.2.2** Davydov(1973)给出了两个例子，指出在某种意义下定理 3.2.1 中 $\alpha(n)$ 的速度不能改进.

**例 3.2.1** 对任一 $\delta>0$ 和 $\epsilon>0$，存在一强平稳可列状态马氏链 $\{X_n,n\geqslant 1\}$，$EX_1=0$，$E|X_1|^{2+\delta}<\infty$ 使得

(i) $\alpha(n)=o(n^{-(1-\epsilon)(1+2/\delta)})\quad(n\to\infty)$，

(ii)[1] 对某 $1<d<2$，$\mathrm{Var}S_n\approx n^d$，

(iii) $S_n$ 吸引于指数为 $\alpha$，$1<\alpha<2$ 的对称稳定律.

**例 3.2.2** 对任给 $\epsilon>0$，存在一强平稳可列状态马氏链 $\{X_n,n\geqslant 1\}$，$EX_1=0$，$|X_1|<C_0$ a.s.（对某 $C_0<\infty$），$\alpha(n)=o(n^{-(1-\epsilon)})$ $(n\to\infty)$ 且例 3.2.1 中的性质(ii)和(iii)成立.

现在我们来研究弱不变原理. 定义 $D[0,1]$ 上随机元如下：

$$W_n(t)=S_{[nt]}/\sigma_n,\qquad 0\leqslant t\leqslant 1.$$

概率测度弱收敛理论告诉我们证明弱不变原理的关键是胎紧性(tightness)的验证. 下述条件之一关于胎紧性是充分的（见 Billinsley1968 §16).

(1) 对任给的 $\epsilon>0,\eta>0$，存在 $\delta$，$0<\delta<1$，和整数 $n_0$ 使对 $0\leqslant t\leqslant 1$，当 $n\geqslant n_0$ 时有

$$\frac{1}{\delta}P\{\sup_{t\leqslant s\leqslant t+\delta}|W_n(s)-W_n(t)|\geqslant\epsilon\}\leqslant\eta; \tag{3.2.5}$$

(2) 对任给的 $\epsilon>0$，存在 $\lambda>1$ 和整数 $n_0$ 使当 $n\geqslant n_0$ 时

$$P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant\lambda\sigma_n\}\leqslant\epsilon/\lambda^2. \tag{3.2.6}$$

Davydov(1968) 首先对有界变量推广中心极限定理到弱不变原

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1) $a\approx b$ 是指 $\lim a/b=1$.

理，但条件(3.2.4)被加强为 $\sum_{n=1}^{\infty}\alpha^{1/2}(n)<\infty$. Oodaira 和 Yoshihara(1972)改进了他的结论，在关于中心极限定理的基本条件下获得了不变原理.

**定理 3.2.2** 设 $EX_1=0$, $|X_i|<C_0<\infty$. 若(3.2.4)被满足且 $\alpha(n)\leqslant c/n\log n$. 那么若 $\sigma>0$，记 $\sigma_n=\sigma\sqrt{n}$，有

$$W_n\Rightarrow W.$$

**证** 由注 3.2.1 即得对任一 $t$, $0\leqslant t\leqslant 1$, $W_n(t)$ 依分布收敛于 $W(t)$. 由 Cramer-Wold 方法，容易看到对任给 $0\leqslant t_1<t_2<\cdots<t_k\leqslant 1$, $(W_n(t_1),\cdots,W_n(t_k))$ 依分布收敛于 $(W(t_1),\cdots,W(t_n))$.

现在来证 $\{W_n\}$ 的胎紧性. 由(3.2.6)只需证明

$$P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant 3\lambda\sigma\sqrt{n}\}\leqslant \varepsilon/\lambda^2, n\geqslant n_0. \tag{3.2.7}$$

令 $p=[\sqrt{n}/(\log n)^{3/8}]$, $k=[n/p]$. 由于 $\{X_n\}$ 的有界性，对充分大 $n$ 有

$$P\{|X_1|+\cdots+|X_{2p}|\geqslant\lambda\sigma\sqrt{n}\}=0. \tag{3.2.8}$$

另外，利用引理 1.2.1 和条件(3.2.4)，得 $\{S_n^2/n, n\geqslant 1\}$ 是一致可积的. 所以对任给 $\varepsilon>0$，存在 $\lambda>1$ 使对每一 $i\geqslant 1$

$$P\{|S_i|\geqslant\lambda\sigma\sqrt{i}\}\leqslant\varepsilon/3\lambda^2. \tag{3.2.9}$$

令 $E_j=\{\max_{1\leqslant i<j}|S_i|<3\lambda\sigma\sqrt{n}\leqslant|S_j|\}$. 我们有

$$\begin{aligned}P\{\max_{1\leqslant i\leqslant n}&|S_i|\geqslant 3\lambda\sigma\sqrt{n}\}\\ &\leqslant P\{|S_n|\geqslant\lambda\sigma\sqrt{n}\}+P(\bigcup_{j=1}^{n}\{E_j\cap(|S_n\\ &\quad -S_j|\geqslant 2\lambda\sigma\sqrt{n})\})\\ &\leqslant P\{|S_n|\geqslant\lambda\sigma\sqrt{n}\}+\sum_{i=0}^{k-2}P(\bigcup_{j=1}^{p}\{E_{ip+j}\cap(|S_n\\ &\quad -S_{ip+j}|\geqslant 2\lambda\sigma\sqrt{n})\})\\ &\quad +\sum_{j=(k-1)p+1}^{n}P\{|S_n-S_j|\geqslant 2\lambda\sigma\sqrt{n}\}\end{aligned} \tag{3.2.10}$$

$$\begin{aligned}
&\leqslant P\{|S_n| \geqslant \lambda\sigma\sqrt{n}\} + \sum_{i=0}^{k-2} P(\bigcup_{j=1}^{p}\{E_{ip+j} \cap (|S_n| \\
&\quad - S_{(i+2)p}| \geqslant \lambda\sigma\sqrt{n})\}) \\
&\quad + \sum_{j=1}^{p} P\{|S_{(i+2)p} - S_{ip+j}| \geqslant \lambda\sigma\sqrt{n}\} \\
&\quad + \sum_{j=(k-1)p+1}^{n} P\{|X_1| + \cdots + |X_{n-j}| \geqslant 2\lambda\sigma\sqrt{n}\} \\
&\leqslant P\{|S_n| \geqslant \lambda\sigma\sqrt{n}\} + \sum_{i=0}^{k-2} P\{(\bigcup_{j=1}^{p} E_{ip+j}) \cap (|S_n| \\
&\quad - S_{(i+2)p}| \geqslant \lambda\sigma\sqrt{n})\} \\
&\quad + 2nP\{|X_1| + \cdots + |X_{2p}| \geqslant \lambda\sigma\sqrt{n}\} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

由(3.2.8)，$I_3 = 0$，从(3.2.9)可得 $I_1 < \epsilon/3\lambda^2$，由于 $\bigcup_{j=1}^{p} E_{ip+j} \in \mathscr{F}_0^{(i+1)p}$，$(|S_n - S_{(i+2)p}| \geqslant \lambda\sigma\sqrt{n}) \in \mathscr{F}_{(i+2)p-1}^{\infty}$，再由(3.2.9)我们得

$$\begin{aligned}
I_2 &\leqslant \sum_{i=0}^{k-2} P\{\bigcup_{j=1}^{p} E_{ip+j}\} P\{|S_n - S_{(i+2)p}| \geqslant \lambda\sigma\sqrt{n}\} + k\alpha(p) \\
&\leqslant \epsilon/3\lambda^2 + k\alpha(p).
\end{aligned}$$

从 $\alpha(n) \leqslant c/n\log n$ 即得

$$k\alpha(p) \leqslant \frac{cn}{n(\log n)^{-3/4}\log(n^{1/2}(\log n)^{-3/8})} \to 0, n \to \infty.$$

把这些估计代入(3.2.10)得证(3.2.7)，定理 3.2.2 证毕．

一些作者讨论并推广了这一定理．最一般结果由 Herrndorf(1985)给出．他去掉了平稳性假设且使矩条件变得比较灵活．记

$$\mathscr{G} = \{g(x): [0,\infty) \to [0,\infty), g(x)\text{ 是凸的}, g(0) = 0,$$
$$g(x)/x^2\text{ 是不减的}, \lim_{x\to\infty} g(x)/x^2 = \infty\}.$$

对每一 $g \in \mathscr{G}$，我们定义它的逆 inv $g$：$(0,\infty) \to (0,\infty)$ 为 $g(\text{inv } g(x)) = x$，定义 $f_g$：$[0,\infty) \to [0,\infty)$ 为 $f_g(0) = 0$ 且

$$f_g(x) = (\text{inv } g(1/x))^2 x \qquad \text{对 } x > 0.$$

**定理 3.2.3** 设 $\{X_n, n \geqslant 1\}$ 是 $\alpha$ 混合序列，对一切 $n \geqslant 1$，$EX_n$

$=0, EX_n^2<\infty$ 且当 $n\to\infty$ 时对某 $\sigma>0$

$$ES_n^2/n \to \sigma^2 \quad n\to\infty. \tag{3.2.11}$$

若存在 $g\in\mathscr{G}$ 使得

$$\sup_{n\geqslant 1} E_g(|X_n|)<\infty, \qquad \sum_{n=1}^{\infty} f_g(\alpha(n))<\infty, \tag{3.2.12}$$

那么 $W_n\Rightarrow W$.

定理 3.2.3 的证明需要下述引理.

**引理 3.2.1** 设 $\xi_1,\cdots,\xi_n$ 是随机变量，记

$$\alpha=\max_{1\leqslant k\leqslant n-1}\sup\{|P(AB)-P(A)P(B)|: A\in\sigma(\xi_1,\cdots,\xi_k), B\in\sigma(\xi_{k+1},\cdots,\xi_n)\}.$$

那么对任给 $\varepsilon>0$

$$P\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{j=1}^{k}\xi_j\right|>2\varepsilon\right\}\leqslant\frac{P\left\{\left|\sum_{j=1}^{n}\xi_j\right|>\varepsilon\right\}+n\alpha}{\min\limits_{1\leqslant k\leqslant n-1}P\left\{\left|\sum_{j=k+1}^{n}\xi_j\right|\leqslant\varepsilon\right\}}. \tag{3.2.13}$$

**证** 令

$$A_1=\{|\xi_1|>2\varepsilon\},$$

$$A_k=\left\{\left|\sum_{j=1}^{k}\xi_j\right|>2\varepsilon,\left|\sum_{j=1}^{l}\xi_j\right|\leqslant 2\varepsilon, 1\leqslant l\leqslant k-1\right\},$$
$$1<k\leqslant n$$

$$B_k=\left\{\left|\sum_{j=k-1}^{n}\xi_j\right|<\varepsilon\right\}, 1\leqslant k<n, B_n=\Omega,$$

$$C=\left\{\left|\sum_{j=1}^{n}\xi_j\right|>\varepsilon\right\}.$$

容易看出 $\bigcup\limits_{k=1}^{n}A_kB_k\subset C$ 且

$$|P(A_kB_k)-P(A_k)P(B_k)|\leqslant\alpha.$$

因此

$$P(C)\geqslant\sum_{k=1}^{n}P(A_kB_k)\geqslant\min_{1\leqslant k\leqslant n}P(B_k)\sum_{k=1}^{n}P(A_k)-n\alpha. \tag{3.2.14}$$

注意到

$$P\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{j=1}^{k}\xi_j\right|>2\varepsilon\right\}=\sum_{k=1}^{n}P(A_k).$$

由它结合(3.2.14)即得(3.2.13).

**引理 3.2.2** 设$\{X_n,n\geqslant 1\}$如定理 3.2.3 且还满足

(3.2.15) $$\sup_{n\geqslant 1,m\geqslant 0}E(S_{m+n}-S_m)^2/n<\infty.$$

假设存在正整数 $p=p(n),q=q(n)$使当 $n\to\infty$时

(3.2.16) $$p=o(n),q=o(p),np^{-1}\alpha(q)=o(1),$$

(3.2.17) $$n^{-1}\sum_{1\leqslant i,j\leqslant n,|i-j|>q}|EX_iX_j|\to 0,$$

(3.2.18) $$p^{-1}\max_{0\leqslant m\leqslant n-p}E\{(S_{m+p}-S_m)^2 \cdot I(|S_{m+p}-S_m|>\varepsilon n^{1/2})\}\to 0,\text{任意 }\varepsilon>0.$$

那么中心极限定理成立.

又若对任给 $\varepsilon>0$

(3.2.19) $$np^{-1}\max_{0\leqslant m\leqslant n-p}P\{\max_{1\leqslant r\leqslant p}|S_{m+r}-S_m|>\varepsilon\sqrt{n}\}\to 0,$$

那么对 $\sigma_n=\sigma\sqrt{n}$ 有 $W_n\Rightarrow W$.

**证** 记 $k=[n/(p+q)]$,由(3.2.16)知当 $n\to\infty$时 $k\to\infty$. 令

$$\xi_j=\sum_{i=j(p+q)+1}^{j(p+q)+p}X_i,\qquad \eta_j=\sum_{i=j(p+q)+p+1}^{(j+1)(p+q)}X_i,0\leqslant j\leqslant k-1,$$

$$\eta_k=\sum_{i=k(p+q)+1}^{n}X_i,$$

$$S'_n=\sum_{j=0}^{k-1}\xi_j,\qquad S''_n=\sum_{j=0}^{k}\eta_j.$$

为证明中心极限定理,只需证明当 $n\to\infty$时

(a) $ES''^2_n/n\to 0$,

(b) $\sum_{0\leqslant i<j\leqslant k-1}|E\xi_i\xi_j|/n\to 0$,

(c) 对$t\in(-\infty,\infty)$一致地,$\left|E\exp(itS'_n)-\prod_{j=0}^{k-1}E\exp(it\xi_j)\right|\to 0$,

(d) 对任给 $\varepsilon>0$,$\sum_{j=0}^{k-1}E(\xi_j^2I(|\xi_j|>\varepsilon\sigma n^{1/2}))/n\to 0$.

从引理 1.2.3 和(3.2.15)我们有

$$ES_n''^2 \leqslant \sum_{j=0}^{k} E\eta_j^2 + 2\sum_{i=1}^{n} \sum_{i+p<j\leqslant n} |EX_iX_j|$$

$$\leqslant c(kq + p + q) + 20n \sum_{i=p+1}^{\infty} f_g(\alpha(i)) \sup_{j\leqslant n} \| X_j \|_g^2.$$

现在从(3.2.12)和(3.2.16)得(a),同样可得(b),对(c),从引理 1.2.1 和(3.2.16)有

$$\left| E\exp(itS_n') - \prod_{j=0}^{k-1} E\exp(it\xi_j) \right| \leqslant 16k\alpha(q) \to 0 \quad n \to \infty.$$

最后从(3.2.18)即得(d).这样,对$\{X_n\}$中心极限定理成立.

设给定的 $0\leqslant t_1<\cdots<t_k\leqslant 1$,我们来证当 $n\to\infty$时

(3.2.20) $\quad (W_n(t_1),\cdots,W_n(t_k)) \xrightarrow{d} (W(t_1),\cdots,W(t_k)) \quad n\to\infty.$

从(3.2.11)和中心极限定理,对任给 $t$,$0\leqslant t\leqslant 1$,$W_n(t)$依分布收敛于 $W(t)$,所以由 Prohorov 对胎紧性的经典刻划知$\{W_n(t_1)\cdots W_n(t_k)\}$是胎紧的. 设 $Q$ 是$\{W_n(t_1),\cdots,W_n(t_k)\}$的某子列的极限分布,$\pi_{t_i}$是 $R^k$ 的点 $x=(x_1,\cdots,x_k)$到 $R$ 的点 $x_{t_i}$的投影映射.按中心极限定理,边际分布 $Q\pi_{t_i}^{-1}$是方差为 $t_i$ 的正态分布.取 $r_n=q/n$,那么 $r_n\to 0$ 且 $\alpha_n(q)\to 0$,其中

$$\alpha_n(q) = \sup\{|P(AB) - P(A)P(B)| : A \in \sigma(X_i, 1 \leqslant i \leqslant m),$$
$$B \in \sigma(X_i, m + q \leqslant i \leqslant n, 1 \leqslant m \leqslant n - q\}.$$

利用(3.2.15)我们得 $E(W_n(t_i+r_n)-W_n(t_i))^2\to 0$ $(n\to\infty)$. 因此 $Q(\pi_{t_1},\pi_{t_2}-\pi_{t_1},\cdots,\pi_{t_k}-\pi_{t_{k-1}})^{-1}$是$(W_n(t_1),W_n(t_2)-W_n(t_1+r_n),\cdots,W(t_k)-W(t_{k-1}+r_n))$的某子列的极限分布.现在 $\alpha_n([nr_n]-1)\to 0$ 可推出在$Q$下 $\pi_{t_1},\pi_{t_2}-\pi_{t_1},\cdots,\pi_{t_k}-\pi_{t_{k-1}}$是独立的,所以 $Q$ 就是$(W(t_1),\cdots,W(t_k))$的分布,这就证明了(3.2.20).

最后,我们来证明序列$\{W_n\}$的胎紧性.这只需证明(3.2.5)的如下变形,即对任给 $\varepsilon>0$,$\eta>0$ 存在 $\delta$,$0<\delta<1$,和整数 $n_0$ 使对 $n\geqslant n_0$ 有

(3.2.21) $\quad \displaystyle\sum_{k=0}^{[1/\delta]} P\{\max_{[nk\delta]<r\leqslant[n(k+1)\delta]} |S_r - S_{[nk\delta]}| > \varepsilon\sigma\sqrt{n}\} < \eta.$

设 $k\in\{0,\cdots,[1/\delta]\}$ 是固定的，$m=m(n)=[([n(k+1)\delta]-[nk\delta])/(p+q)]$，

$$\xi_j=\sum_{i=[nk\delta]+j(p+q)+1}^{[nk\delta]+j(p+q)+p} X_i,$$

$$\eta_j=\sum_{i=[nk\delta]+j(p+q)+p+1}^{[nk\delta]+(j+1)(p+q)} X_i,\quad j=0,1,\cdots,m-1.$$

那么我们有

$$(3.2.22)\qquad P\Big\{\max_{[nk\delta]<r\leqslant[n(k+1)\delta]}|S_r-S_{[nk\delta]}|>\varepsilon\sigma\sqrt{n}\Big\}$$

$$\leqslant(m+1)\max_{0\leqslant l\leqslant n-(p+q)}P\Big\{\max_{1\leqslant r\leqslant p+q}|S_{l+r}-S_l|>\varepsilon\sigma\sqrt{n}/3\Big\}$$

$$+P\Big\{\max_{0\leqslant r\leqslant m-1}\Big|\sum_{j=0}^{r}\xi_j\Big|>\varepsilon\sigma\sqrt{n}/3\Big\}$$

$$+P\Big\{\max_{0\leqslant r\leqslant m-1}\Big|\sum_{j=0}^{r}\eta_j\Big|>\varepsilon\sigma\sqrt{n}/3\Big\}$$

$$=:I_1+I_2+I_3.$$

从(3.2.19)即得当 $n\to\infty$ 时

$$I_1\leqslant cnp^{-1}\max_{0\leqslant l\leqslant n-(p+q)}P\Big\{\max_{1\leqslant r\leqslant p+q}|S_{l+r}-S_l|>\varepsilon\sigma\sqrt{n}/3\Big\}\to 0.$$

由引理 1.2.3，(3.2.15)和(3.2.12)，我们得

$$\max_{J\subset\{0,\cdots,m-1\}}E\Big(\sum_{j\in J}\eta_j\Big)^2/(\sigma^2n)$$

$$\leqslant\sum_{0\leqslant j\leqslant m-1}E\eta_j^2/(\sigma^2n)+2\sum_{1\leqslant i\leqslant n}\sum_{i+p\leqslant j\leqslant n}|EX_iX_j|/(\sigma^2n)$$

$$\leqslant cmq/(\sigma^2n)+20\sigma^{-2}\sum_{j>p}f_g(\alpha(j))\sup_{j\leqslant n}\|X_j\|_g^2\to 0.$$

类似地

$$\max_{J\subset\{0,\cdots,m-1\}}E\Big(\sum_{j\in J}\xi_j\Big)^2/(\sigma^2n)$$

$$\leqslant\sum_{0\leqslant j\leqslant m-1}E\xi_j^2/(\sigma^2n)+2\sum_{1\leqslant i\leqslant n}\sum_{i+q\leqslant j\leqslant n}|EX_iX_j|/(\sigma^2n)$$

$$\leqslant cmq/(\sigma^2n)+20\sigma^{-2}\sum_{j>q}f_g(\alpha(j))\sup_{j\leqslant n}\|X_j\|_g^2.$$

利用 $m$ 的定义和(3.2.12)，当 $n\to\infty$ 时，后一表示式右端收敛于

$c\sigma^{-2}\delta$. 选 $\delta_0(\epsilon)>0$ 使

$$(18/\epsilon)^2 c\sigma^{-2}\delta_0(\epsilon) < 1/2.$$

现设 $\delta<\delta_0(\epsilon)$. 由 Chebyshev 不等式对充分大 $n$ 我们有

$$\min_{0\leqslant r\leqslant m-2} P\left\{\left|\sum_{j=r+1}^{m-1}\xi_j\right| \leqslant \epsilon\sigma\sqrt{n}/6\right\} \geqslant \frac{1}{2}.$$

应用引理 3.2.1 得

$$I_2 \leqslant 2P\left\{\left|\sum_{j=0}^{m-1}\xi_j\right| > \epsilon\sigma\sqrt{n}/6\right\} + 2m\alpha_n(q+1).$$

从(3.2.16)可得当 $n\to\infty$时 $m\alpha_n(q+1)\to 0$, 由于

$$E\left(\sum_{j=0}^{m-1}\eta_j\right)^2/(\sigma^2 n) \to 0,$$

$$E(S_{[n(k+1)\delta]} - S_{[nk\delta]} - \sum_{j=0}^{m-1}(\xi_j+\eta_j))^2/(\sigma^2 n) \to 0 \quad n\to\infty.$$

我们得

$$\limsup_{n\to\infty} P\left\{\max_{0\leqslant r\leqslant m-1}\left|\sum_{j=0}^{r}\xi_j\right| > \epsilon\sigma\sqrt{n}/3\right\}$$

$$\leqslant 2\limsup_{n\to\infty} P\{|S_{[n(k+1)\delta]} - S_{[nk\delta]}| > \epsilon\sigma\sqrt{n}/7\}.$$

再次应用 Chebyshev 不等式和引理 3.2.1 得

$$\limsup_{n\to\infty} P\left\{\max_{0\leqslant r\leqslant m-1}\left|\sum_{j=0}^{r}\eta_j\right| > \epsilon\sigma\sqrt{n}/3\right\}$$

$$\leqslant 2\limsup_{n\to\infty} P\left\{\left|\sum_{j=0}^{m-1}\eta_j\right| > \epsilon\sigma\sqrt{n}/6\right\}$$

$$+ 2\limsup_{n\to\infty} m\alpha_n(p+1) = 0.$$

对 $k=0,\cdots,[1/\delta]$, 综合这些结果, 我们得到

$$\limsup_{n\to\infty}\sum_{k=0}^{[1/\delta]} P\left\{\max_{[nk\delta]<r\leqslant[n(k+1)\delta]}|S_r - S_{[nk\delta]}| > \epsilon\sigma\sqrt{n}\right\}$$

$$\leqslant 2\sum_{k=0}^{[1/\delta]}\limsup_{n\to\infty} P\{|S_{[n(k+1)\delta]} - S_{[nk\delta]}| > \epsilon\sigma\sqrt{n}/7\}$$

$$\leqslant 2\left(\frac{1}{\delta}+1\right)P\{|N(0,\delta)| > \epsilon/7\} \to 0 \quad \delta\to 0.$$

引理 3.2.2 证毕.

**定理 3.2.3 的证明**

假设某 $g\in\mathscr{G}$ 满足条件(3.2.12),那么 $K:=\sup\limits_{n\geqslant1}\|X_n\|_g<\infty$.

注意到引理 3.2.2 的条件(3.2.18)和(3.2.19)都可由

(3.2.23) $$p^{-1}\max_{0\leqslant m\leqslant n-p}E\Big(\Big(\sum_{i=m+1}^{m+p}|X_i|\Big)^2 I\Big(\sum_{i=m+1}^{m+p}|X_i|>\varepsilon\sqrt{n}\Big)\Big)\to 0\quad n\to\infty$$

推出. 从 $g(x)/x^2$ 的单调性, $g$ 的下凸性及当 $0<\|X\|_g<\infty$ 时 $Eg(|X|/\|X\|_g)\leqslant1$, 我们有

$$\begin{aligned}&E\Big(\Big(\sum_{i=m+1}^{m+p}|X_i|\Big)^2 I\Big(\sum_{i=m+1}^{m+p}|X_i|>\varepsilon\sqrt{n}\Big)\Big)\\&\leqslant E_g\Big(\sum_{i=m+1}^{m+p}|X_i|/Kp\Big)\varepsilon^2 n/g(\varepsilon\sqrt{n}/Kp)\\&\leqslant\varepsilon^2 n/g(\varepsilon\sqrt{n}/Kp).\end{aligned}$$

这样从

(3.2.24) $$p^{-1}n/g(\varepsilon\sqrt{n}/Kp)\to0\quad n\to\infty.$$

可推出(3.2.23). 从引理 1.2.3, 对 $m\geqslant0, n\geqslant1$ 我们有

$$E(S_{m+n}-S_m)^2\leqslant n\sup_{j\geqslant1}EX_j^2+2nK^2\sum_{k=1}^{\infty}f_g(\alpha(k)).$$

因 $\sup\limits_{j\geqslant1}EX_j^2<\infty$, 我们得(3.2.15). 现在为完成定理 3.2.3 的证明只需构作序列 $p(n), q(n)$ 使(3.2.16), (3.2.17) 和(3.2.24) 被满足. 由于 $k\to f_g(\alpha(k))$ 是非增的, 假设 $\sum\limits_{k=1}^{\infty}f_g(\alpha(k))<\infty$ 可推出当 $k\to\infty$ 时 $f_g(\alpha(k))k\to0$, 所以我们可选 $a:[0,\infty)\to[0,\infty)$ 是连续, 严格减并使当 $x\to\infty$ 时

(3.2.25) $$f_g(a(x))x\to0,$$

(3.2.26) $$a(x)\geqslant\alpha(k)\quad 对所有整数\ k\geqslant x.$$

由于 $x\to x\ \mathrm{inv}\ g(1/a(x))$ 是 $(0,\infty)$ 到 $(0,\infty)$ 上的同胚映射, 我们可由

$$x(n)\text{inv } g(1/a(x(n))) = n^{1/4}$$

定义 $x(n)\in(0,\infty)$，那么 $x(n)\to\infty(n\to\infty)$. (3.2.25)蕴含着存在 $L=L(n)\geqslant n^{-1/4}$，$L(n)\to 0(n\to\infty)$使得当 $n\to\infty$时

(3.2.27) $$L(n)^{-2}\sup_{t\geqslant x(n)}(\text{inv } g(1/a(t)))^2 a(t)t\to 0.$$

现在由

$$y(n)\text{inv } g(1/a(y(n))) = L(n)n^{1/2}$$

定义 $y=y(n)\in(0,\infty)$，显然 $y(n)\geqslant x(n)$且 $y(n)=o(n^{1/2})$. 令 $q(n)=\min\{j,j\geqslant y(n)\}$并选序列 $p(n)$使当 $n\to\infty$时

(3.2.28) $$p(n)/y(n)\to\infty,\quad p(n)/n\to 0,$$

$$L(n)p(n)/y(n)\to 0.$$

现在 $p=p(n)=o(n)$，$q=q(n)=o(p)$成立. 因 $q(n)\to\infty$，从定理的假设并应用引理 1.2.3 可得(3.2.17)被满足. 利用(3.2.26)，$y(n)$的定义和(3.2.27)我们得

(3.2.29) $$nq^{-1}\alpha(q)\leqslant ny^{-1}\alpha(q)\leqslant ny^{-1}\alpha(y)$$

$$= L^{-2}(\text{inv } g(1/a(y)))^2 a(y)y\to 0\quad n\to\infty.$$

由此和 $q=o(p)$可推得 $np^{-1}\alpha(q)=o(1)$. 从(3.2.28)，对所有充分大 $n$ 我们有 $p\geqslant y$ 和 $y\varepsilon/KpL\geqslant 1$，对这样的 $n$，由 $g$ 的下凸性和 $y(n)$的定义我们得

$$p^{-1}n/g(\varepsilon\sqrt{n}/Kp)\leqslant y^{-1}npLK/(\varepsilon yg(L\sqrt{n}/y))$$

$$= y^{-1}na(y)pLK/(\varepsilon y).$$

由(3.2.28)和(3.2.29)，上式右端趋于 0，因此(3.2.24)成立，定理 3.2.3 证毕.

下述推论立即可得. 设对某 $\delta>0$，$g(x)=x^{2+\delta}$，我们有

**推论 3.2.1** 设$\{X_n,n\geqslant 1\}$是 $\alpha$ 混合序列，$EX_n=0$，对某 $\delta>0$

$$\sup_n E|X_n|^{2+\delta}<\infty,\qquad \sum_{n=1}^{\infty}\alpha(n)^{\delta/(2+\delta)}<\infty.$$

假设条件(3.2.11)被满足，那么 $W_n\Rightarrow W$.

**推论 3.2.2** 设$\{X_n,n\geqslant 1\}$是 $\alpha$ 混合序列，$EX_n=0$. 若对某 $a>0$

(3.2.30)　$\sup_n EX_n^2|\log|X_n||^a<\infty$，　$\sum_{n=1}^{\infty}|\log\alpha(n)|^{-a}<\infty$，

特别地对某 $a>1$ 和 $b>1$

(3.2.31)　$\sup_n EX_n^2|\log|X_n||^a<\infty, \alpha(n)=O(b^{-n})$；

那么

$$W_n\Rightarrow W.$$

Herrndorf(1985)也给出了两个例子指出在(3.2.31)中 $a>1$ 不能以 $a=1$ 代替. 例子的构造不在此给出了.

**注 3.2.3**　Doukhan, Massart 和 Rio(1994)通过 $\alpha$ 混合函数 $\alpha(t)$和$|X_1|$的尾分布函数讨论了 $\alpha$ 混合序列的弱不变原理，他们给出了另一类充分条件：

设$\{X_n,n\geq1\}$是强平稳 $\alpha$ 混合序列，$EX_1=0$，且满足：

(3.2.32)　$$\int_0^1 \text{inv}\ \alpha(u)[\text{inv}\ G(u)]^2du<\infty$$

其中 $G(u)=P\{|X_1|>u\}$. 那么级数 $\sigma^2=\sum_{n=1}^{\infty}EX_1X_n$ 绝对收敛，且 $Z_n/\sigma\Rightarrow W$，其中 $Z_n(t)=\sum_{k=1}^{[nt]}X_k/\sqrt{n}$.

特别地，若$|X_1|\leq C<\infty$，条件(3.2.32)等价于(3.2.4)，故此时定理 3.2.2 是它的特例.

## §3.3　方差无穷时的中心极限定理与弱不变原理

具有无穷方差的混合序列的中心极限定理首先被林正炎(1981,1982)所讨论，他在独立情形的类似条件下证得下述定理. 设$\{X_n\}$是强平稳 $\alpha$ 混合序列

**定理 3.3.1**　假设 $EX_1=0$ 且下列条件被满足：

(i) 存在两个正整数序列 $p=p(n),q=q(n)$满足

(3.3.1)　$p=o(n), q=o(p), k\alpha(q)\to0\quad n\to\infty$,

其中 $k=k(n)=[n/(p+q)]$.

(ii) 存在常数列$\{C_n\}$,$C_n\uparrow\infty$且

$$k\int_{|X_1|>C_k}dP=o\left(\frac{1}{p}\right),\tag{3.3.2}$$

$$C_k^2/\left(k\int_{|X_1|<C_k}X_1^2dP\right)=o\left(\frac{1}{p}\right),\tag{3.3.3}$$

$$\int_{|X_1|<C_k,|X_i|<C_k}X_1X_idP\Big/\int_{|X_1|<C_k}X_1^2dP=o\left(\frac{1}{p}\right)\quad n\to\infty\tag{3.3.4}$$

关于 $i$ 一致地成立,那么$\{X_n\}$服从中心极限定理.

此外,若存在常数 $a>0$ 使 $p\leqslant k^a$ 且

$$\lim_{n\to\infty}k^{2+a}\alpha(q)=0.\tag{3.3.5}$$

那么存在常数 $B_n>0$ 使得$\{W_n(t)=S_{[nt]}/B_n,0\leqslant t\leqslant 1\}_{n\geqslant 1}$弱收敛于 $W$.

**证** 定义 $\xi_j$, $j=0,1,\cdots,k-1$ 和 $\eta_j$, $j=0,1,\cdots,k$ 如引理 3.2.2 中. 从(3.3.2)和(3.3.3)式我们得

$$C_k^2\int_{|X_1|>C_k}dP\Big/\int_{|X_1|<C_k}X_1^2dP=o(p^{-2}).\tag{3.3.6}$$

因此

$$\begin{aligned}0&\leqslant\int_{|\xi_0|<pC_k}\xi_0^2dP-\int_{\cap_{j=1}^p(|X_j|<C_k)}\xi_0^2dP\\&=\int_{(|\xi_0|<pC_k)\cap\cup_{j=1}^p(|X_j|\geqslant C_k)}\xi_0^2dP\\&\leqslant p^3C_k^2\int_{|X_1|\geqslant C_k}dP=o\left(p\int_{|X_1|<C_k}X_1^2dP\right),\end{aligned}$$

即

$$\begin{aligned}\int_{|\xi_0|<pC_k}\xi_0^2dP=\int_{\cap_{j=1}^p(|X_j|<C_k)}&\Big(\sum_{j=1}^pX_j^2\\&+2\sum_{1\leqslant i<j\leqslant p}X_iX_j\Big)dP+o\left(p\int_{|X_1|<C_k}X_1^2dP\right).\end{aligned}\tag{3.3.7}$$

进一步,再从(3.3.6)得

$$0\leqslant\int_{|X_j|<C_k}X_j^2dP-\int_{\cap_{i=1}^p(|X_i|<C_k)}X_j^2dP$$

$$= \int_{(|X_j|<C_k)\cap\bigcup_{j\neq i\leq p}(|X_i|\geq C_k)} X_j^2 dP$$

$$\leq pC_k^2\int_{|X_1|\geq C_k} dP = o\left(p^{-1}\int_{|X_1|<C_k} X_1^2 dP\right).$$

所以

$$(3.3.8)\qquad \sum_{j=1}^{p}\int_{\bigcap_{i=1}^{p}(|X_i|<C_k)} X_j^2 dP$$

$$= p\int_{|X_1|<C_k} X_1^2 dP + o\left(\int_{|X_1|<C_k} X_1^2 dP\right).$$

类似地

$$\int_{\bigcap_{j=1}^{p}(|X_j|<C_k)} X_i X_j dP = \int_{|X_i|<C_k,|X_j|<C_k} X_i X_j dP$$

$$+ o\left(p^{-1}\int_{|X_1|<C_k} X_1^2 dP\right).$$

利用条件(3.3.4),我们得

$$(3.3.9)\qquad \sum_{1\leq i<j\leq p}\int_{\bigcap_{j=1}^{p}(|X_j|<C_k)} X_i X_j dP = o\left(p\int_{|X_1|<C_k} X_1^2 dP\right).$$

现在从(3.3.7)、(3.3.8)和(3.3.9)可推得

$$(3.3.10)\qquad \int_{|\xi_0|<pC_k} \xi_0^2 dP = (1+o(1))p\int_{|X_1|<C_k} X_1^2 dP.$$

此外

$$(3.3.11)\qquad \int_{|\xi_0|>pC_k} dP \leq \int_{\bigcup_{j=1}^{p}(|X_j|>C_k)} dP \leq p\int_{|X_1|>C_k} dP.$$

那么利用条件(3.3.3)和(3.3.2)我们得

$$(3.3.12)\qquad \frac{k}{(pC_k)^2}\int_{|\xi_0|<pC_k} \xi_0^2 dP \to \infty,$$

$$(3.3.13)\qquad k\int_{|\xi_0|>pC_k} dP \to 0.$$

按(3.3.10),当 $n\to\infty$ 时 $\int_{|\xi_0|<pC_k}\xi_0^2 dP\to\infty$. 仿照 Gnedenko 和 Kolmogorov(1955) 定理 35.1 的证明,我们有

$$\left(\int_{|\xi_0|<pC_k}\xi_0 dP\right)^2 = o\left(\int_{|\xi_0|<pC_k}\xi_0^2 dP\right),$$

即(3.3.12)等价于

$$(3.3.14)\quad \frac{k}{(pC_k)^2}\left\{\int_{|\xi_0|<pC_k}\xi_0^2dP-\left(\int_{|\xi_0|<pC_k}\xi_0dP\right)^2\right\}\to\infty.$$

设 $\xi'_j, j=0,1,\cdots,k-1$ 是相互独立且与 $\xi_0$ 具有相同分布的随机变量. 验证 Gnedenko 和 Kolmogorov(1954)定理 26.4 的证明，在条件(3.3.13)和(3.3.14)下，存在正的常数列 $B_k^{(p)}$ 使得当 $n\to\infty$时

$$(3.3.15)\quad \frac{1}{B_k^{(p)}}\sum_{j=0}^{k-1}\xi'_j\xrightarrow{d}N(0,1),$$

事实上，由(3.3.10)我们可取

$$(3.3.16)\quad B_k^{(p)^2}=k\left\{\int_{|\xi_0|<pC_k}\xi_0^2dP-\left(\int_{|\xi_0|<pC_k}\xi_0dP\right)^2\right\}$$

$$=(1+o(1))k\int_{|\xi_0|<pC_k}\xi_0^2dP$$

$$=(1+o(1))kp\int_{|X_1|<C_k}X_1^2dP.$$

利用引理 1.2.1 和(3.3.1)我们有

$$(3.3.17)\quad \left|E\exp\left(it\sum_{j=0}^{k-1}\xi_j/B_k^{(p)}\right)-\prod_{j=1}^{k-1}E\exp(it\xi'_j/B_k^{(p)})\right|$$

$$\leqslant 4k\alpha(q)\to 0.$$

把它与(3.3.15)相结合即得当 $n\to\infty$时

$$(3.3.18)\quad \frac{1}{B_k^{(p)}}\sum_{j=0}^{k-1}\xi_j\xrightarrow{d}N(0,1).$$

对于 $\eta_j, j=0,1,\cdots,k-1$ 重复上述讨论，有 $B_k^{(q)}>0$ 使得

$$(3.3.19)\quad \frac{1}{B_k^{(q)}}\sum_{j=0}^{k-1}\eta_j\xrightarrow{d}N(0,1)\quad n\to\infty.$$

类似于(3.3.16)，$B_k^{(q)^2}=(1+o(1))kq\int_{|X_1|<C_k}X_1^2dP$. 所以

$$(3.3.20)\quad (B_k^{(q)}/B_k^{(p)})^2=(1+o(1))q/p=o(1).$$

这样当 $n\to\infty$时有

$$(3.3.21)\quad \frac{1}{B_k^{(p)}}\sum_{j=0}^{k-1}\eta_j\xrightarrow{P}0.$$

此外,注意到$n-k(p+q)\leqslant p+q$且利用(3.3.16)和(3.3.3),对任给的$\epsilon>0$我们有

$$(3.3.22)\quad P\{|\eta_k/B_k^{(p)}|>\epsilon\}\leqslant(p+q)E|X_1|/\epsilon B_k^{(p)}$$

$$\leqslant 2pE|X_1|/\epsilon\left(kp\int_{|X_1|<C_k}X_1^2dP\right)^{1/2}=o(1/C_k)\to 0.$$

设$B_n=B_k^p$. 那么从(3.3.18),(3.3.21)和(3.3.22)得

$$S_n/B_n\xrightarrow{d}N(0,1)\quad n\to\infty.$$

现在我们转向弱不变原理的证明. 令$k_a=[k^{2+a}]$,对任给正整数$N$存在$n$使$k_a(p+q)\leqslant N<(k_a+1)(p+q)$. 定义$\xi_j$和$\eta_j$如上,只是它们的指标扩充到$j=0,1,\cdots,k_a-1$. 重复(3.3.18)的证明在$k_a\alpha(q)\to 0$的条件下,存在常数$B_m^{(p)}>0(m=k,k+1,\cdots,k_a)$使得

$$(3.3.23)\qquad \frac{1}{B_m^{(p)}}\sum_{j=0}^{m-1}\xi_j\xrightarrow{d}N(0,1).$$

其中$B_m^{(p)}$是(对固定$p$)相对于独立具有与$\xi_0$相同分布的随机变量的正则化常数. 若我们记对应于Gnedenko和Kolmogorov(1954)定理26.4中$\{\xi_j,j=0,1,\cdots,m\}$的常数列为$\{C'_m\}$,那么类似于(3.3.16)有

$$(3.3.24)\qquad B_m^{(p)^2}=(1+o(1))mp\int_{|X_1|<C'_m}X_1^2dP.$$

类似于(3.3.19)

$$(3.3.25)\qquad B_{k_a}^{(q)-1}\sum_{j=0}^{k_a-1}\eta_j\xrightarrow{d}N(0,1)\quad n\to\infty.$$

定义$B_N=B_{k_a}^{(p)}$,令

$$S_{[Nt]}=\sum_{j=0}^{[(k_a-1)t]}\xi_j+\sum_{j=0}^{[(k_a-1)t]}\eta_j+\eta_{Nt},$$

其中$\eta_{N,t}=X_{([(k_a-1)t]+1)(p+q)+1}+\cdots+X_{[Nt]}$,其项数

$$[Nt]-([(k_a-1)t]+1)(p+q)$$
$$\leqslant(k_a+1)(p+q)t-(k_a-1)(p+q)t\leqslant 2(p+q).$$

定义

$$W'_N(t)=B_N^{-1}\sum_{j=0}^{[(k_a-1)t]}\xi_j,\quad W'_N(t)=B_N^{-1}\Big(\sum_{j=0}^{[(K_a-1)t]}\eta_j+\eta_{N_i}\Big).$$

回顾(3.3.17),我们知道 $\xi_j(\eta_j)$的正则化和与 $\xi'_j(\eta'_j)$的正则化和具有相同的正则化常数,有同样的收敛性,其中 $\xi'_j, j=1,\cdots,k_a-1$,$(\eta'_j, j=1,\cdots,k_a-1)$是独立的.

因此由 Gnedenko 和 Kolmogorov(1954)定理 26.2,对任给 $\epsilon>0, 0\leqslant t\leqslant 1$(3.3.23)推出

$$[mt]\int_{|x|\geqslant\epsilon}dF_m^{(p)}(x)\to 0,$$

$$[mt]\Big\{\int_{|x|<\epsilon}x^2dF_m^{(p)}(x)-\Big(\int_{|x|<\epsilon}xdF_m^{(p)}(x)\Big)^2\Big\}\to t,$$

其中 $F_m^{(p)}(x)$是 $\xi_0/B_m^{(p)}$ 的分布,也即

$$[mt]\int_{|y|\geqslant\epsilon/\sqrt{t}}dF_m^{(p)}(\sqrt{t}\,y)\to 0,$$

$$[mt]\Big\{\int_{|y|<\epsilon/\sqrt{t}}y^2dF_m^{(p)}(\sqrt{t}\,y)$$
$$-\Big(\int_{|y|<\epsilon/\sqrt{t}}ydF_m^{(p)}(\sqrt{t}\,y)\Big)^2\Big\}\to 1.$$

那么由刚才所引用的同一个定理我们得

(3.3.26) $$W'_N(t)\xrightarrow{d}W(t).$$

类似地,我们也可证明

(3.3.27) $$W'_N(t)-W'_N(s)\xrightarrow{d}W(t)-W(s),\quad 0\leqslant s<t\leqslant 1.$$

对于$\{\eta_j\}$,(3.3.25)的一个类似结果是

(3.3.28) $$\frac{1}{B_{k_a}^{(q)}}\sum_{j=0}^{[(k_a-1)t]}\eta_j\xrightarrow{d}W(t).$$

借助类似于(3.3.20)的结果,(3.3.28)推出关于 $t$ 一致地有

$$\frac{1}{B_N}\sum_{j=0}^{[(k_a-1)t]}\eta_j\xrightarrow{P}0.$$

仿照(3.3.22),我们也有关于 $t$ 一致地

$$\eta_{N,1}/B_N \xrightarrow{P} 0.$$

由此推得

$$W''_N \xrightarrow{P} 0.$$

这样就可用 $W'_N$ 来代替 $W_N$.

我们先来考察 $W'_N$ 的有限维分布的收敛性．由(3.3.26)知一维分布的收敛性.对于 2 维情形我们只需证明

$$(W'_N(s), W'_N(t) - W'_N(s)) \xrightarrow{d} (W(s),$$
$$W(t) - W(s)), 0 \leqslant s < t \leqslant 1,$$

换句话说，只需证明对于任给 Borel 集 $A_1, A_2$

(3.3.29) $$P\{W'_N(s) \in A_1, W'_N(t) - W'_N(s) \in A_2\}$$
$$\to P\{W(s) \in A_1\} P\{W(t) - W(s) \in A_2\}.$$

按 $\alpha$ 混合性，我们有

$$|P\{W'_N(s) \in A_1, W'_N(t) - W'_N(s) \in A_2\}$$
$$- P\{W'_N(s) \in A_1\} P\{W'_N(t) - W'_N(s) \in A_2\}|$$
$$\leqslant \alpha(q).$$

把它与(3.3.26)和(3.3.37)结合起来就得(3.3.29)．3 维或更高维情形同法可得，这就得证有限维分布的收敛性成立．

最后，我们来证 $W'_N$ 的胎紧性．由(3.2.5)只需证明对任给的 $\varepsilon, \eta > 0$，存在 $\delta, 0 < \delta < 1$，和整数 $n_0$ 使对 $0 \leqslant t \leqslant 1$

$$P\{\sup_{t \leqslant s \leqslant t+\delta} |W'_N(s) - W'_N(t)| \geqslant \varepsilon\} \leqslant \delta_\eta, \quad n \geqslant n_0.$$

等价地

(3.3.30) $$P\{\max_{0 \leqslant j \leqslant 2k_q\delta} \Big|\sum_{i=0}^{j} \xi_i\Big| \geqslant \varepsilon B_N\} \leqslant \delta_\eta, \quad n \geqslant n_0.$$

由中心极限定理．当 $\delta$ 充分小时我们有

$$P\{\Big|\sum_{i=0}^{[2k_q\delta]} \xi_i\Big| \geqslant \varepsilon B_N/2\} \to P\{|W(2\delta)| \geqslant \varepsilon/2\}$$
$$= P\{|W(1)| \geqslant \varepsilon/(2\sqrt{2\delta})\} \leqslant \frac{16\delta}{\varepsilon^3}\sqrt{2\delta} E|W(1)|^3 \leqslant \frac{\delta\eta}{4}.$$

因此，存在 $n_0$ 使得

$$P\left\{\left|\sum_{i=0}^{[2k_a\delta]}\xi_i\right|\geqslant \varepsilon B_N/2\right\}\leqslant\frac{\delta\eta}{2},\quad n\geqslant n_0.$$

所以若我们能证明对充分大 $N$ 和每一 $j,0\leqslant j\leqslant 2k_a\delta-1$ 有

$$P\left\{\left|\sum_{i=0}^{j}\xi_i\right|\leqslant \varepsilon B_N/2\right\}\geqslant\frac{1}{2},\tag{3.3.31}$$

从引理 2.2.1 即得

$$P\left\{\max_{0\leqslant j\leqslant 2k_a\delta}\left|\sum_{i=0}^{j}\xi_i\right|\geqslant \varepsilon B_N\right\}\leqslant 2P\left\{\left|\sum_{i=0}^{[2k_a\delta]}\xi_i\right|\geqslant \varepsilon B_N/2\right\}\leqslant\delta\eta.$$

现在余下来只需证明(3.3.31)成立. 我们先考察 $j\leqslant k$ 的情形. 利用(3.3.24)并注意到 $p\leqslant k^a$，我们有

$$P\left\{\left|\sum_{i=0}^{j}\xi_i\right|\geqslant \varepsilon B_N/2\right\}\leqslant 2jpE|X_1|/\varepsilon B_N$$

$$\leqslant 3kpE|X_1|/\varepsilon\left(k_a p\int_{|X_1|<C_{k_a}}X_1^2dP\right)^{1/2}\to 0.$$

假设 $k<j\leqslant 2k_a\delta$. 回顾(3.3.23)，由定理 I 1，我们可写

$$B_j^{(p)^2}=jh^{(p)}(j).$$

借助缓变函数的性质，

$$\limsup_{N\to\infty}\frac{B_j^{(p)^2}}{B_N^2}=\limsup_{N\to\infty}\frac{jh^{(p)}(j)}{k_a h^{(p)}(k_a)}\leqslant 2\delta.$$

所以存在 $n_1$ 使当 $n\geqslant n_1$ 时有 $B_N^2/B_j^{(p)^2}\geqslant 1/3\delta$. 让 $\delta$ 充分小有

$$P\{|W(1)|\geqslant\varepsilon/2\sqrt{3\delta}\}<1/4.$$

那么

$$P\left\{\left|\sum_{i=0}^{j}\xi_i\right|\geqslant \varepsilon B_N/2\right\}\leqslant P\left\{\left|\sum_{i=0}^{j}\xi_i\right|\geqslant \varepsilon B_{j+1}^{(p)}/2\sqrt{3\delta}\right\}$$

$$\to P\{|W(1)|\geqslant\varepsilon/2\sqrt{3\delta}\}<1/4,$$

它蕴含着(3.3.31). 定理 3.3.1 证毕.

# 第四章　$\rho$混合序列的弱收敛

Ibragimov(1975)首先对强平稳$\rho$混合序列在某些条件下证明了中心极限定理和弱不变原理成立,这就是

**定理 4.0.1**　设$\{x_n,n\geqslant1\}$是强平稳$\rho$混合序列,$EX_1=0$,$EX_1^2<\infty$且

(i) $\sigma_n^2=ES_n^2\to\infty$,

(ii) $\sum\limits_{n=1}^{\infty}\rho(2^n)<\infty$.

那么$S_n/\sigma_n$依分布收敛于$\Phi(x)$.

**定理 4.0.2**　设$\{x_n,n\geqslant1\}$是强平稳$\rho$混合序列,$EX_1=0$,对某$\delta>0$,$E|X_1|^{2+\delta}<\infty$且条件(i)被满足.那么对$W_n(t)=S_{[nt]}/\sigma_n$,$0\leqslant t\leqslant1$有$W_n\Rightarrow W$.

Peligrad(1982)在2阶矩条件下给出一个弱收敛结果,然而在混合速度上作了较强的限制,即要求$\sum\rho^{1/2}(2^n)<\infty$. Peligrad(1986)提出了五个未决问题,其中之一就是在中心极限定理同样的充分条件下证明弱不变原理.邵启满(1988b)给出了一个正面的答案,这些将被介绍于§4.1中.

当$S(>2)$阶矩有限时,Bradley(1984)提出了在较灵活的矩假设下寻求最低容许的$\rho$混合速度以保证中心极限定理成立的问题.假设$\{x_n,n\geqslant1\}$是强平稳$\rho$混合序列,$EX_1=0$,$\sigma_n^2=ES_n^2\to\infty$且$EX_1^2g(X_1)<\infty$,其中$g:[0,\infty)\to[0,\infty)$满足:对某$\delta>0$,$g(x)$和$x^\delta/g(x)$都是不减的函数. Bradley(1984)问:对任给$d>0$,当

$$\exp\Big(d\sum_{i=1}^{[\log n]}\rho(2^i)\Big)=O(g(n^{1/2}))$$

时,中心极限定理是否成立? Peligrad(1987)证明了一个更精确的结果.

**定理 4.0.3** 设$\{x_n,n\geqslant 1\}$和 $g(x)$如上. 若对某 $0<\varepsilon^*<1$

$$\exp\left\{(2+\varepsilon^*)\sum_{i=1}^{[\log n]}\rho(2^i)\right\}=O(g(n^{1/2})),$$

那么 $S_n/\sigma_n$ 依分布收敛于 $\Phi(x)$.

Peligrad(1987)猜测在定理 4.0.3 条件下有 $W_n\Rightarrow W$. 邵启满(1989b)给出了肯定的回答,它将被讨论于 § 4.2 中. 在邵启满(1989$a$)中,对于非平稳的 $\rho$ 混合序列给出了一个较一般的结果,我们将在 § 4.3 中讨论. 对强平稳 $\rho$ 混合、具有无穷方差的随机变量序列的弱不变原理将被介绍于 § 4.4 中.

在相依随机变量弱不变原理的研究中,属于 Billingsley(1968)的下述基本结果常被用到. 设

$$\mathscr{B}=\sigma\{(-\infty,x),-\infty<x<\infty\}$$

是 $R$ 的 Borel $\sigma$-域.

**定理 4.0.4** 设$\{W_n,n\geqslant 1\}$是 $D[0,1]$中的随机元序列,满足下述条件:

(i)$\{W_n(t),0\leqslant t\leqslant 1\}$具有渐近独立增量,即对任给 $B_i\in\mathscr{B},i=1,\cdots,r$ 和 $0\leqslant s_1\leqslant t_1<\cdots<s_r\leqslant t_r\leqslant 1$ 有

$$\lim_{n\to\infty}\{P(W_n(t_i)-W_n(s_i)\in B_i,i=1,\cdots,r)-\prod_{i=1}^{r}P(W_n(t_i)-W_n(s_i)\in B_i)\}=0,$$

(ii)对每一 $t$,$\{W_n^2(t),n\geqslant 1\}$一致可积,

(iii)当 $n\to\infty$时 $EW_n(t)\to 0$,$EW_n^2(t)\to t$,

(iv)对任给 $\varepsilon,\eta>0$ 存在 $\delta>0$ 和正整数 $n_0$ 使当 $n\geqslant n_0$ 时,

$$P\{w(W_n,\delta)\geqslant\varepsilon\}\leqslant\delta,$$

其中 $w(x,\delta)=\sup\limits_{|s-t|\leqslant\delta}|x(s)-x(t)|$. 那么 $W_n\Rightarrow W$.

## § 4.1 2 阶矩有限时的弱不变原理

邵启满(1988b)证明了下述定理.

**定理 4.1.1** 设$\{x_n, n\geqslant 1\}$是$\rho$混合序列，$EX_n=0$，$EX_n^2<\infty$且

(i) $\lim\limits_{n\to\infty} ES_n^2/n=\sigma^2>0$，

(ii) $\{x_n^2, n\geqslant 1\}$一致可积，

(iii) $\sum\limits_{n=1}^{\infty}\rho(2^n)<\infty$.

那么

$$W_n \Rightarrow W,$$

其中

$$W_n(t) = S_{[nt]}/\sigma\sqrt{n}, \qquad 0\leqslant t\leqslant 1.$$

定理 4.1.1 的证明需要下述引理.

**引理 4.1.1**(Peligrad 1982). 设$\{x_n, n\geqslant 1\}$是$\alpha$混合序列，$EX_n=0$，$EX_n^2<\infty$且满足定理 4.1.1 的(i)和

(ii′)对每一$t\in[0,1]$，$\{W_n^2(t), n\geqslant 1\}$一致可积，

(iv)对任给$\varepsilon>0$存在$\lambda>2$和正整数$n_0$使对每一$n\geqslant n_0$和一切$k\geqslant 1$有

$$P\left\{\max_{1\leqslant i\leqslant n}|S_k(i)|\geqslant\lambda\sigma\sqrt{n}\right\}\leqslant\varepsilon/\lambda^2. \tag{4.1.1}$$

那么

$$W_n\Rightarrow W.$$

引理 4.1.1 是定理 4.0.4 的推论.

**引理 4.1.2**(Moricz 1982). 设$\{x_n, n\geqslant 1\}$是随机变量序列. 假设非负函数$f(k,m)$满足：对一切$k\geqslant 0, m\geqslant 1, l\geqslant 1$

$$f(k,m)+f(k+m,l)\leqslant f(k,m+l). \tag{4.1.2}$$

又函数$g(t,s)$对每一变量是不减的. 若

$$E|S_k(m)|^r\leqslant f(k,m)g^r(f(k,m),m)\quad r\geqslant 1, \tag{4.1.3}$$

那么

$$E\max_{1\leqslant m\leqslant n}|S_k(m)|^r\leqslant\frac{5}{2}f(k,n)\left\{\sum_{i=0}^{[\log n]}g\left(\frac{f(k,n)}{2^i},\left[\frac{n}{2^i}\right]\right)\right\}^r. \tag{4.1.4}$$

引理 4.1.2 的证明不在此陈述.

**定理 4.1.1 的证明**

我们仅需验证引理 4.1.1 中的条件(ii′)和(iv).

1)我们来证明$\{S_k^2(n)/n, n\geqslant 1, k\geqslant 0\}$是一致可积的. 设 $N>0$ 待下面确定. 记

$$X_i^N = X_i I(|X_i| < N) - EX_i I(|X_i| < N), \overline{X}_i^N = X_i - X_i^N,$$

$$S_k^N(n) = \sum_{i=k+1}^{k+n} X_i^N, \qquad \overline{S}_k^N(n) = \sum_{i=k+1}^{k+n} \overline{X}_i^N,$$

$$E_a U = \int_{U>a} U dP.$$

显然

$$S_k(n) = S_k^N(n) + \overline{S}_k^N(n),$$

(4.1.5) $\quad E_a S_k^2(n)/n \leqslant 4E_{a/4}(S_k^N(n))^2/n + 4E(\overline{S}_k^N(n))^2/n.$

从引理 2.2.2 即得对每一 $n$ 和某 $K>0$

$$\sup_{k\geqslant 1} E(\overline{S}_k^N(n))^2/n \leqslant K \sup_{k\geqslant 1} E(\overline{X}_k^N)^2.$$

由于$\{X_n^2, n\geqslant 1\}$是一致可积的,对任给 $\varepsilon>0$ 有 $N$ 使得$\sup_{k\geqslant 1} E(\overline{X}_k^N)^2 \leqslant \varepsilon/(8K)$. 这样对每一 $n\geqslant 1$ 和 $k\geqslant 1$ 有

(4.1.6) $\quad E(\overline{S}_k^N(n))^2/n \leqslant \varepsilon/8.$

另一方面,从引理 2.2.4,存在常数 $K_1 = K_1(N, \delta, \rho) > 0$ 使对每一 $n$

$$\sup_{k\geqslant 1} E(S_k^N(n))^{2+\delta}/n^{1+\delta/2} \leqslant K_1.$$

那么,对充分大的 $a$ 我们得

(4.1.7) $\quad E_{a/4}(S_k^N(n))^2/n \leqslant \frac{c}{a^{\delta/2}} E_{a/4}(S_k^N(n))^{2+\delta}/n^{1+\delta/2} < \frac{\varepsilon}{8}.$

把(4.1.6)和(4.1.7)代入(4.1.5)得证$\{S_k^2(n)/n, n\geqslant 1, k\geqslant 1\}$是一致可积的.

2)我们来证(4.1.1)成立. 设

$$p = [\exp(2C\log^{1/3} n)], \qquad r = [n/p],$$

其中常数 $C$ 被确定于 $\delta=1$ 时的引理 2.2.4 中. 记

$$X_i^n = X_i I(|X_i| < n^{1/2}/p) - EX_i I(|X_i| < n^{1/2}/p),$$

$$\overline{X}_i^n = X_i - X_i^n,$$

$$S_k^n(l) = \sum_{j=k+1}^{k+l} X_j^n, \qquad \overline{S}_k^n(l) = \sum_{j=k+1}^{k+l} \overline{X}_j^n,$$

$$Y_i = \sum_{j=k+1+(2i-1)r}^{k+2ir} X_j^n, \qquad i = 1,2,\cdots,p_1 := [p/2],$$

$$Z_i = \sum_{j=k+1-2ir}^{k+(2i+1)r} \overline{X}_j^n, \qquad i = 0,1,\cdots,p_2 := [(p-1)/2],$$

$$T_l(i) = \sum_{j=l+1}^{l+i} Y_j, \qquad \overline{T}_l(i) = \sum_{j=l}^{l+i} Z_j,$$

$$T(i) = T_0(i), \qquad \overline{T}(i) = \overline{T}_0(i).$$

显然地 $S_k(i) = S_k^n(i) + \overline{S}_k^n(i)$. 不失一般性可设 $\sigma = 1$.

首先我们来证对充分大 $n$

(4.1.8) $$P\left\{\max_{1\leqslant i\leqslant n} |S_k^n(i)| \geqslant \lambda n^{1/2}\right\} \leqslant \varepsilon/6\lambda^2.$$

从 $\delta=1$ 时的引理 2.2.4 即得

$$\max_{1\leqslant k\leqslant n} |ES_k^n(i)|^3 \leqslant C\{n^{3/2}\sigma_0^3 + n \exp(C\log^{1/3}n)a_0\}$$
$$\leqslant cn^{3/2},$$

其中 $\sigma_0^2 = \sup_n E(X_i^{(n)})^2, a_0 = \sup_n E|X_i^{(n)}|^3$. 应用引理 4.1.2 我们有

$$\sup_k E \max_{1\leqslant i\leqslant n} |S_k^n(i)|^3 = O(n^{3/2}).$$

因此 $\{\max_{1\leqslant i\leqslant n} S_k^n(i)^2/n, n\geqslant 1, k\geqslant 1\}$ 是一致可积的. 所以对任给 $\varepsilon > 0$ 存在 $\lambda \geqslant 2$ 和充分大 $n_0$ 使对 $n \geqslant n_0$ 时(4.1.8)被满足 .

其次,我们来证

(4.1.9) $$P\left\{\max_{1\leqslant i\leqslant n} |\overline{S}_k^n(i)| \geqslant 5\lambda n^{1/2}\right\} \leqslant 5\varepsilon/6\lambda^2.$$

注意到(4.1.9)式左边不超过

(4.1.10) $$P\left\{\max_{1\leqslant i\leqslant p_1} |T(i)| \geqslant 2\lambda n^{1/2}\right\} + P\left\{\max_{1\leqslant i\leqslant p_2} |\overline{T}(i)| \geqslant 2\lambda n^{1/2}\right\}$$
$$+ \sum_{j=0}^{p} P\left\{\max_{1\leqslant i\leqslant r} |\overline{S}_{k+jr}^n(i)| \geqslant \lambda n^{1/2}\right\}$$
$$=: I_1 + I_2 + I_3.$$

由于 $\{x_n^2, n\geqslant 1\}$ 一致可积,不失一般性我们可设 $\sup_{n\geqslant 1} \| X_n \|_2 \leqslant 1$. 这样我们有

$$P\left\{\max_{1\leqslant i\leqslant r} |\overline{S}_{k+jr}^n(i)| \geqslant \lambda n^{1/2}\right\}$$

$$\leqslant P\left\{\sum_{i=k+1+jr}^{k+(j+1)r}\left(|\overline{X}_i^n|-E|\overline{X}_i^n|\right)\geqslant \lambda n^{1/2}/2\right\}$$

$$=O\left(\frac{r}{n\lambda^2}\sup_m E(\overline{X}_m^n)^2\right).$$

那么由条件(ii)得

$$I_3=O\left(\frac{pr}{\lambda^2 n}\sup_m E(\overline{X}_m^n)^2\right)\leqslant\frac{\varepsilon}{6\lambda^2}. \tag{4.1.11}$$

现在估计 $I_1$. 记

$$\mathscr{G}_i=\sigma(Y_1,\cdots,Y_i),u_i=E(Y_i|\mathscr{G}_{i-1}),$$

$$U_l(i)=\sum_{j=l+1}^{l+i}u_j,T_l^*(i)=T_l(i)-U_l(i),$$

$$U(i)=U_0(i),T^*(i)=T(i)-U(i).$$

容易看出

$$I_1\leqslant P\left\{\max_{1\leqslant i\leqslant p_1}|T^*(i)|\geqslant\lambda n^{1/2}\right\} \tag{4.1.12}$$

$$+P\left\{\max_{1\leqslant i\leqslant p_1}|U(i)|\geqslant\lambda n^{1/2}\right\}=:I_{11}+I_{12}.$$

由于$\{T^*(i),i=1,\cdots,p_1\}$是鞅,对充分大 $n$ 有

$$I_{11}\leqslant\varepsilon/6\lambda^2. \tag{4.1.13}$$

为估计 $I_{12}$,我们来证存在一个与 $l,i,k$ 和 $n$ 无关的常数 $C_0$ 使得

$$EU_l^2(i)\leqslant C_0 ir\rho^2(r)(\log 2i)^2. \tag{4.1.14}$$

从引理 2.2.2,存在一个与 $l,i,k$ 和 $n$ 无关的常数 $C_1>0$ 使得

$$ET_l^2(i)\leqslant C_1 ir. \tag{4.1.15}$$

对 $i$ 用归纳法,可以证明(4.1.14)对 $C_0=C_1/(\log\frac{3}{2})^2$ 成立. 从 $\rho$ 混合的定义我们有

$$EU_l^2(1)=Eu_{l+1}^2=EY_{l+1}u_{l+1}\leqslant\rho(r)\|Y_{l+1}\|_2\|u_{l+1}\|_2.$$

把它与(4.1.15)相结合就得 $i=1$ 时的(4.1.14)式.

对 $i\geqslant 2$,假设对 $j<i$(4.1.14)成立. 令 $i_1=[i/2]$,$i_2=i-i_1$,我们有

$$EU_l^2(i)=EU_l^2(i_1)+EU_{l+i_1}^2(i_2)+2EU_l(i_1)U_{l+i_1}(i_2)$$

$$\leqslant EU_l^2(i_1) + EU_{l+i_1}^2(i_2) + 2\rho(r) \| U_l(i_1) \|_2 \| T_{l+i_1}^*(i_2) \|_2.$$

由归纳假设和(4.1.15),我们得

$$EU_l^2(i) \leqslant C_0 i_1 r \rho^2(r)(\log(2i_1))^2 + C_0 i_2 r \rho^2(r)(\log(2i_2))^2$$
$$+ 2\rho^2(r) r i_1^{1/2} i_2^{1/2} C_0^{1/2} C_1^{1/2} \log(2i_1)$$
$$\leqslant C_0 i r \rho^2(r)\left((\log(2i_2))^2 + 2\left(\log \frac{3}{2}\right)\log(2i_2)\right)$$
$$\leqslant C_0 i r \rho^2(r)(\log(2i))^2.$$

这就证明了对每一 $i$,(4.1.14)成立.从(4.1.14)和引理 4.1.2 我们得

$$E \max_{1\leqslant i\leqslant p_1} U_0^2(i) \leqslant c p_1 r (\log (2p_1))^4 \rho^2(r) \leqslant cn(\log n)^{-1/2}.$$

这样对充分大 $n$ 存在 $\lambda\geqslant 2$ 使得

$$I_{12} \leqslant \varepsilon/6\lambda^2. \tag{4.1.16}$$

把它与(4.1.13)相结合得到

$$I_1 \leqslant \varepsilon/3\lambda^2. \tag{4.1.17}$$

同样可证

$$I_2 \leqslant \varepsilon/3\lambda^2. \tag{4.1.18}$$

从(4.1.11),(4.1.17)和(4.1.18)即得(4.1.9)成立.这就证明了定理 4.1.1.

从定理 4.1.1 和定理 2.1.5,我们即有下述推论.

**推论 4.1.1** 设$\{x_n,n\geqslant 1\}$是强平稳 $\rho$ 混合序列,$EX_1=0$, $EX_1^2<\infty$且 $\sum_{n=1}^{\infty}\rho(2^n)<\infty$. 若 $\sigma_n^2=ES_n^2\to\infty$,那么

$$\lim_{n\to\infty}\sigma_n^2/n = \sigma^2.$$

其次,若 $\sigma>0$,那么

$$W_n\Rightarrow W,$$

其中 $W_n(t)=S_{[nt]}/\sigma n^{1/2}$,$0\leqslant t\leqslant 1$.

## §4.2 当高于 2 阶矩有限时的弱不变原理

设$\{x_n,n\geqslant 1\}$是强平稳 $\rho$ 混合序列,$EX_1=0$,$EX_1^2<\infty$且 $\sigma_n^2=$

$ES_n^2 \to \infty$. 设 $g: [0,\infty) \to [0,\infty)$ 是不减函数且对某 $0<\delta<1$, $x^\delta/g(x)$ 也是一个不减函数.

**定理 4.2.1**(Peligrad 1987, 邵启满 1989b)　设 $\{x_n, n\geqslant 1\}$ 如上, 且满足

(i) $EX_1^2 g(|X_1|)<\infty$,

(ii) 对某 $0<\varepsilon^*<1$, $\exp\left((2+\varepsilon^*)\sum_{k=1}^{[\log n]}\rho(2^k)\right)=O(g(n^{1/2}))$.

那么

$$W_n \Rightarrow W.$$

通过简单的计算, 我们有下述推论:

**推论 4.2.1**　设 $\{x_n, n\geqslant 1\}$ 如上. 假设对某 $\varepsilon>0$ 和 $a>0$

$$EX_1^2(\log^+|X_1|)^{2a/(1-\varepsilon)}<\infty,$$

且对每一充分大的 $n$

$$\rho(n)\leqslant a/\log n.$$

那么

$$W_n \Rightarrow W.$$

**推论 4.2.2**　设 $\{x_n, n\geqslant 1\}$ 如上. 假设对某 $0<\beta<1$, $\varepsilon>0$ 和 $a>0$

$$EX_1^2\exp\left(\frac{2a(1+\varepsilon)}{1-\beta}(2\log^+|X_1|)^{1-\beta}\right)<\infty,$$

且对每一充分大的 $n$

$$\rho(n)\leqslant a/(\log n)^\beta.$$

那么

$$W_n \Rightarrow W.$$

**推论 4.2.3**　设 $\{x_n, n\geqslant 1\}$ 如上. 假设对某 $r>0$, $\varepsilon>0$ 和 $a>0$

$$EX_1^2\exp\left(\frac{4a\log^+|X_1|}{(1-\varepsilon)(\log^+\log^+|X_1|)^r}\right)<\infty,$$

且对每一充分大的 $n$

$$\rho(n)\leqslant a/(\log\log n)^r.$$

那么

$$W_n \Rightarrow W.$$

定理 4.2.1 的证明需要下面的引理,引理 4.2.1 是定理 3.1.2 和 Billingsley(1968)定理 8.4 的直接推论.

**引理 4.2.1** 设$\{x_n, n\geqslant 1\}$如上.那么$W_n$弱收敛于$W$的充要条件是$\{S_n^2/\sigma_n^2, n\geqslant 1\}$是一致可积的且对任给的$\varepsilon>0$存在$\lambda>1$使得

$$P\left\{\max_{1\leqslant i\leqslant n}|S_i| > \lambda\sigma_n\right\}\leqslant \varepsilon/\lambda^2. \tag{4.2.1}$$

**引理 4.2.2**(Peligrad 1987). 设$\{x_n, n\geqslant 1\}$如上且$EX_n^4<\infty$.那么对任给的$\varepsilon>0$存在$C=C(\varepsilon,\rho(\cdot))$使对每一$n\geqslant 1$

$$ES_n^4 \leqslant C(n^{1+\varepsilon}EX_1^4 + \sigma_n^4).$$

**证** 记$a_m = \| S_m \|_4$. 显然

$$a_{2m} \leqslant \| S_m + S_{k+m}(m) \|_4 + 2k\, a_1.$$

利用 Schwarz 不等式和$\rho$混合的定义,我们有

$$\begin{aligned}
&E|S_m + S_{k+m}(m)|^4\\
&\leqslant 2a_m^4 + 6E|S_mS_{k+m}(m)|^2 + 8a_m^2(E|S_mS_{k+m}(m)|^2)^{1/2}\\
&\leqslant 2(1 + 7\rho^{1/2}(k))a_m^4 + 8a_m^2\sigma_m^2 + 6\sigma_m^4\\
&\leqslant (2^{1/4}(1 + 7\rho^{1/2}(k))^{1/4}a_m + 2\sigma_m)^4.
\end{aligned}$$

由此即得

$$a_{2m} \leqslant 2^{1/4}(1 + 7\rho^{1/2}(k))^{1/4}a_m + 2\sigma_m + 2ka_1.$$

设$0<\varepsilon<1/3$,$k$是充分大整数使$1+7\rho^{1/2}(k)\leqslant 2^\varepsilon$. 通过递推法对每一整数$r\geqslant 1$有

$$a(2^r) \leqslant 2^{r(1+\varepsilon)/4}a_1 + 2\sum_{i=1}^{r}2^{(i-1)(1+\varepsilon)/4}(\sigma(2^{r-i}) + ka_1).$$

由此得出

$$a(2^r) \leqslant c(2^{r(1+\varepsilon)/4}a_1 + \sigma(2^r)).$$

这就蕴含了引理的结论.

**定理 4.2.1 的证明.**

若$\sum \rho(2^n)<\infty$,从定理 2.1.4 可知定理 4.1.1 的条件被满

足. 所以定理 4.2.1 的结论成立. 现在我们来处理 $\sum \rho(2^n)=\infty$ 的情形. 由条件(ii)此时 $g(x)\to\infty(x\to\infty)$. 不失一般性我们可设对充分大 $n$ 有

$$\rho(n) \geqslant (\log n)^{-1}(\log \log n)^{-2}. \tag{4.2.2}$$

1) 首先我们来证 $\{S_n^2/\sigma_n^2, n\geqslant 1\}$ 是一致可积的. 容易看到条件(ii)蕴含着对充分大 $n$

$$g(n^{1/2}) \geqslant \exp\left\{2\sum_{i=1}^{[\log n]}\rho(2^i)/(1-\varepsilon^*)\right\}. \tag{4.2.3}$$

令

$$T = \operatorname{inv} g\left\{\exp\left\{2\sum_{i=1}^{[(1-\varepsilon)\log n]}\rho(2^i)/(1-\varepsilon^*)\right\}\right\}, \tag{4.2.4}$$

其中 $0<\varepsilon<\varepsilon^*<1$. 记

$$X_{i1} = X_i I(|X_i| \leqslant T) - EX_i I(|X_i| \leqslant T),$$
$$X_{i2} = X_i I(|X_i| > T) - EX_i I(|X_i| > T),$$
$$S_{n1} = \sum_{i=1}^n X_{i1}, \qquad S_{n2} = \sum_{i=1}^n X_{i2},$$
$$\sigma_{n1}^2 = \operatorname{Var} S_{n1}, \qquad \sigma_{n2}^2 = \operatorname{Var} S_{n2}.$$

由引理 2.2.2 并注意到函数 $g(x)$ 是不减的，我们有

$$\sigma_{n2}^2 \leqslant C_1 \frac{n}{g(T)} EX_1^2 g(|X_1|) I(|X_1| > T) \cdot \exp\left\{\sum_{i=1}^{[(1-\varepsilon)\log n]}\rho(2^i)/(1-\varepsilon)\right\},$$

其中 $C_1=C_1(\varepsilon)$. 从 $T$ 的定义和引理 2.2.3 我们得

$$\sigma_{n2}^2 \leqslant \frac{C_1}{C}\sigma_n^2 EX_1^2 g(|X_1|) I(|X_1| > T). \tag{4.2.5}$$

显然 $\operatorname{inv} g(x)\to\infty(x\to\infty)$. 从(4.2.4)和(4.2.5)即得

$$\sigma n2 = o(\sigma_n), \tag{4.2.6}$$

这就蕴含着

$$\sigma_{n1}/\sigma_n \to 1, \quad n\to\infty. \tag{4.2.7}$$

对序列 $\{x_{n1}, n\geqslant 1\}$ 应用引理 4.2.2，存在常数 $K_1=K_1(\rho(\cdot),\varepsilon)$ 使对每一 $n\geqslant 1$

(4.2.8) $$E|S_{n1}|^4 \leqslant K_1(n^{1+\varepsilon/2}T^2EX_1^2 + \sigma_{n1}^4).$$

从引理 2.2.3 并注意到到 $\exp\left(d\sum_{i=1}^{[\log n]}\rho(2^i)\right)$ 当 $n\to\infty$ 时是缓变函数，即得存在 $C=C(\rho(\cdot),\varepsilon)$ 使对每一 $n\geqslant 1$

$$\sigma_n^4 \geqslant Cn^{2-\varepsilon/2}.$$

这样从(4.2.7)和(4.2.8)我们有

$$E(|S_{n1}|/\sigma_n)^4 \leqslant K_2(T^2/n^{1-\varepsilon}+1),$$

其中 $K_2=K_2(\rho(\cdot),\varepsilon)$.

从(4.2.3)即得对充分大的 $n$

$$g([n^{1-\varepsilon}]^{1/2}) \geqslant \exp\left(2\sum_{i=1}^{[(1-\varepsilon)\log n]}\rho(2^i)/(1-\varepsilon^*)\right).$$

结合(4.2.4)我们得 $T/n^{(1-\varepsilon)/2}$ 被 1 所界. 所以

(4.2.9) $$\sup_n E(|S_{n1}|/\sigma_n)^4 < \infty.$$

由此从(4.2.6)和(4.2.9)我们证明了 $\{S_n^2/\sigma_n^2, n\geqslant 1\}$ 是一致可积的.

2) 其次我们来证对任给 $\varepsilon>0$，存在 $\lambda>1$ 使得

(4.2.10) $$P\left\{\max_{1\leqslant i\leqslant n}|S_i| \geqslant 6\lambda\sigma_n\right\} \leqslant 6\varepsilon/\lambda^2.$$

记

(4.2.11) $$T_1 = \exp\left(\frac{40}{\delta}\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\right), J = n^{1/2}/T_1,$$

$$X_{i1} = X_iI(|X_i| \leqslant J) - EX_iI(|X_i| \leqslant J),$$

$$X_{i2} = X_iI(|X_i| > J) - EX_iI(|X_i| > J),$$

$$S_{n1}(k) = \sum_{i=1}^k X_{i1}, \qquad S_{n2}(k) = \sum_{i=1}^k X_{i2},$$

$$\sigma_{n1}^2(k) = ES_{n1}^2(k), \qquad \sigma_{n2}^2(k) = ES_{n2}^2(k).$$

显然地 $S_k=S_{n1}(k)+S_{n2}(k)$ 且

$$P\left\{\max_{1\leqslant i\leqslant n}|S_i| \geqslant 6\lambda\sigma_n\right\}$$

$$\leqslant P\left\{\max_{1\leqslant i\leqslant n}|S_{n1}(i)| \geqslant \lambda\sigma_n\right\} + P\left\{\max_{1\leqslant i\leqslant n}|S_{n2}(i)| \geqslant 5\lambda\sigma_n\right\}.$$

首先我们注意到

$$\log T_1 = \frac{40}{\delta}\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)$$

$$\leqslant \frac{40}{\delta}\rho^{-\delta/(2+\delta)}\left(\frac{n}{T_1^2}\right)\sum_{i=1}^{[\log n/T_1^2]}\rho^{(2^i)} + \frac{90}{\delta}\rho^{2/(2+\delta)}\left(\frac{n}{T_1^2}\right)\log T_1.$$

因此对每一充分大 $n$ 我们有

$$\log T_1 \leqslant \frac{50}{\delta}\rho^{-\delta/(2+\delta)}\left(\frac{n}{T_1^2}\right)\sum_{i=1}^{[\log(n/T_1^2)]}\rho(2^i). \tag{4.2.12}$$

和

$$\sum_{i=1}^{[\log n]}\rho\, 2^i \leqslant \left(1+\frac{\varepsilon^*}{12}\right)\sum_{i=1}^{[\log(n/T_1^2)]}\rho(2^i). \tag{4.2.13}$$

由此和条件(ii),以及 $g(x)$ 的不减性我们有

$$g(J) \geqslant \exp\left(\frac{2+\varepsilon^*}{1+\varepsilon^*/12}\sum_{i=1}^{[\log n]}\rho(2^i)\right). \tag{4.2.14}$$

由引理 2.2.2,引理 2.2.3 和(4.2.14)式,对充分大 $n$,当 $k\leqslant n$ 时

$$\sigma_{n2}^2(k) \leqslant C_1 kEX_1^2 I(|X_1|>J)\exp\left(\sum_{i=1}^{[\log n]}\left(1+\frac{\varepsilon^*}{4}\right)\rho(2^i)\right)$$

$$\leqslant \frac{C_1C_2^{-1}\sigma_k^2 EX_1^2 g(|X_1|)}{g(J)}\exp\left(\left(2+\frac{3\varepsilon^*}{4}\right)\sum_{i=1}^{[\log n]}\rho(2^i)\right)$$

$$\leqslant C_1C_2^{-1}\sigma_k^2 EX_1^2 g(|X_1|)\exp\left(\frac{-\varepsilon^*}{52}\sum_{i=1}^{[\log n]}\rho(2^i)\right).$$

由此并注意到 $\sum\rho(2^i)=\infty$,我们得

$$\max_{1\leqslant k\leqslant n}\frac{\sigma_{n2}(k)}{\sigma_k} = o(1), n\to\infty.$$

因此容易看到对 $k=1,2,\cdots,n$ 和充分大 $n$ 有

$$\sigma_{n1}^2(k) \leqslant 2\sigma_k^2 \tag{4.2.15}$$

由引理 2.2.4 和(4.2.15)有

$$E|S_{n1}(k)|^{2+\delta} \leqslant c\Big(\sigma_k^{2+\delta} + kE|X_1|^{2+\delta}I(|X_1|<J)$$

$$\cdot\exp\Big(30\sum_{i=1}^{[\log n]}\rho(2^i)\Big)\Big).$$

从上式并由引理 2.2.2,条件(i),(ii)(4.2.11),(4.2.14)和引理 4.1.2 我们有

$$
\begin{aligned}
&E\max_{1\leqslant k\leqslant n}|S_{n1}(k)|^{2+\delta}\\
&\leqslant C\Big(\sigma_n^{2+\delta}+n(\log n)^{2+\delta}E|X_1|^{2+\delta}I(|X_1|<J)\\
&\quad\cdot\exp\Big(30\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big)\Big)\\
&\leqslant C\Big(\sigma_n^{2+\delta}+\frac{n^{2+\delta}(\log n)^{2+\delta}EX_1^2g(|X_1|)}{g(J)T^\delta}C_1\\
&\quad\cdot\exp\Big(35\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big)\Big)\\
&\leqslant c\sigma_n^{2+\delta}(1+EX_1^2g(|X_i|)).
\end{aligned}
$$

因此存在 $\lambda>1$ 使对每一充分大 $n$ 有

$$
P\Big\{\max_{1\leqslant k\leqslant n}|S_{n1}(k)|\geqslant\lambda\sigma_n\Big\}\leqslant\epsilon/\lambda^2. \tag{4.2.16}
$$

现在我们来估计 $P\Big\{\max_{1\leqslant k\leqslant n}|S_{n2}(k)|\geqslant 5\lambda\sigma_n\Big\}$. 设

$$
p=\exp\Big(\frac{50}{\delta}\sum_{r=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big),
$$

$$
r=\Big[\frac{n}{p}\Big],p_1=\Big[\frac{p}{2}\Big],p_2=\Big[\frac{p-1}{2}\Big].
$$

令

$$
Y_i=\sum_{j=1+(2i-1)r}^{2ir}X_{j2},\qquad i=1,2,\cdots,p_1;
$$

$$
Z_i=\sum_{j=1+2ir}^{(2i+1)r}X_{j2},\qquad i=0,1,\cdots,p_2;
$$

$$
T_1(i)=\sum_{j=1}^{i}Y_j,\qquad T_2(i)=\sum_{j=0}^{i}Z_j.
$$

注意到 $\{X_{j2},1\leqslant j\leqslant n\}$ 是平稳的，我们有

$$
\begin{aligned}
&P\Big\{\max_{1\leqslant k\leqslant n}|S_{n2}(k)|\geqslant 5\lambda\sigma_n\Big\}\\
&\leqslant P\Big\{\max_{1\leqslant k\leqslant p_1}|T_1(k)|\geqslant 2\lambda\sigma_n\Big\}+P\Big\{\max_{1\leqslant k\leqslant p2}|T_2(k)|\geqslant 2\lambda\sigma_n\Big\}\\
&\quad+(p+1)P\Big\{\max_{1\leqslant k\leqslant r}|S_{n2}(k)|\geqslant\lambda\sigma_n\Big\}\\
&=:I_1+I_2+I_3.
\end{aligned}
$$

借助引理 2.2.2 和 2.2.3，对每一充分大 $n$ 有

$$P\left\{\max_{1\leqslant k\leqslant r}|S_{n2}(k)|\geqslant\lambda\sigma_n\right\}$$

$$\leqslant P\left\{\sum_{i=1}^{r}(|X_{n2}(i)|-E|X_{n2}(i)|)\geqslant\lambda\sigma_n-2\sum_{i=1}^{r}E|X_{n2}(i)|\right\}$$

$$\leqslant P\left\{\sum_{i=1}^{r}(|X_{n2}(i)|-E|X_{n2}(i)|)\geqslant\lambda\sigma_n-2r\frac{EX_1^2g(|X_1|)}{g(J)J}\right\}$$

$$\leqslant P\left\{\sum_{i=1}^{r}(|X_{n2}(i)|-E|X_{n2}(i)|)\geqslant\lambda\sigma_n/2\right\}$$

$$\leqslant 4Cr\sigma_n^{-2}\exp\left((1+\varepsilon^*/4)\sum_{i=1}^{[\log n]}\rho(2^i)\right)EX_1^2I(|X_1|>J)\cdot\lambda^{-2}.$$

因此从(4.2.14)得

$$I_3\leqslant 4Cn\sigma_n^{-2}\exp\left((1+\varepsilon^*/4)\sum_{i=1}^{[\log n]}\rho(2^i)\right)\tag{4.2.17}$$

$$\cdot EX_1^2I(|X_1|>J)\cdot\lambda^{-2}$$

$$\leqslant\frac{4C}{C'\lambda^2g(J)}\exp\left((2+3\varepsilon^*/4)\sum_{i=1}^{[\log n]}\rho(2^i)\right)EX_1^2g(|X_1|)$$

$$\leqslant\frac{4C}{C'\lambda^2}\exp\left(-\frac{\varepsilon^*}{52}\sum_{i=1}^{[\log n]}\rho(2^i)\right)EX_1^2g(|X_1|)\leqslant\varepsilon/\lambda^2.$$

为估计 $I_1$，设

$$\mathscr{F}_0=(\Omega,\Phi),\qquad \mathscr{F}_k=\sigma(X_i,1\leqslant i\leqslant 2rk);$$

$$u_k=E(Y_k|\mathscr{F}_{k-1}),\qquad k=1,2,\cdots,p_1;$$

$$U_i(k)=\sum_{j=i+1}^{i+k}u_j,\qquad T^*(k)=T_1(k)-U_0(k).$$

显然地

$$I_1\leqslant P\left\{\max_{1\leqslant i\leqslant p_1}|T^*(i)|\geqslant\lambda\sigma_n\right\}+P\left\{\max_{1\leqslant i\leqslant p_1}|U_0(i)|\geqslant\lambda\sigma_n\right\}$$

$$=:I_{11}+I_{12}.$$

因为 $\{T^*(i),i=1,2,\cdots,p_1\}$ 是鞅，我们有

$$I_{11}\leqslant\frac{16}{\lambda^2\sigma_n^2}\sum_{i=1}^{p_1}EY_i^2.$$

类似于求 $I_3$ 的估计，对任给 $\lambda>1$ 及充分大 $n$，我们也有

$$I_{11} \leqslant \varepsilon/\lambda^2. \tag{4.2.18}$$

最后，对 $k$ 用归纳法，我们来证对每一 $i,k,n$

$$EU_i^2(k) \leqslant C_1 k\rho^2(r)\log^2(2k)EX_1^2 I(|X_1| > J) \cdot r \cdot \exp\left((1+\varepsilon^*/4)\sum_{j=1}^{[\log n]}\rho(2^j)\right). \tag{4.2.19}$$

当 $k=1$，由 $\rho$ 混合的定义

$$EU_i^2(1) = EE^2(Y_{i+1}|\mathscr{F}_i) = E(Y_{i+1}E(Y_{i+1}|\mathscr{F}_i)) \leqslant \rho(r)\,\|Y_{i+1}\|_2 \cdot \|E(Y_{i+1}|\mathscr{F}_i)\|_2.$$

这样对 $k=1$ 和 $i+1\leqslant p_1$，由引理 2.2.2 得(4.2.19)成立. 当 $k\geqslant 2$ 时，归纳假设对每一小于 $k$ 的整数(4.2.19)成立. 令 $k_1=[k/2]$，$k_2=k-k_1$，那么

$$\begin{aligned}EU_i^2(k) &= EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1)U_{i+k_1}(k_2)\\ &= EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1)\sum_{j=i+k_1+1}^{i+k} Y_j\\ &\leqslant EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2\,\|U_i(k_1)\|_2 \cdot \Big\|\sum_{j=i+k_1+1}^{i+k} Y_j\Big\|_2\,\rho(r).\end{aligned}$$

由归纳假设和引理 2.2.2 有

$$\begin{aligned}EU_i^2(k) &\leqslant C_1(k_1\log^2 2k_1 + k_2\log^2 2k_2 + 2(k_1k_2)^{1/2}\log 2k_1)\\ &\quad\cdot \rho^2(r)\cdot r\cdot \exp\left(\left(1+\frac{\varepsilon^*}{4}\right)\sum_{j=1}^{[\log n]}\rho(2^j)\right)EX_1^2 I(|X_1|>J)\\ &\leqslant C_1 k(\log 2k)^2 r\cdot \exp\left(\left(1+\frac{\varepsilon^*}{4}\right)\sum_{j=1}^{[\log n]}\rho(2^j)\right)EX_1^2 I\\ &\quad(|X_1|>J)\cdot \rho^2(r),\end{aligned}$$

这就证明了(4.2.19)成立.

从(4.2.19)并应用引理 4.1.2 我们得

$$\begin{aligned}&E\max_{1\leqslant i\leqslant p_1} U_0^2(i)\\ &\leqslant 3C_1 r p_1\rho^2(r)\log^4(2p_1)\cdot\exp\left((1+\varepsilon^*/4)\sum_{j=1}^{[\log n]}\rho(2^j)\right)\\ &\quad EX_1^2 I(|X_1|>J)\end{aligned}$$

$$\leqslant \frac{3C_1\sigma_n^2\rho^2(n/p_1)\log^4(2p_1)}{C_2 g(J)}\exp\Big((2+3\varepsilon^*/4)\sum_{j=1}^{[\log n]}\rho(2^j)\Big)$$
$$EX_1^2 g(|X_1|).$$
$$\leqslant \frac{3C_1\sigma_n^2\rho^2(n/p_1)\log^4(2p_1)}{C_2}\exp\Big(-\frac{\varepsilon^*}{52}\sum_{j=1}^{[\log n]}\rho(2^j)\Big)$$
$$EX_1^2 g(|X_1|).$$

由(4.2.12)

$$\rho^2\Big(\frac{n}{p_1}\Big)\log^4(2p_1)\leqslant\Big(\frac{50}{\delta}\Big)^4\rho^{2/3}\Big(\frac{n}{T_1^2}\Big)\sum_{j=1}^{[\log n]}\rho(2^j).$$

因此对任给 $\lambda>1$ 和每一充分大 $n$ 我们得

$$I_{12}\leqslant\varepsilon/\lambda^2.$$

所以

(4.2.20) $$I_1\leqslant\varepsilon/\lambda^2.$$

类似地，我们有

(4.2.21) $$I_2\leqslant 2\varepsilon/\lambda^2.$$

现在从(4.2.16)，(4.2.17)和(4.2.20)，(4.2.21)即得(4.2.10)．定理 4.2.1 证毕．

## §4.3 当高于 2 阶矩有限时的一般化结果

邵启满(1989a)给出了定理 4.2.1 的一个一般化结果，其中强平稳性条件被去掉了．

**定理 4.3.1** 设$\{x_n, n\geqslant 1\}$是 $\rho$ 混合序列，$EX_n=0$，$EX_n^2\geqslant C>0$．假设 $g:[0,\infty)\to[0,\infty)$是不减函数且 $x^\delta/g(x)$，$0<\delta<1$，也是不减的．如果下述条件被满足：

(i) $\{X_n^2 g(|x_n|), n\geqslant 1\}$是一致可积的，

(ii) $\sigma_n^2 := ES_n^2 = nh(n)$，其中 $h(n)$是缓变函数，

(iii) $\sup\limits_{m\geqslant 0, n\geqslant 1} ES_m^2(n)/\sigma_n^2<\infty$，

(iv) $\lim\limits_{n\to\infty}\inf\limits_{m\geqslant 0} ES_m^2(n)=\infty$，

(v) 对某 $0<\varepsilon^*<1$, $\exp\left((2+\varepsilon^*)\sum_{k=1}^{[\log n]}\rho(2^k)\right)=O(g(n^{1/2}))$, 那么 $W_n$ 弱收敛于 $W$.

定理 4.3.1 的证明需要下述引理.

**引理 4.3.1** 设 $f(k,m)$ 是满足 (4.1.2) 的非负函数. 假设存在 $\alpha>0$, $r\geqslant1$ 使得

(4.3.1) $E|S_k(l)|^r\leqslant f(k,l)\{f^\alpha(k,l)w_1(f(k,l))+w_2(f(k,l))\}$,

其中 $w_2$ 是非负不减函数, $w_1$ 是非负数使对某 $a>0$, $0<\beta<\alpha$ 和任何 $t>0$ 满足

(4.3.2) $$\max_{0\leqslant s\leqslant t}s^\beta w_1(s)\leqslant at^\beta w_1(t)$$

那么我们有

(4.3.3) $$E\max_{1\leqslant i\leqslant n}|S_k(i)|^r$$
$$\leqslant 2^{r-1}f(k,n)\left\{a\left(1-\left(\frac{1}{2}\right)^{(\alpha-\beta)/r}\right)^{-r}f^\alpha(k,n)w_1(f(k,n))\right.$$
$$\left.+w_2(f(k,n))\log^r(2n)\right\}.$$

**证** 从 (4.1.2) 和 (4.3.2), 对 $l\leqslant n$ 有

$$f^\beta(k,l)w_1(f(k,l))\leqslant af^\beta(k,n)w_1(f(k,n))$$

所以由 (4.3.1) 对每一 $k\geqslant0$, $1\leqslant l\leqslant n$

$$E|S_k(l)|^r\leqslant f(k,l)\{f^{\alpha-\beta}(k,l)$$
$$\cdot af^\beta(k,n)w_1(f(k,n))+w_2(f(k,n))\}$$

从引理 4.1.2 和 $w_2(\cdot)$ 的单调性即得

$$E\max_{1\leqslant l\leqslant n}|S_k(l)|^r$$
$$\leqslant\frac{5}{2}f(k,n)\left\{\sum_{i=0}^{[\log n]}w_2^{1/r}(f(k,n))\right.$$
$$\left.+(af^\beta(k,n)w_1(f(k,n)))^{1/r}\sum_{i=0}^{[\log n]}\left(\frac{f(k,n)}{2^i}\right)^{(\alpha-\beta)/r}\right\}^r$$
$$\leqslant\frac{5}{2}f(k,n)\{w_2^{1/r}(f(k,n))\log(2n)$$
$$+(af^\alpha(k,n)w_1(f(k,n)))^{1/r}\left(1-\left(\frac{1}{2}\right)^{(\alpha-\beta)/r}\right)^{-1}\}^r$$

$$\leqslant 5 \cdot 2^{r-2} f(k,n)\left\{ a f^{\alpha}(k,n) w_1(f(k,n))\left(1-\left(\frac{1}{2}\right)^{(\alpha\beta)/r}\right)^{-r} \right.$$

$$\left. + w_2(f(k,n))\log^r(2n)\right\}.$$

得证引理 4.3.1 成立.

**引理 4.3.2** 设$\{x_n, n\geqslant 1\}$是$\rho$混合序列,$EX_n=0$,且$q_1, q_2\geqslant 2$. 假设非负函数$h(n)$满足:对每一$n\geqslant 1$和某$0<\theta<2^{1-2/(q_1\wedge 3)}$有

$$\max\left(h\left(\left[\frac{n}{2}\right]\right), h\left(n-\left[\frac{n}{2}\right]\right)\right)\leqslant \theta h(n) \tag{4.3.4}$$

且当$q_1>3$时对某$a>0$和某$\alpha$,$0<\alpha<q_1-2$满足

$$h(n)\geqslant \frac{1}{a}\exp\left\{-a\sum_{i=0}^{[\log n]}\rho^{2/q_1}(2^i)\right\}, \tag{4.3.5}$$

$$\max_{1\leqslant i\leqslant n} i^{\alpha} h(i)\leqslant a n^{\alpha} h(n).$$

又设对整数$1\leqslant k\leqslant n$, $l\geqslant 0$和数$x>0$, $0<B\leqslant A<\infty$满足

$$4n\max_{l<i\leqslant l+n} E|X_i|I(|X_i|\geqslant A)\leqslant x, \tag{4.3.6}$$

$$48k\max_{l<i\leqslant l+n} E|X_i|I(|X_i|\geqslant B)\leqslant x, \tag{4.3.7}$$

$$\operatorname{Var}\left(\sum_{i=j+1}^{j+m} X_i I(|X_i|\leqslant B)\right)\leqslant m h(m)\max_{j<i\leqslant j+m} EX_i^2 I(|X_i|\leqslant B). \tag{4.3.8}$$

那么对任给的$\varepsilon>0$,存在$K=K(\varepsilon, q_1, q_2, a, \theta, \alpha, \rho(\cdot))$使得

$$P\left\{\max_{1\leqslant i\leqslant n}|S_l(i)|\geqslant x\right\}\leqslant \sum_{i=l+1}^{l+n} P(|X_i|\geqslant A)$$

$$+K\left\{x^{-q_1}\left[\left(nh(n)\max_{l<i\leqslant l+n} EX_i^2 I(|X_i|\leqslant B)\right)^{q_1/2}\right.\right.$$

$$\left.+n\exp\left\{K\sum_{i=0}^{[\log n]}\rho^{2/q_1}(2^i)\right\}\log^{q_1}(2n)\max_{1<i\leqslant l+n} E|X_i|^{q_1} I(|X_i|\leqslant B)\right]$$

$$+x^{-q_2}\left[\left(n\exp\left\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}\max_{1<i\leqslant l+n} EX_i^2 I(B<|X_i|<A)\right)^{q_2/2}\right.$$

$$\left.+n\exp\left\{K\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\right\}\max_{1<i\leqslant l+n} E|X_i|^{q_2} I(B<|X_i|<A)\right]$$

$$+x^2 n\rho^2(n)\log^4\left[\frac{n}{k}\right]$$

$$\cdot \exp\left\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}\max_{l<i\leqslant l+n} EX_i^2 I(B<|X_i|<A)\right\}.$$

**证** 为简单计，假设 $X_n(n\geqslant 1)$ 有相同分布. 记

$$X_{i1}=X_iI(|X_i|\leqslant B)-EX_iI(|X_i|\leqslant B),$$

$$X_{i2}=X_iI(B<|X_i|<A)-EX_iI(B<|X_i|<A),$$

$$X_{i3}=X_iI(|X_i|\geqslant A)-EX_iI(|X_i|\geqslant A),$$

$$S_{ik}(i)=\sum_{j=l+1}^{l+i}X_{jk},\qquad k=1,2,3.$$

容易看出

$$P\left\{\max_{1\leqslant i\leqslant n}|S_l(i)|\geqslant x\right\} \tag{4.3.9}$$

$$\leqslant P\left\{\max_{1\leqslant i\leqslant n}|S_{l1}(i)|\geqslant\frac{x}{4}\right\}+P\left\{\max_{1\leqslant i\leqslant n}|S_{l2}(i)|\geqslant\frac{x}{4}\right\}$$

$$+P\left\{\max_{1\leqslant i\leqslant n}|S_{l3}(i)|\geqslant\frac{x}{2}\right\}$$

$$=:I_1+I_2+I_3.$$

由(4.3.6)，我们有

$$I_3\leqslant P\left\{\sum_{i=l+1}^{l+n}|X_i|I(|X_i|\geqslant A)\geqslant\frac{x}{2}\right. \tag{4.3.10}$$

$$\left.-\sum_{i=l+1}^{l+n}E|X_i|I(|X_i|\geqslant A)\right\}$$

$$\leqslant P\left\{\sum_{i=l+1}^{l+n}|X_i|I(|X_i|\geqslant A)\geqslant\frac{x}{4}\right\}$$

$$\leqslant\sum_{i=l+1}^{l+n}P(|X_i|\geqslant A)$$

由引理 2.2.6，存在 $K_1=K_1(q_1,a,\theta,\rho(\cdot))$ 使得

$$E\left|\sum_{i=j+1}^{j+n}X_{i1}\right|^{q_1}\leqslant K_1\{(nh(n)EX_1^2I(|X_1|\leqslant B))^{q_1/2}$$

$$+n\exp\left[K_1\sum_{i=0}^{[\log n]}\rho^{2/q_1}(2^i)\right]E|X_1|^{q_1}I(|X_1|\leqslant B)\}.$$

从引理 4.3.1 即得存在 $K_2=K_2(k_1,\alpha,q_1)$ 使得

$$E\max_{1\leqslant i\leqslant n}|S_{l1}(i)|^{q_1}\leqslant K_2\left\{(nh(n)EX_1^2I(|X_1|\leqslant B))^{q_1/2}\right.$$

$$-n\exp\left\{K_2\sum_{i=0}^{[\log n]}\rho^{2/q_1}(2^i)\right\}(\log(2n))^{q_1}E|X_1|^{q_1}I(|X_1|\leqslant B)\right\}.$$

那么我们有

$$(4.3.11)\quad I_1\leqslant K_2x^{-q_1}\{(nh(n)EX_1^2I(|X_1|\leqslant B))^{q_1/2}$$

$$+n\exp\left\{K_2\sum_{i=0}^{[\log n]}\rho^{2/q_1}(2^i)\right\}(\log(2n))^{q_1}$$

$$\cdot E|X_1|^{q_1}I(|X_1|\leqslant B)\}.$$

为估计 $I_2$，令

$$Y_i=\sum_{j=2ik+l+1}^{(2i+1)k+l}X_{j2},\qquad i=0,1,\cdots,p_1,$$

$$Z_i=\sum_{j=(2i+1)k+l+1}^{2(i+1)k+l}X_{j2},\qquad i=0,1,\cdots,p_2,$$

其中 $p_1=\left[\left(\frac{n}{k}-1\right)/2\right]$，$p_2=\left[\left(\frac{n}{k}-2\right)/2\right]$. 记

$$W_i=\sum_{j=0}^{i}Y_j,\qquad W_i^*=\sum_{j=0}^{i}Z_j$$

容易看出

$$(4.3.12)\quad I_2\leqslant P\left\{\max_{0\leqslant i\leqslant p_1}|W_i|\geqslant\frac{x}{12}\right\}+P\left\{\max_{0\leqslant i\leqslant p_2}|W_i^*|\geqslant\frac{x}{12}\right\}$$

$$+P\left\{\max_{0\leqslant i\leqslant[n/k]}\max_{ik+1\leqslant j<(i+1)k}\left|\sum_{v=l+ik+1}^{l+j}X_{v2}\right|\geqslant\frac{x}{12}\right\}$$

$$=:I_{21}+I_{22}+I_{23}.$$

从条件(4.3.7)和引理 2.2.5 即得

$$I_{23}\leqslant 2\left[\frac{n}{k}\right]\max_{0\leqslant i\leqslant[n/k]}P\left\{\sum_{j=l+ik+1}^{l+(i+1)k}\right.$$

$$(|X_j|I(B<|X_j|<A)-E|X_j|I(B<|X_j|<A))$$

$$\left.\geqslant\frac{x}{12}-2\sum_{j=l+ik+1}^{l+(i+1)k}E|X_j|I(B<|X_j|<A)\right\}$$

$$\leqslant\frac{2n}{k}\max_{0\leqslant i\leqslant[n/k]+}P\left\{\sum_{j=l+ik+1}^{l+(i+1)k}(|X_j|I(B<|X_j|<A)\right.$$

$$\left.-E|X_j|I(B<|X_j|<A))\geqslant x/24\right\}$$

$$\leqslant C\frac{n}{k}x^{-q_2}\left\{\left(k\exp\left((1+\varepsilon)\sum_{i=0}^{[\log k]}\rho(2^i)\right)EX_1^2I(B<|X_j|<A)\right)^{q_2/2}\right.$$

$$+ k\exp\left\{C\sum_{i=0}^{[\log k]}\rho^{2/q_2}(2^i)\right\}E|X_1|^{q_2}I(B<|X|<A)\Big\}$$

$$\leqslant Cx^{-q_2}\Big\{\Big(n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}EX_1^2I(B<|X_1|<A)\Big)^{q_2/2}$$

$$+ n\exp\Big\{C\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\Big\}E|X_1|^{q_2}I(B<|X_1|<A)\Big\}.$$

其次，我们来估计 $I_{21}$（$I_{22}$的估计同法可得）. 记 $\mathscr{F}_{-1}=\{\phi,\Omega\}$，$\mathscr{F}_i=\sigma(X_j: j\leqslant l+(2i+1)k)$，$i=0,1,\cdots,p_1$. 令

$$U_i=Y_i-E(Y_i|\mathscr{F}_{i-1}),\ G_i=\sum_{j=0}^{i}U_j,$$

$$H_i=\sum_{j=1}^{i}E(Y_j|\mathscr{F}_{j-1}),\ i=0,1,\cdots,p_1.$$

容易看出

$$(4.3.13)\quad I_{21}\leqslant P\Big\{\max_{1\leqslant i\leqslant p_1}|G_i|\geqslant x/24\Big\}+P\Big\{\max_{1\leqslant i\leqslant p_1}|H_i|\geqslant x/24\Big\}$$

$$=:I_{21}(1)+I_{21}(2).$$

由于$\{V_i,\mathscr{F}_i,i\geqslant 1\}$是鞅差序列，由极大值不等式（见 Brown 1971），我们有

$$(4.3.14)\quad I_{21}(1)\leqslant\frac{24}{x}E|G_{p_1}|I\Big(|G_{p_1}|\geqslant\frac{x}{48}\Big)$$

$$\leqslant\frac{48}{x}\Big\{E|W_{p_1}|I\Big(|W_{p_1}|$$

$$\geqslant\frac{x}{96}\Big)+E|H_{p_1}|I\Big(|H_{p_1}|\geqslant\frac{x}{96}\Big)\Big\}$$

$$\leqslant(96/x)^{q_2}E|W_{p_1}|^{q_2}+(96/x)^{q_2}E|H_{p_1}|^{q_2}.$$

从引理 2.2.5 即得

$$(4.3.15)\quad E|W_{p_1}|^{q_2}\leqslant C\Big\{\Big(n\exp\Big[(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big]$$

$$\cdot EX_1^2I(B<|X_1|<A)\Big)^{q_2/2}$$

$$+n\exp\Big[C\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\Big]E|X_1|^{q_2}I(B<|X_1|<A)\Big\}$$

下面用归纳法来证存在常数 $K'$ 使得

(4.3.16) $$E\Big(\sum_{j=i+1}^{i+m} E(Y_i|\mathscr{F}_{j-1})\Big)^2$$
$$\leqslant K' mk\rho^2(k)(\log(2m))^2$$
$$\cdot\exp\Big[(1+\varepsilon)\sum_{j=0}^{[\log m]}\rho(2^j)\Big]EX_1^2I(B<|X_1|<A).$$

从引理 2.2.1 即得存在常数 $C'$ 使对 $mk\leqslant n$ 有

(4.3.17) $$E\Big(\sum_{j=i+1}^{i+m} Y_j\Big)^2 \leqslant C' mk\exp\Big[(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\Big]$$
$$\cdot EX_1^2I(B<|X_1|<A).$$

当 $m=1$ 时由 $\rho$ 混合的定义得

$$E(E(Y_{i+1}|\mathscr{F}_i))^2=E(Y_{i+1}E(Y_{i+1}|\mathscr{F}_i))$$
$$\leqslant\rho(k)\,\|Y_{i+1}\|_2\,\|E(Y_{i+1}|\mathscr{F}_i)\|_2$$

即

$$E(E(Y_{i+1}|\mathscr{F}_i))^2\leqslant\rho^2(k)EY_{i+1}^2.$$

取 $K'=C'/\log^2(3/2)$ 即得 $m=1$ 时(4.3.16)成立.

假设(4.3.16)对小于 $m$ 的整数成立.我们来证对 $m$(4.3.16)也成立.记 $m_1=[m/2]$,$m_2=m-m_1$,我们有

$$E\Big(\sum_{j=i+1}^{i+m} E(Y_j|\mathscr{F}_{j-1})\Big)^2$$
$$=E\Big(\sum_{j=i+1}^{i-m_1} E(Y_j|\mathscr{F}_{j-1})\Big)^2+E\Big(\sum_{j=i+m_1+1}^{i+m} E(Y_j|\mathscr{F}_{j-1})\Big)^2$$
$$+2E\Big(\sum_{j=i+1}^{i+m_1} E(Y_j|\mathscr{F}_{j-1})\Big)\Big(\sum_{j=i+m_1+1}^{i+m} E(Y_j|\mathscr{F}_{j-1})\Big)$$
$$\leqslant E\Big(\sum_{j=i+1}^{i-m_1} E(Y_j|\mathscr{F}_{j-1})\Big)^2+E\Big(\sum_{j=i+m_1+1}^{i+m} E(Y_j|\mathscr{F}_{j-1})\Big)^2$$
$$+2\rho(k)E\Big\|\sum_{j=i+1}^{i+m_1} E(Y_j|\mathscr{F}_{j-1})\Big\|_2\Big\|\sum_{j=i+m_1+1}^{i+m} Y_j\Big\|_2.$$

由归纳假设和(4.3.17)可以推出

$$E\Big(\sum_{j=i+1}^{i+m} E(|Y_j|\mathscr{F}_{j-1})\Big)^2$$

$$\leqslant K'\left\{m_1\log^2(2m_1)+m_2\log^2(2m_2)+2\left(\log\frac{3}{2}\right)\right.$$

$$\left.\cdot(m_1m_2)^{1/2}\log(2m_1)\right\}$$

$$\cdot\rho^2(k)k\exp\left\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\right\}EX_1^2I(B<|X_1|<A)$$

$$\leqslant K'mk\rho^2(k)\log^2(2m)\exp\left\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\right\}$$

$$\cdot EX_1^2I(B<|X_1|<A).$$

这就证明了(4.3.16). 其次,从(4.3.16)和引理 4.3.1 可知,存在常数 $K''$使得

(4.3.18) $$E\max_{0\leqslant i\leqslant p1}H_i^2$$

$$\leqslant K''p_1k\rho^2(k)(\log(2p_1))^4\exp\left\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\right\}$$

$$\cdot EX_1^2I(B<|X_1|<A).$$

这样我们有

(4.3.19) $$I_{21}(2)\leqslant 288K''x^{-2}n\rho^2(k)\log^4[n/k]$$

$$\cdot\exp\left\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\right\}EX_1^2I(B<|X_1|<A).$$

从(4.3.13),(4.3.14),(4.3.15),(4.3.18)和(4.3.19)即得对某 $K_3>0$

$$I_{21}\leqslant K_3x^{-q2}\left\{\left(n\exp\left\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\right\}\right.\right.$$

$$\left.\cdot EX_1^2I(B<|X_1|<A)\right)^{q2/2}$$

$$+n\exp\left\{K_3\sum_{j=0}^{[\log n]}\rho^{2/q_2}(2^j)\right\}$$

$$\left.\cdot E|X_1|^{q_2}I(B<|X_1|<A)\right\}$$

$$+K_3x^{-2}n\rho^2(k)\log^4\left[\frac{n}{k}\right]$$

$$\cdot\exp\left\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\right\}EX_1^2I(B<|X_1|<A).$$

引理 4.3.2 证毕.

**引理 4.3.3** 设 $0<\delta\leqslant1$. 假设非负函数 $h(n)$ 满足下述条件：存在整数 $n_0>0$, $0<\theta<2^{\delta/(2+\delta)}$, $0<\delta'<\delta$ 和 $a>0$ 使对任一 $n\geqslant n_0$ 有

(4.3.20) $$h\left(\left[\frac{n}{2}\right]\right)\vee h\left(n-\left[\frac{n}{2}\right]\right)\leqslant\theta h(n),$$

(4.3.21) $$\max_{1\leqslant i\leqslant n} i^{\delta'}h(i)\leqslant an^{\delta'}h(n).$$

设 $\{X_n,n\geqslant1\}$ 是 $\rho$ 混合序列, $EX_n=0$, $EX_n^2<\infty$ 且对每一 $k\geqslant0$, $n\geqslant1$ 满足

(4.3.22) $$ES_k^2(n)\leqslant nh(n)\max_{k<i\leqslant k+n}EX_i^2.$$

那么对任给 $\epsilon>0$ 存在 $K=K(\epsilon,\delta,\delta',n_0,a,\rho(\cdot))$ 使对任一 $n\geqslant k\geqslant0$, $l\geqslant0$ 和 $B>0$ 有

(4.3.23) $$E\max_{1\leqslant i\leqslant n}|S_l(i)|^2I(\max_{1\leqslant i\leqslant n}|S_l(i)|\geqslant x)$$

$$\leqslant K\Big\{x^{-\delta}\Big\{\Big(nh(n)\max_{l<i\leqslant l+n}EX_i^2+n\exp\Big\{(1+\epsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}$$

$$\cdot\max_{l<i\leqslant l+n}EX_i^2(|X_i|\geqslant B)\Big)^{\frac{2+\delta}{2}}$$

$$+n\exp\Big\{K\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big\}\log^{2+\delta}(2n)$$

$$\cdot\max_{l<i\leqslant l+n}E|X_i|^{2+\delta}I(|X_i|<B)\Big\}$$

$$+n\exp\Big\{(1+\epsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\Big(1+\rho^2(k)\log^4\Big[\frac{n}{k}\Big]\Big)$$

$$\cdot\max_{l<i\leqslant l+n}EX_i^2I(|X_i|\geqslant B)$$

$$+nk\max_{l<i\leqslant l+n}(E|X_i|I(|X_i|>B))^2\Big\}.$$

**证** 记

$$X_{i_1}=X_iI(|X_i|<B)-EX_iI(|X_i|<B),\quad S_{l1}(i)=\sum_{j=l+1}^{l+i}X_{j1},$$

$$X_{i_2}=X_iI(|X_i|\geqslant B)-EX_iI(|X_i|\geqslant B),\quad S_{l_2}(i)=\sum_{j=l+1}^{l+i}X_{j2}.$$

容易看出

(4.3.24) $E\max\limits_{1\leqslant i\leqslant n}S_l^2(i)I(\max\limits_{1\leqslant i\leqslant n}|S_l(i)|\geqslant x)$

$\leqslant 4E\max\limits_{1\leqslant i\leqslant n}S_{l1}^2(i)I(\max\limits_{1\leqslant i\leqslant n}|S_{l1}(i)|\geqslant x/2)+4E\max\limits_{1\leqslant i\leqslant n}S_{l2}^2(i)$

$\leqslant 8x^{-\delta}E\max\limits_{1\leqslant i\leqslant n}|S_{l1}(i)|^{2+\delta}+4E\max\limits_{1\leqslant i\leqslant n}S_{l2}^2(i)$

$=: 8I_1+4I_2.$

从引理 2.2.2 对每一 $n\geqslant 1, l\geqslant 0$ 我们有

$$ES_{l_2}^2(n)\leqslant Cn\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\max_{l\leqslant i\leqslant l+n}EX_i^2I(|X_i|\geqslant B),$$

其中 $C=C(\varepsilon)$. 所以由(4.3.22)我们得

(4.3.25) $ES_{l1}^2(n)\leqslant 2ES_l^2(n)+2ES_{l_2}^2(n)$

$$\leqslant 2nh(n)\max_{l\leqslant i\leqslant l+n}EX_i^2$$

$$+2Cn\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\max_{l\leqslant i\leqslant l+n}EX_i^2I(|X_i|\geqslant B).$$

从引理 2.2.6, 存在 $K_1$ 使对每一 $l\geqslant 0, n\geqslant 1$ 有

$$E|S_{l1}(n)|^{2+\delta}$$

$$\leqslant K_1\Big\{\Big(nh(n)\max_{l<i\leqslant l+n}EX_i^2$$

$$+n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\max_{l<i\leqslant l+n}EX_i^2I(|X_i|\geqslant B)\Big)^{(2+\delta)/2}$$

$$+n\exp\Big\{K_1\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big\}$$

$$\cdot\max_{l<i\leqslant l+n}E|X_i|^{2+\delta}I(|X_i|\leqslant B)\Big\}.$$

由引理 4.3.1 我们得

(4.3.26) $I_1\leqslant x^{-\delta}c\Big\{\Big(nh(n)\max\limits_{l<i\leqslant l+n}EX_i^2$

$$+n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\max_{l<i\leqslant l+n}EX_i^2I(|X_i|>B)\Big)^{\frac{2+\delta}{2}}$$

$$+n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big\}$$

$$\max_{l<i\leqslant l+n}E|X_i|^{2+\delta}I(|X_i|\leqslant B)\log^{2+\delta}(2n)\Big\}.$$

下面我们来估计 $I_2$. 记

$$Y_i = \sum_{j=l+2ik+1}^{l+(2i+1)k} X_{j2}, W_i = \sum_{j=0}^{i} Y_j,\ i = 0,1,\cdots,p_1,$$

$$Z_i = \sum_{j=l+(2i+1)k+1}^{l+2(i+1)k} X_{j2}, W_i^* = \sum_{j=0}^{i} Z_j, i = 0,1,\cdots,p_2,$$

其中 $p_1 = \left[\left(\frac{n}{k} - 1\right)\Big/2\right], p_2 = \left[\frac{n}{2k}\right] - 1$. 我们有

(4.3.27) $$I_2 \leqslant 8\Big\{E \max_{0\leqslant i\leqslant p_1} W_i^2 + E \max_{0\leqslant i\leqslant p_2} W_i^{*2} + E \max_{0\leqslant i\leqslant [n/k]} \max_{ik+1\leqslant j\leqslant (i+1)k} \Big|\sum_{v=j+ik+1}^{l+j} X_{v2}\Big|^2\Big\} =: 8(I_{21} + I_{22} + I_{23}).$$

从引理 4.3.2 的证明容易看出

(4.3.28) $$I_{21}+I_{22} \leqslant cn\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\Big(1+\rho^2(k)\log^4\Big[\frac{n}{k}\Big]\Big) \cdot \max_{l<i\leqslant l+n} EX_i^2 I(|X_i|>B).$$

且从引理 2.2.2 有

(4.3.29) $$\begin{aligned} I_{23} &\leqslant \sum_{0\leqslant i\leqslant [n/k]} E \max_{ik+1\leqslant j\leqslant (i+1)k} \Big|\sum_{v=l+ik+1}^{l+j} X_{v2}\Big|^2 \\ &\leqslant 32 \sum_{0\leqslant i\leqslant [n/k]} \Big(E\Big|\sum_{v=i+ik+1}^{l+(i+1)k} \{|X_v|I(|X_v| \geqslant B) - E|X_v|I(|X_v| \geqslant B)\}\Big|^2 + k^2 \\ &\qquad \max_{l+ik+1\leqslant j\leqslant l+(i+1)k} (E|X_j|I(|X_j| \geqslant B)^2)\Big) \\ &\leqslant c\Big\{n\exp\Big\{(1 + \varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\} \max_{l<i\leqslant l+n} EX_i^2 I(|X_i| \geqslant B) \\ &\qquad + nk \max_{l<i\leqslant l+n} (E|X_i|I(|X_i| \geqslant B))^2\Big\}. \end{aligned}$$

把(4.3.26)—(4.3.29)代入(4.3.24)就证明了(4.3.23). 证毕.

**定理 4.3.1 的证明.**

由条件(ii), 我们只需证明

(a) 对任给 $0\leqslant t\leqslant 1$, $\{S_{[nt]}^2/\sigma_n^2, n\geqslant 1\}$是一致可积的,

(b) 对任给 $\varepsilon>0$, 存在 $\lambda>1$ 和整数 $n_0$ 使对 $n\geqslant n_0$, $0\leqslant k\leqslant n\lambda^2/\varepsilon$

有

(4.3.30) $$P\left\{\max_{1\leq i\leq n}\left|\sum_{j=k+1}^{k+i}x_j\right|\geq\lambda\sigma_n\right\}\leq\varepsilon/\lambda^2.$$

为此我们只需证明对任给 $\varepsilon>0$，存在 $\lambda>1$ 和整数 $n_0$ 使对每一 $n\geq n_0$，$l\geq 0$ 有

(4.3.31) $$E\max_{1\leq i\leq n}S_l^2(i)I(\max_{1\leq i\leq n}|S_l(i)|\geq\lambda(nh(n))^{1/2})/nh(n)\leq\varepsilon.$$

不失一般性我们假设对 $n\geq 16$ 有

(4.3.32) $$\rho(n)\geq 1/(\log n(\log\log n)^2).$$

事实上，若令 $\rho^*(n)=\rho(n)\vee(\log n)^{-1}(\log\log n)^{-2}$，容易验证 $\rho^*(n)$也满足条件(V)。

由条件(i)，(ii)，(iii)和引理 4.3.3，存在常数 $K$ 使对每一 $n$，$1\leq k\leq n$，$B>0$ 和 $\lambda>0$ 我们有

(4.3.33) $$E\max_{1\leq i\leq n}S_l^2(i)I\left(\max_{1\leq i\leq n}|S_l(i)|\geq\lambda(nh(n))^{1/2}\right)/nh(n)$$

$$\leq K\left\{\left(\left\{nh(n)+n\exp\left(\left(1+\frac{\varepsilon}{4}\right)\sum_{i=0}^{[\log n]}\rho(2^i)\right)\right.\right.\right.$$

$$\left.\cdot\max_{l<i\leq l+n}EX_i^2I(|X_i|>B)\right\}^{\frac{2+\delta}{2}}$$

$$+n\exp\left(K\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\right)(\log(2n))^{2+\delta}$$

$$\cdot\max_{l<i\leq l+n}E|X_i|^{2+\delta}I(|X_i|\leq B)\Big)/(\lambda^\delta(nh(n))^{\frac{2+\delta}{2}})$$

$$+n\exp\left(\left(1+\frac{\varepsilon}{4}\right)\sum_{i=0}^{[\log n]}\rho(2^i)\right)\max_{l<i\leq l+n}EX_i^2I(|X_i|>B)$$

$$\left.1-\rho^2(k)\log^2\left[\frac{n}{k}\right]\right)/nh(n)$$

$$\left.+nk\max_{l<i\leq l+n}(E|X_i|I(|X_i|\geq B))^2/nh(n)\right\}$$

$$=:K(I_1+I_2+I_3).$$

令

$$T=\exp\left\{\frac{3k}{\delta}\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\right\},$$

$$B=n^{1/2}/T,\ k=[n/T^2]+1.$$

我们先来估计 $I_1$. 注意到

$$\log T = \frac{3k}{\delta}\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)$$

$$\leqslant \frac{3k}{\delta}\rho^{-\delta/(2+\delta)}\left(\frac{n}{T^2}\right)\sum_{i=0}^{[\log nT^{-2}]}\rho(2^i) + \frac{3k}{\delta}\rho^{2/(2+\delta)}\left(\frac{n}{T^2}\right)\sum_{i=[\log nT^{-2}]+1}^{[\log n]} 1$$

$$\leqslant \frac{3k}{\delta}\rho^{-\delta/(2+\delta)}\left(\frac{n}{T^2}\right)\sum_{i=0}^{[\log nT^{-2}]}\rho(2^i) + \frac{7k}{\delta}\rho^{2/(2+\delta)}\left(\frac{n}{T^2}\right)\log T.$$

所以对充分大 $n$ 我们有

$$\log T \leqslant \frac{3k}{\delta}\left(1+\frac{\varepsilon}{24}\right)\rho^{-\delta/(2+\delta)}\left(\frac{n}{T^2}\right)\sum_{i=0}^{[\log nT-2]}\rho(2^i) \tag{4.3.34}$$

和

$$\sum_{i=0}^{[\log n]}\rho(2^i) \leqslant \left(1+\frac{\varepsilon}{12}\right)\sum_{i=0}^{[\log nT-2]}\rho(2^i). \tag{4.3.35}$$

由条件(V)，存在 $C_1>0$ 使对每一 $n\geqslant 1$ 有

$$g(n^{1/2}) \geqslant C_1\exp\left\{(2+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}. \tag{4.3.36}$$

它与(4.3.35)和条件(V)相结合，对充分大 $n$ 得

$$g(B) \geqslant C_1\exp\left\{\frac{2+\varepsilon}{1+\varepsilon/12}\sum_{i=0}^{[\log n]}\rho(2^i)\right\}. \tag{4.3.37}$$

由引理 2.2.3 和条件(iii)，存在常数 $C_2>0$ 使对充分大的 $n$

$$ES_n^2 \geqslant C_2 n\exp\left\{-\left(1+\frac{\varepsilon}{4}\right)\sum_{i=0}^{[\log n]}\rho(2^i)\right\},$$

因此，由条件(ii)我们有

$$h(n) \geqslant C_2\exp\left\{-\left(1+\frac{\varepsilon}{4}\right)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}. \tag{4.3.38}$$

由 $g(x)$ 和 $x^\delta/g(x)$ 的单调性，我们得

$$\max_{i\geqslant 1}EX_i^2 I(|X_i|\geqslant B) \leqslant \frac{1}{g(B)}\cdot\max_{i\geqslant 1}EX_i^2 g(|X_i|)I(|X_i|\geqslant B), \tag{4.3.39}$$

$$\max_{i\geqslant 1}E|X_i|^{2+\delta}I(|X_i|\leqslant B) \leqslant \frac{B^\delta}{g(B)}\max_{i\geqslant 1}EX_i^2 g(|X_i|). \tag{4.3.40}$$

结合(4.3.37)，(4.3.38)，(4.3.40)和(4.3.32)我们推及对充分大

$n$ 有

(4.3.41) $$n\exp\left\{\left(1+\frac{\varepsilon}{4}\right)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}\max_{i\geq 1}EX_i^2I(|X_i|\geq B)$$
$$\leq\frac{nh(n)}{C_1C_2}\max_{i\geq 1}EX_i^2g(|X_i|)I(|X_i|\geq B),$$

(4.3.42) $$n\exp\left\{K\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\right\}\log^{2+\delta}(2n)$$
$$\max_{i\geq 1}E|X_i|^{2+\delta}I(|X_i|\leq B)$$
$$\leq n\exp\left\{2K\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\right\}B^{\delta}\max_{i\geq 1}EX_i^2g(|X_i|)/g(B)$$
$$\leq n^{(2-\delta)/2}\max_{i\geq 1}EX_i^2g(|X_i|)\exp\left\{-K\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\right\}$$
$$\leq (nh(n))^{(2+\delta)/2}\max_{i\geq 1}EX_i^2g(|X_i|).$$

由条件(i)我们有$\max_{i\geq 1}EX_i^2g(|X_i|)<\infty$. 所以对充分大 $n$ 和充分大 $\lambda$ 有

(4.3.43) $$I_1\leq\varepsilon/(3K).$$

现在来估计 $I_2$. 从(4.3.34)和 $k$ 的定义,我们有

(4.3.44) $$\left(\log\left[\frac{n}{k}\right]\right)^4\rho^2(k)\leq 2(\log T)^4\rho^2(k)$$
$$\leq\left(\frac{3k}{8}\right)^4\left(1+\frac{\varepsilon}{24}\right)^4\left(\rho\left(\frac{n}{T^2}\right)\right)^{2-\frac{4\delta}{2+\delta}}\left(\sum_{i=0}^{[\log n]}\rho(2^i)\right)^4$$
$$\leq\left(\frac{3k}{8}\right)^4\left(\sum_{i=0}^{[\log n]}\rho(2^i)\right)^4.$$

由(4.3.44),(4.3.39),(4.3.37)和(4.3.38),我们得

(4.3.45) $$n\exp\left\{\left(1+\frac{\varepsilon}{4}\right)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}\max_{i\geq 1}EX_i^2I(|X_i|\geq B)$$
$$\cdot\left(1+\rho^2(k)\log^4\left[\frac{n}{k}\right]\right)$$
$$\leq\frac{nh(n)}{g(B)h(n)}\exp\left\{\left(1+\frac{\varepsilon}{4}\right)\sum_{i=0}^{[\log n]}\rho(2^i)\right\}$$
$$\cdot\left(1+\left(\frac{3k}{\delta}\right)^4\left(\sum_{i=0}^{[\log n]}\rho(2^i)\right)^4\right)\max_{i\geq 1}EX_i^2g(|X_i|)I(|X_i|\geq B)$$

$$\leqslant \frac{nh(n)}{C_1C_2}\exp\left\{-\frac{\varepsilon}{8}\sum_{i=0}^{[\log n]}\rho(2^i)\right\}$$

$$\cdot\left(1+\left(\frac{3k}{\delta}\right)^4\left(\sum_{i=0}^{[\log n]}\rho(2^i)\right)^4\right)\max_{i\geqslant 1}EX_i^2g(|X_i|)I(|X_i|\geqslant B)$$

$$\leqslant C_3nh(n)\max_{i\geqslant 1}EX_i^2g(|X_i|)I(|X_i|\geqslant B),$$

这里我们用到了下述结果：

$$\left(\sum_{i=0}^{[\log n]}\rho(2^i)\right)^4\exp\left\{-\frac{\varepsilon}{8}\sum_{i=0}^{[\log n]}\rho(2^i)\right\}=O(1).$$

结合(4.3.45)和条件(i)，我们有

$$I_2\leqslant\varepsilon/3K. \tag{4.3.46}$$

最后，我们考察 $I_3$. 注意到

$$\max_{i\geqslant 1}E|X_i|I(|X_i|\geqslant B)\leqslant\frac{1}{Bg(B)}\max_{i\geqslant 1}EX_i^2g|X_i|I(|X_i|\geqslant B).$$

因此从(4.3.37)和(4.3.38)即得

$$kn\max_{i\geqslant 1}(E|X_i|I(|X_i|\geqslant B))^2$$

$$\leqslant\frac{kn}{B^2g^2(B)}\max_{i\geqslant 1}(EX_i^2g(|X_i|)I(|X_i|\geqslant B))^2$$

$$\leqslant\frac{2nh(n)}{C_1^2C_2}\max_{i\geqslant 1}(EX_i^2g(|X_i|)I(|X_i|\geqslant B))^2.$$

由条件(i)，对充分大 $n$ 我们有

$$I_3\leqslant\varepsilon/(3K). \tag{4.3.47}$$

把(4.3.43)，(4.3.46)和(4.3.47)代入(4.3.33)，我们证明了(4.3.31). 证毕.

从定理 4.3.1 我们有下述直接推论.

**推论 4.3.1** 设$\{X_n,n\geqslant 1\}$是强平稳 $\rho$ 混合序列，$EX_1=0$，$EX_1^2<\infty$，$\sigma_n^2=ES_n^2\to\infty$. 若下述条件之一被满足：

(i) $EX_1^2g(|X_1|)<\infty$，且对某 $0<\varepsilon<1$，

$$\exp\left\{(2+\varepsilon)\sum_{j=1}^{[\log n]}\rho(2^j)\right\}=O(g(n^{1/2})),$$

(ii) 对某 $\varepsilon>0$，$a>0$，$EX_1^2(\log|X_1|)^{2a/(1-\varepsilon)}<\infty$ 且 $\rho(n)\leqslant a/\log n$，

(iii)对某 $0<\beta<1,\varepsilon>0,a>0$

$$EX_1^2\exp\left\{\frac{2a(1+\varepsilon)}{1-\beta}(2\log|X_1|)^{1-\beta}\right\}<\infty$$

且

$$\rho(n)\leqslant a/(\log n)^{\beta},$$

(iv)对某 $r>0,\varepsilon>0,a>0$

$$EX_1^2\exp\left(\frac{4a(1+\varepsilon)\log|X_1|}{(\log\log|X_1|)^r}\right)<\infty$$

$$\rho(n)\leqslant a(\log\log n)^{-r},$$

那么

$$W_n\Rightarrow W.$$

## §4.4　方差无穷时的弱收敛

Bradley(1988)对方差无穷的强平稳 $\rho$ 混合序列建立了中心极限定理. 邵启满(1993$a$)在同样假设下证明了弱收敛.

**定理 4.4.1**　设 $\{X_n,n\geqslant1\}$ 是强平稳 $\rho$ 混合非退化随机变量序列, $EX_1=0$. 假设

(i) $H(x):=EX_1^2I(|X_1|\leqslant x)$ 当 $x\to\infty$ 时是缓变的,

(ii) $\rho(1)<1$,

(iii) $\sum_{n=1}^{\infty}\rho(2^n)<\infty$.

那么存在正数列 $\{A_n,n\geqslant1\}$, $A_n\to\infty(n\to\infty)$ 使得

$$W_n\Rightarrow W.$$

其中 $W_n(t)=S_{[nt]}/A_n,0\leqslant t\leqslant1$.

**注 4.4.1**　事实上, 邵启满(1993a)证明了一个更一般化结果: 设 $g:(-\infty,\infty)\to[0,\infty)$ 是不减连续的偶函数且对任一 $\delta>0$, 对充分大 $x$, $x^{\delta}/g(x)$ 是不减的, 设

$$e(x,\varepsilon)=\exp\left\{\varepsilon\sum_{i=1}^{[\log x]}\rho(2^i)\right\},x(\delta)=\exp\left\{\sum_{i=0}^{[\log x]}\rho^{1-\delta}(2^i)\right\}.$$

假设定理 4.4.1 中的条件(i),(ii)和

(i)′当 $x\to\infty$ 时 $G(x):=EX_1^2g(X_1)I(|X_1|\leqslant x)$ 是缓变的，

(iv)对某 $0<\varepsilon<1$，$G(x)e(x^2,2+\varepsilon)=O(H(x)g(x))$，

(v)对某 $0<\delta<1$，当 $x\to\infty$ 时 $g(x)=O(g(x/x(\delta))$ 或 $G(x)=O(G(x/x(\delta)))$.

那么我们也有定理 4.4.1 的结论成立.

显然地，条件(i)和(iii)蕴含着条件(i)′，(iv)和(v)(取 $g(x)\equiv 1$ 即可). 如所周知，即使在有有限 2 阶矩情形，混合速度(iii)一般认为是不可减弱的. 然而下述例子是有兴趣的：设 $X_1$ 有密度函数 $p(x)=a(1+|x|^3)^{-1}, x\in R^1$，其中 $a^{-1}=\int_{-\infty}^{\infty}(1+|x|^3)^{-1}dx$. 设 $g(x)=\exp(\log(1+|x|^3)^{\alpha})$，(某 $0<\alpha<1$). 容易看出当 $x\to\infty$ 时

$$H(x)\sim 2a\log(1+|x|^3)/3,$$

$$G(x)\sim 2a(\log(1+|x|^3))^{1-\alpha}g(x)/3\alpha.$$

若 $\rho(n)\leqslant\alpha/(5\log n)$，容易验证注 4.4.1 中条件都被满足，但定理 4.4.1 中条件(iii)不成立. 因此在某些无穷方差的特殊情形，即使有限方差情形，条件(iii)本质上可能并不是不可减弱的.

为证明定理 4.4.1，我们引入某些记号. 设 $M^*$ 是正整数使得

(4.4.1) $$\sup_{x>0}H(x)/x^2>1/M^*.$$

对每一 $n\geqslant M^*$ 定义

(4.4.2) $$t_n=\sup\{x>0: H(x)/x^2\geqslant 1/n\}.$$

显然当 $n\to\infty$ 时 $t_n$ 单调地趋向 $\infty$. 通过一平凡的讨论，对 $n\geqslant M^*$ 有

$$t_n^2=nH(t_n).$$

由条件(i)，对任给 $0<\varepsilon<1/2$ 和充分大 $n$ 有

(4.4.3) $$n^{1-\varepsilon}\leqslant t_n^2\leqslant n^{1+\varepsilon}.$$

我们需要关于 $\{t_n\}$ 的一些性质.

**引理 4.4.1** 若条件(iii)被满足，那么对任给 $0<a<1$

(4.4.4) $$t_{[na]}^2/t_n^2\to a\qquad n\to\infty.$$

**证** (4.4.2)蕴含着

(4.4.5) $$t_n^2H(t_{[na]})/(t_{[na]}^2H(t_n))\to 1/a\qquad n\to\infty.$$

我们来证存在 $M>0$ 使得

(4.4.6) $$\limsup_{n\to\infty} t_n^2/t_{[na]}^2 \leqslant M.$$

事实上，若不然，那么存在子列$\{n_k\}$使得$\lim_{k\to\infty} t_{n_k}^2/t_{[n_k a]}^2=\infty$. 那么利用附录 $A$ 中缓变函数的性质 $A5$ 我们得

$$\lim_{k\to\infty} t_{n_k}^2 H(t_{[n_k a]})/(t_{[n_k a]}^2 H(t_{[n_k a]}t_{n_k}/t_{[n_k a]}))=\infty,$$

这与(4.4.5)矛盾. 由(4.4.2)和(4.4.6)即得

$$n/[na]\leqslant t_n^2/t_{[na]}^2\leqslant nH(Mt_{[na]})/([na]H(t_{[na]})),$$

由此得证(4.4.4).

**引理 4.4.2** 我们有

$$\lim_{k\to\infty} nP(|X_1|>t_n)=0,$$

$$\lim_{n\to\infty}(n/H(t_n))^{1/2}E|X_1|I(|X_1|>t_n)=0.$$

此引理的证明可在 Bradley(1988)中找到，不在此详述.

**定理 4.4.1 的证明.**

对某 $\varepsilon>0$，令 $C_1=C\exp\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\}$，其中 $C$ 如引理 2.2.2 中定义，且 $C_2=C'\exp\{-(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\}$，其中 $C'$ 如引理 2.2.3 中定义. 对 $n\geqslant M^*$ 定义

(4.4.7) $$X_k^{(n)}=X_kI(|X_k|\leqslant t_n)-EX_kI(|X_k|\leqslant t_n)\qquad k\geqslant 1.$$

记

$$S_m^{(n)}=X_1^{(n)}+\cdots+X_m^{(n)}\qquad m\geqslant 1.$$

由 $C_1$ 和 $C_2$ 的定义，我们有

(4.4.8) $$C_2 mEX_1^{(n)^2}\leqslant ES_m^{(n)^2}\leqslant C_1 mEX_1^{(n)^2}.$$

令 $A_n(m)=\|S_m^{(n)}\|_2$，$A_n=A_n(n)$. 注意到 $EX_1=0$，容易验证

(4.4.9) $$E(X_1^{(n)})^2\sim H(t_n)\qquad n\to\infty.$$

所以存在 $0<C'_2<C'_1<\infty$ 使得

(4.4.10) $$C'_2 nH(t_n)\leqslant A_n^2\leqslant C'_1 nH(t_n);$$

现在我们分两步叙述定理的证明.

**第一步** 我们来证明当 $n\to\infty$时，$S_n^{(n)}/A_n\xrightarrow{d}N(0,1)$. 为此只需证明对任一 $t\in R$ 有

(4.4.11) $$\lim_{n\to\infty} E\{\exp(itS_n^{(n)}/A_n)\} = \exp(-t^2/2).$$

由于 $t=0$ 情形是平凡的，所以只需证明 $t\neq 0$ 时上式成立. 固定 $t\neq 0$. 设 $J$ 是正整数，将在后面确定. 定义正整数 $p^*$ 和 $L^*$ 使得

(4.4.12) $$\frac{4C_1}{C_2 p^*}\sum_{j=1}^{\infty} 2^{J-j/2} \leqslant \frac{\varepsilon^2}{4},$$

且对每一 $j\geqslant 2^{L^*}$ 有

(4.4.13) $$|(1-t^2/(2j))^j - \exp(-t^2/2)| \leqslant \varepsilon/3.$$

设 $N^*\geqslant M^*$ 是正整数使得

(4.4.14) $$N^* \geqslant 2p^* \cdot 2^{L^*},$$

(4.4.15) $$E(X_1^{(n)})^2 > 0 \qquad n\geqslant N^*,$$

而且由(4.4.9)对每一 $n\geqslant N^*$ 有

(4.4.16) $$\frac{128p^{*2}t^2}{C_2 nE(X_1^{(n)})^2}(EX_1 I(|X_1|\leqslant t_n))^2 + \frac{400p^{*4}(t^2\vee |t|^3)}{C_2\wedge C_2^{3/2}} E\left\{\frac{X_1^2 I(|X_1|\leqslant t_n)}{nE(X_1^{(n)})^2} \wedge \frac{|X_1|^3 I(|X_1|\leqslant t_n)}{(nE(X_1^{(n)})^2)^{3/2}}\right\} \leqslant \varepsilon/(3n).$$

设 $N\geqslant N^*$ 是任意固定的整数. 那么为证(4.4.11)只需证明

(4.4.17) $$|E\exp(itS_N^{(N)}/A_N) - \exp(-t^2/2)| \leqslant \varepsilon.$$

回顾(4.4.14)，设 $L$ 是正整数使得

$$p^* \leqslant N/2^L \leqslant 2p^*.$$

注意到 $L\geqslant L^*$，设 $p$ 是正整数使得

(4.4.18) $$p2^L \leqslant N < (p+1)2^L.$$

容易看出

(4.4.19) $$p^* \leqslant p < 2p^*.$$

把自然数集 $M$ 分割成互不相交的连贯的数组，相继的数组之间无空隙. 数组的次序为 $G(1), G(2), \cdots$ 且

(4.4.20) $$\operatorname{Card} G(j) = \begin{cases} p, & \text{若 } j \text{ 是奇数}, \\ [2^{J+l/2}], & \text{其中整数 } l \text{ 使对偶数 } j, \\ & j/2^l \text{ 是一个奇数}. \end{cases}$$

下面我们仅处理数组 $G(1), G(2), \cdots, G(2^{L+1}-1)$. 对每一 $l=1$,

$\cdots,L$，恰好存在 $2^{L-l}$ 个整数 $j\in\{1,2,\cdots,2^{L+1}-1\}$ 使 $j/2^l$ 是一奇数. 所以

(4.4.21) $$\mathrm{Card}\{G(2)\cup G(4)\cup\cdots\cup G(2^{L+1}-2)\} = \sum_{l=1}^{L} 2^{L-l}[2^{J+l/2}].$$

因此由(4.4.18)

(4.4.22) $$\begin{aligned} N &\leqslant 2^L p+\sum_{l=1}^{L}2^{L-l}\leqslant 2^L p+\sum_{l=1}^{L}2^{L-l}[2^{J+l/2}] \\ &= \mathrm{Card}\{G(1)\cup G(2)\cup\cdots\cup G(2^{L+1}-1)\} \\ &\leqslant N+\sum_{l=1}^{L}2^{L-l}[2^{J+l/2}]. \end{aligned}$$

对每一 $j=1,2,\cdots,2^{L+1}-1$ 定义

$$U(j)=\sum_{k\in G(j)}X_k^{(N)}.$$

其次，对偶数 $j_1,j_2$ 使 $0\leqslant j_1\leqslant j_2\leqslant 2^{L+1}$，定义

$$V(j_1,j_2)=U(j_1+1)+U(j_1+3)+\cdots+U(j_2-1).$$

若 $1\leqslant j\leqslant 2^l-1$，那么对使 $j/2^m$ 为奇数的整数 $m$，我们有 $m<l$ 且 $(2^l+j)/2^m$ 是奇数，因此由(4.4.20)可得 $\mathrm{Card}G(2^l+j)=\mathrm{Card}G(j)$. 若记 $u=\mathrm{Card}\{G(1)\cup G(2)\cup\cdots\cup G(2^l)\}$ 且对正整数集 $G$ 用记号 $u+G=\{u+g:g\in G\}$. 对 $j$ 用归纳法我们有 $G(2^l+j)=u+G(j)$，$j=1,2,\cdots,2^l-1$. 特别地，若我们记

$$G=G(1)\cup G(3)\cup\cdots\cup G(2^l-1)$$

和

$$G^*=G(2^l-1)\cup G(2^l+3)\cup\cdots\cup G(2^{l+1}-1),$$

那么

$$G^*=u+G, V(2^l,2^{l+1})=\sum_{k\in G^*}X_k^{(N)}=\sum_{k\in G}X_{k+u}^{(N)},$$

$$V(0,2^l)=\sum_{k\in G}X_k^{(N)}.$$

序列 $\{X_k^{(N)},k\geqslant 1\}$ 的平稳性蕴含着下述有用的性质：

(4.4.23) 对每一 $l=1,\cdots,L$，$V(0,2^l)$ 和 $L(2^l,2^{l+1})$ 具有相同分布.

因此通过一简单计算对每一 $l=1,\cdots,L$ 有

(4.4.24)
$$2(1-\rho([2^{J+l/2}]))EV(0,2^l)^2 \leqslant EV(0,2^{l+1})^2 \leqslant 2(1+\rho([2^{J+l/2}]))EV(0,2^l)^2.$$

而且对每一 $l=1,\cdots,L$ 有

(4.4.25)
$$\begin{aligned}&|E\exp\{itV(0,2^{l+1})\}-(E\exp\{itV(0,2^l)\})^2|\\ &\leqslant \rho([2^{J+l/2}])E|\exp\{itV(0,2^l)\}-1|^2\\ &\leqslant \rho([2^{J+l/2}])t^2EV(0,2^l)^2\\ &\leqslant \rho([2^{J+l/2}])t^2C_12^{l-1}EU(1)^2.\end{aligned}$$

由(4.4.15)，事实 $N\geqslant N^*$ 以及(4.4.8)，对所有的 $m\geqslant 1$，$E(S_m^{(N)})^2>0$. 我们将一直记住这一结论. 由(4.4.24)和归纳法，有

$$2^L\Big\{\prod_{l=1}^{L}(1-\rho([2^{J+l/2}]))\Big\}EU_1^2$$
$$\leqslant EV(0,2^{L+1})^2\leqslant 2^L\Big\{\prod_{l=1}^{L}(1-\rho([2^{J+l/2}]))\Big\}EU_1^2.$$

因此对充分大的 $J$

(4.4.26) $$1-\varepsilon/2\leqslant \|V(0,2^{L+1})\|_2/(2^{L/2}\|U_1\|_2)\leqslant 1+\varepsilon/2.$$

由(4.4.21)和(4.4.8)

$$E(U(2)+U(4)+\cdots+U(2^{L+1}-2))^2$$
$$\leqslant C_1\Big(\sum_{l=1}^{L}2^{L+J-l/2}\Big)E(X_1^{(n)})^2.$$

又由(4.4.22)，$U(1)+U(2)+\cdots+U(2^{L+1}-1)-S_N^{(N)}$ 至多是 $\Big[\sum_{l=1}^{L}2^{L+J-l/2}\Big]$ 个不同的 $X_i^{(N)}$ 的和，因此

$$E(U(1)+U(2)+\cdots+U(2^{L+1}-1)-S_N^{(N)})^2$$
$$\leqslant C_1\Big(\sum_{l=1}^{L}2^{L+J-l/2}\Big)E(X_1^{(N)})^2.$$

由此利用(4.4.8)，(4.4.12)和(4.4.19)有

(4.4.27)
$$\|V(0,2^{L+1})-S_N^{(N)}\|_2$$
$$\leqslant \|U(1)+U(2)+\cdots+U(2^{L+1}-1)-S_N^{(N)}\|_2$$

$$+ \| U(2) + U(4) + \cdots + U(2^{L+1}-2) \|_2$$

$$\leqslant 2C_1^{1/2}\Big(\sum_{l=1}^{L} 2^{L+J-l/2}\Big)^{1/2} \| X_1^{(N)} \|_2$$

$$\leqslant 2C_1^{1/2}2^{L/2}\Big(\sum_{l=1}^{L} 2^{J-l/2}\Big)^{1/2} \| U(1) \|_2/(C_2 p)^{1/2}$$

$$\leqslant 2^{L/2} \| U(1) \|_2 \varepsilon/2.$$

现在我们回到(4.4.17). 由(4.4.26)和(4.4.27)我们有

(4.4.28)
$$\left|1-\frac{A_N}{2^{L/2}\| U(1) \|_2}\right|$$
$$\leqslant\left|1-\frac{\| V(0,2^{L+1}) \|_2}{2^{L/2}\| U(1) \|_2}\right|$$
$$+\left|\frac{\| V(0,2^{L+1}) \|_2-\| S_N^{(N)} \|_2}{2^{L/2}\| U(1) \|_2}\right|$$
$$\leqslant\frac{\varepsilon}{2}+\frac{\| V(0,2^{L+1})-S_N^{(N)} \|_2}{2^{L/2}\| U(1) \|_2}$$
$$\leqslant\varepsilon.$$

(4.4.28)和(4.4.27)蕴含着(4.4.17)等价于

(4.4.29)
$$D:=|E\exp\{itV(0,2^{L+1})/(2^{L/2}\| U(1) \|_2)\}$$
$$-\exp(-t^2/2)|\leqslant\varepsilon.$$

显然地

$$D\leqslant|E\exp\{itV(0,2^{L+1})/(2^{L/2}\| U(1) \|_2)\}$$
$$-(E\exp\{i(t/2^{L/2})U(1)/\| U(1) \|_2\})^{2^L}|$$
$$+|(E\exp\{i(t/2^{L/2})U(1)/\| U(1) \|_2\})^{2^L}$$
$$-(1-(1/2)t^2/2^L)^{2^L}|$$
$$+|(1-(1/2)t^2/2^L)^{2^L}-\exp(-t^2/2)|$$
$$=:e_1+e_2+e_3.$$

运用(4.4.25)和初等不等式

(4.4.30)
$$\left|\prod_{k=1}^{m} y_k-\prod_{k=1}^{m} z_k\right|\leqslant\sum_{k=1}^{m}|y_k-z_k|,$$

其中 $y_1,\cdots,y_m,z_1,\cdots,z_m$ 是闭单位圆中的复数,对任何 $T$ 我们有

$$|(E\exp\{iTV(0,2^{l+1})\})^{2^{L-l}}-(E\exp\{iTV(0,2^l)\})^{2^{L-l+1}}|$$

$$\leqslant 2^{L-l}\rho([2^{J+l/2}])T^2C_1 2^{l-1}EU(1)^2$$

因此由归纳法

$$|E\exp\{iTV(0,2^{L+1})\} - (E\exp\{iTV(0,2)\})^{2^L}|$$
$$\leqslant 2^L T^2 C_1 EU(1)^2 \sum_{l=1}^{L}\rho([2^{J+l/2}]).$$

设 $T=t/(2^{L/2}\|U(1)\|_2)$ 并注意到 $U(1)=V(0,2)$，我们有

$$e_1 \leqslant t^2 C_1 \sum_{l=1}^{L}\rho([2^{J+l/2}]) \leqslant \varepsilon/3 \tag{4.4.31}$$

对充分大的常数 $J$ 成立.

为估计 $e_2$，定义事件 $F_k=\{|X_k^{(N)}|=\max_{1\leqslant j\leqslant p}|X_j^{(N)}|\}, k=1,\cdots,p$. 令 $s=t/(2^{L/2}\|U(1)\|_2)$. 由(4.4.8)和(4.4.18)，

$$s^2 \leqslant t^2/(2^L C_2 pE(X_1^{(N)})^2) \leqslant 2t^2/(C_2 NE(X_1^{(N)})^2).$$

现在我们需要下一事实：对任何实数 $x$ 和 $r$，

$$|x-r|^2 \wedge |x-r|^3 \leqslant 4r^2+8(x^2\wedge|x|^3).$$

利用这个事实和(4.4.19)，(4.4.16)和(4.4.18)我们有

$$\begin{aligned}
&E(|sU(1)|^2 \wedge |sU(1)|^3)\\
&\leqslant \sum_{k=1}^{p} EI(F_k)(|spX_k^{(N)}|^2 \wedge |spX_k^{(N)}|^3)\\
&\leqslant p^4 E(|sX_1^{(N)}|^2 \wedge |sX_1^{(N)}|^3)\\
&\leqslant p^4\{4s^2(EX_1 I(|X_1|\leqslant t_N))^2\\
&\quad + 8E\{(s^2X_1^2 I(|X_1|\leqslant t_N)) \wedge (|s|^3|X_1|^3 I(|X_1|\leqslant t_N))\}\}\\
&\leqslant (2p^*)^4\left\{\frac{8t^2}{C_2NE(X_1^{(N)})^2}(EX_1I(|X_1|\leqslant t_N))^2\right.\\
&\quad \left.+ 8E\left\{\frac{2t^2X_1^2I(|X_1|\leqslant t_N)}{C_2NE(X_1^{(N)})^2} \wedge \frac{2^{3/2}|t|^3|X_1|^3I(|X_1|\leqslant t_N)}{C_2^{3/2}N^{3/2}(E(X_1^{(N)})^2)^{3/2}}\right\}\right\}\\
&\leqslant \frac{128p^{*4}t^2}{C_2NE(X_1^{(N)})^2}(EX_1I(|X_1|\leqslant t_n))^2\\
&\quad + \frac{400p^{*4}(t^2 \vee |t|^3)}{C_2 \wedge C_2^{3/2}}E\left\{\frac{X_1^2I(|X_1|\leqslant t_N)}{NE(X_1^{(N)})^2}\right.\\
&\quad \left.\wedge \frac{|X_1|^3I(|X_1|\leqslant t_N)}{N^{3/2}(E(X_1^{(N)})^2)^{3/2}}\right\} \leqslant \varepsilon/(3N) \leqslant \varepsilon(3\cdot 2^L).
\end{aligned}$$

因此注意到(4.4.30)我们得

(4.4.32) $$e_2 \leqslant 2^L \left| E\exp(isU(1)) - \left(1 - \frac{1}{2}s^2 EU(1)^2\right) \right| \leqslant 2^L E(|sU(1)|^2 \wedge |sU(1)|^3) \leqslant \varepsilon/3.$$

其中我们利用了关于特征函数的不等式(见 Bradley 1988 $p$. 331).

对于 $e_3$,由(4.4.13)并注意到 $L \geqslant L^*$,显然有

(4.4.33) $$e^3 \leqslant \varepsilon/3.$$

从(4.4.31),(4.4.32)和(4.4.33)即得(4.4.29).这就证明了中心极限定限成立.

**第二步** 现在我们来证明

(4.4.34) $$W_n \Rightarrow W \qquad n \to \infty.$$

由引理 4.4.2 和(4.4.10),对任给 $\varepsilon > 0$ 我们有

$$\lim_{n\to\infty} P\{\sup_{0\leqslant t\leqslant 1} |S_{[nt]} - S_{[nt]}^{(n)}| > \varepsilon A_n\} = 0.$$

因此为证定理只需证明

(4.4.35) $$W_n^* \Rightarrow W \qquad n \to \infty,$$

其中 $W_n^* = S_{[nt]}^{(n)}/A_n$. 由定理 4.0.4,只需证明对任一 $0 < t \leqslant 1$ 有

(4.4.36) $$A_n^2([nt])/A_n^2 \to t \qquad n \to \infty,$$

(4.4.37) $$\{W_n^{*2}(t), n \geqslant 1\}$$

一致可积,且对任何 $\varepsilon > 0$ 存在常数 $\lambda > 1$ 使对一切充分大 $n$

(4.4.38) $$P\{\max_{0\leqslant i\leqslant n} |S_i^{(n)}| \geqslant \lambda A_n\} < \varepsilon/\lambda^2.$$

我们先来证明当 $m \to \infty$ 时关于 $n \geqslant M^*$ 一致地有

(4.4.39) $$A_n^2([mt])/A_n^2(m) \to t.$$

由它即得(4.4.36)成立.先考察 $t = 1/p$ 的情形,其中 $p \geqslant 2$ 是一整数.设 $q = [m/p]$,

$$Y_i = \sum_{j=iq+1}^{(i+1)q} X_j^{(n)} \qquad i = 0, 1, \cdots, p-1,$$

$$Y_p = \sum_{j=pq+1}^{m} X_j^{(n)}.$$

注意到

$$A_n^2(m) = pA_n^2(q) + \sum_{i\neq j} EY_iY_j + EY_p^2.$$

对 $i\neq j$ 和整数 $k\geq 1$，由 Minkowski 不等式

$$|EY_iY_j|\leq 2\|Y_i\|_2\|S_k^{(n)}\|_2+\rho(k)\|Y_i\|_2\|Y_j\|_2.$$

因此由(4.4.8)对每一充分大 $m$ 有

(4.4.40) $$|A_n^2(m)/A_n^2(q)-p|$$
$$\leq(p+1)^2(\|S_k^{(n)}\|_2/A_n(q)+\rho(k)+\|Y_p\|_2^2/A_n^2(q))$$
$$\leq(p+1)^2((k^2+p^2)m^{-1/3}+\rho(k)).$$

选 $k$ 使 $(p+1)^2\rho(k)<\varepsilon/2$. 那么对充分大 $m$，对 $n\geq M^*$ 一致地成立

$$|A_n^2(m)/A_n^2([m/p])-p|<\varepsilon.$$

所以当 $m\to\infty$ 时关于 $n\geq M^*$ 一致地有

(4.4.41) $$A_n^2(m)/A_n^2([m/p])\to p.$$

若 $t$ 是有理数，即 $t=q/p$, $p,q$ 是整数且 $q<p$，那么由(4.4.41)当 $m\to\infty$ 时关于 $n\geq M^*$ 一致地有

(4.4.42) $$A_n^2([mq/p])/A_n^2(m)$$
$$=\frac{A_n^2([mq/p])}{A_n^2(mq)}\cdot\frac{A_n^2(mq)}{A_n^2(m)}\to q/p=t.$$

若 $t$ 是无理数，则对任给 $0<\delta<1/2$，取有理数 $t_1$ 使得

$$\varepsilon/4<t-t_1<\varepsilon/2.$$

由 Minkowski 不等式

(4.4.43) $$|A_n([mt])-A_n([mt_1])|\leq A_n([mt]-[mt_1]).$$

设 $p=[m/([mt]-[mt_1])]$. 那么对 $m>20/\varepsilon$

$$\frac{1}{2}\varepsilon^{-1}<m/(mt-mt_1+1)-1\leq p\leq m/(mt-mt_1-1)<5\varepsilon^{-1}.$$

类似于(4.4.40)的证明，有

$$(p-(p+1)^2\rho(k))A_n^2([mt]-[mt_1])$$
$$\leq A_n^2(m)+(p+1)^2A_n^2([mt]-[mt_1])A_n^2(k)$$
$$+(p+1)^2E(X_1^{(n)})^2.$$

取 $k$ 使得 $\rho(k)<\varepsilon/24$. 那么当 $m$ 充分大时

(4.4.44) $$A_n^2([mt]-[mt_1])/A_n^2(m)$$
$$\leq 6p^{-1}(1+3(p+1)^4(A_n^2(k)+E(X_1^{(n)})^2)/A_n^2(m))$$
$$\leq 13\varepsilon.$$

把(4.4.44)与(4.4.43)和(4.4.42)相结合就得(4.4.39). 因此(4.4.36)被证明了.

现在转向(4.4.37). 在第一步中已证对任何 $0<t<1$

$$S_{[nt]}^{(n)}/A_{[nt]} \xrightarrow{d} N(0,1) \qquad n\to\infty.$$

从(4.4.8)和(4.4.4)我们有

$$\begin{aligned}
&E(S_{[nt]}^{(n)} - S_{[nt]}^{([nt])})^2 A_{[nt]}^{-2} \\
&= A_{[nt]}^{-2} Var\Big(\sum_{i=1}^{[nt]} X_i I(t_{[nt]} < |X_i| \leqslant t_n)\Big) \\
&\leqslant c_2 C_1^{-1} E(X_1^2 I(t_{[nt]} < |X_1| \leqslant t_n))/E(X_1^2 I(|X_1| \leqslant t_{[nt]})) \\
&= C_2 C_1^{-1}(H(t_n) - H(t_{[nt]}))/H(t_{[nt]}) \\
&\quad \to 0 \qquad\qquad n\to\infty.
\end{aligned}$$

即得

$$A_n([nt])/A_{[nt]} \to 1 \qquad n\to\infty.$$

所以当 $n\to\infty$ 时 $S_{[nt]}^{(n)}/A_n([nt]) \xrightarrow{d} N(0,1)$. 由关于一致可积性的熟知结果(如参见 Billingsley1968 定理 5.4)$(S_{[nt]}^{(n)}/A_n([nt]))^2$ 是一致可积的,且由(4.4.36)$\{(S_{[nt]}^{(n)}/A_n)^2, n\geqslant 1\}$也是一致可积的.

最后我们证明(4.4.38). 设 $l_n = \exp\Big\{\sum_{i=0}^{[\log n]} \rho(2^i)^{2/3}\Big\}$. 定义

$$X_{i1}^{(n)} = X_i I(|X_i| \leqslant t_n/l_n) - EX_i I(|X_i| \leqslant t_n/l_n),$$

$$X_{i2}^{(n)} = X_i I(t_n/l_n < |X_i| \leqslant t_n) - EX_i I(t_n/l_n < |X_i| \leqslant t_n),$$

$$S_{k1}^{(n)} = \sum_{i=1}^{k} X_{i1}^{(n)}, \qquad S_{k2}^{(n)} = \sum_{i=1}^{k} X_{i2}^{(n)}.$$

显然地

$$\begin{aligned}
(4.4.45)\qquad & P\{\max_{1\leqslant i\leqslant n}|S_i^{(n)}| \geqslant 6\lambda A_n\} \\
&\leqslant P\{\max_{1\leqslant i\leqslant n}|S_{i1}^{(n)}| \geqslant \lambda A_n\} + P\{\max_{1\leqslant i\leqslant n}|S_{i2}^{(n)}| \geqslant 5\lambda A_n\} \\
&=: p_1 + p_2.
\end{aligned}$$

注意到对任何整数 $K>0$

$$\sum_{i=1}^{[\log n]} \rho^{4/5}(2^i) \leqslant K + \rho(2^K)^{2/15} \sum_{i=1}^{[\log n]} \rho(2^i)^{2/3}.$$

因此，当 $l_n\to\infty(n\to\infty)$ 时

$$\sum_{i=1}^{[\log n]}\rho^{4/5}(2^i)=o\Big(\sum_{i=1}^{[\log n]}\rho^{2/3}(2^i)\Big).$$

所以，从引理 2.2.5，(4.4.2)和性质 A4，即得

$$E|S_{k1}^{(n)}|^{5/2}\leqslant C\Big\{k^{5/4}(E(X_{11}^{(n)})^2)^{5/4}+k\exp\Big\{C\cdot\sum_{i=1}^{[\log k]}\rho^{4/5}(2^i)\Big\}E|X_{11}^{(n)}|^{5/2}\Big\}$$

$$\leqslant C\Big\{(kH(t_n/l_n))^{5/4}+k\exp\Big\{C\sum_{i=1}^{[\log k]}\rho^{4/5}(2^i)\Big\}(t_n/l_n)^{1/2}H(t_n/l_n)\Big\}$$

$$\leqslant c\{(kH(t_n)/l_n)^{5/4}+kn^{1/4}H(t_n)^{5/4}/l_n\}.$$

利用引理 4.1.2 我们有

$$E\max_{1\leqslant k\leqslant n}|S_{k1}^{(n)}|^{5/2}\leqslant c\{(nH(t_n))^{5/4}/l_n\}(\log n)^{5/2}.$$

不失一般性我们假设 $\rho(2^i)\geqslant 1/(i\log^2 i)$. 那么回顾(4.4.10)我们得

(4.4.46) $$E\max_{1\leqslant k\leqslant n}|S_{k1}^{(n)}|^{5/2}\leqslant c(nH(t_n))^{5/4}\leqslant cA_n^{5/2}.$$

由此存在 $\lambda>1$ 使对每一充分大 $n$ 有

(4.4.47) $$p_1\leqslant\varepsilon/\lambda^2.$$

现在来估计 $p_2$. 设

$$r_1=[n/l_n],r_2=[n/l_n^2],r=r_1+r_2,$$

$$d_1=[(n-r_1)/r],d_2=[n/r],$$

$$Y_i=\sum_{j=ir+1}^{ir+r1}X_{j2}^{(n)},i=0,1,\cdots,d_1,$$

$$Z_i=\sum_{j=ir+r1+1}^{(i+1)r}X_{j2}^{(n)},i=0,1,\cdots,d_2,$$

$$T_i(1)=\sum_{j=0}^{i}Y_j\quad 和\quad T_i(2)=\sum_{j=0}^{i}Z_j,$$

$$Y_i^*=\sum_{j=ir+1}^{ir+r1}(|X_j|I(t_n/l_n<|X_j|\leqslant t_n)-E|X_j|I(t_n/l_n<|X_j|\leqslant t_n))$$

$$i=0,1,\cdots,d_1.$$

容易看出

$$(4.4.48)\quad P\{\max_{1\leq i\leq n}|S_{i2}^{(n)}|\geq 5\lambda A_n\}$$

$$\leq P\{\max_{1\leq i\leq d_1}|T_i(1)|\geq 2\lambda A_n\}+P\{\max_{1\leq i\leq d_2}|T_i(2)|\geq 2\lambda A_n\}$$

$$+P\left\{\max_{1\leq i\leq d_1}\sum_{j=ir+1}^{ir+r_1}|X_{j2}^{(n)}|\geq \frac{1}{2}\lambda A_n\right\}$$

$$+2l_nP\{\max_{1\leq i\leq r_2}|S_{i2}^{(n)}|\geq\frac{1}{2}\lambda A_n\}$$

$$=:I_1+I_2+I_3+I_4.$$

由(4.4.2)和(4.4.10)，存在 $\lambda_0>0$ 使得

$$r_1E|X_1|I(t_n/l_n<|X_1|\leq t_n)$$

$$\leq r_1l_nt_n^{-1}H(t_n)\leq\lambda_0A_n.$$

因此对 $\lambda>8\lambda_0$

$$(4.4.49)\qquad I_3\leq P\{\max_{1\leq i\leq d_1}|Y_i^*|\geq\frac{1}{4}\lambda A_n\}.$$

且由引理 2.2.2 和(4.4.10)对充分大 $n$ 有

$$(4.4.50)\; I_4\leq 2l_nP\Big\{\sum_{i=1}^{r_2}(|X_i|I(t_n/l_n<|X_i|\leq t_n)$$

$$-E|X_i|I(t_n/l_n<|X_i|\leq t_n))\geq\frac{1}{4}\lambda A_n\Big\}$$

$$\leq cl_n(\lambda A_n)^{-2}r_2EX_i^2I(t_n/l_n<|X_i|\leq t_n)$$

$$\leq c(\lambda A_n)^{-2}nl_n^{-1}H(t_n)$$

$$\leq c\lambda^{-2}l_n^{-1}\leq\varepsilon/\lambda^2.$$

为估计 $I_1$，设

$$G_{-1}=(\Omega,\Phi),G_k=\sigma(X_i,1\leq i\leq r_1+kr),$$

$$U_0(0)=0,U_i(k)=\sum_{j=1}^{k}E(Y_{j+1}|G_{j+i-1})\quad k=0,1,\cdots,d_1,$$

$$T^*(k)=T_k(1)-U_0(k).$$

显然

$$(4.4.51)\quad I_1\leq P\{\max_{1\leq i\leq d_1}|T^*(i)|\geq\lambda A_n\}+P\{\max_{1\leq i\leq d_1}|U_0(i)|\geq\lambda A_n\}$$

$$=:I_1^{(1)}+I_1^{(2)}.$$

注意到$\{T^*(i),G_i,i=0,1,\cdots,d_1\}$是鞅且应用关于鞅的极大不等

式，我们有

(4.4.52) $I_1^{(1)} \leqslant 4(\lambda A_n)^{-2} ET^*(d_1)^2 I(|T^*(d_1)| \geqslant \lambda A_n)$.

下面我们来证对每一 $i,k$ 和 $n$，由对 $k$ 运用归纳法有

(4.4.53) $EU_i^2(k) \leqslant C_1 k r_1 \rho(r_2)^2 \log^2(2k) EX_1^2 I(t_n/l_n < |X_1| \leqslant t_n)$.

若 $k=1$，从 $\rho$ 混合的定义

$$EU_i^2(1) = E(Y_{i+1} E(Y_{i+1}|G_i)) \leqslant \rho(r_2) \| Y_{i+1} \|_2 \| E(Y_{i+1}|G_i) \|_2.$$

这样由(4.4.8)的一个变形，(4.4.53)对 $k=1$ 成立. 若 $k \geqslant 2$，假设(4.4.53)对每一小于 $k$ 的整数成立. 令 $k_1=[k/2]$，$k_2=k-k_1$. 那么由归纳假设

$$\begin{aligned} EU_i^2(k) &= EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1)U_{i+k_1}(k_2) \\ &\leqslant EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2\rho(r_2) \| U_i(k_1) \|_2 \left\| \sum_{j=k_1+1}^{k} Y_{j+i} \right\|_2 \\ &\leqslant C_1 \{ k_1 \log^2(2k_1) + k_2 \log^2(2k_2) + 2k_1^{1/2} k_2^{1/2} \log(2k_2) \} \\ &\quad \cdot r_1 \rho(r_2)^2 EX_1^2 I(t_n/l_n < |X_1| \leqslant t_n) \\ &\leqslant C_1 k r_1 \rho(r_2)^2 \log^2(2k) EX_1^2 I(t_n/l_n < |X_1| \leqslant t_n). \end{aligned}$$

这就证明了(4.4.53). 从它和引理 4.1.2，我们有

$$\begin{aligned} E \max_{1 \leqslant i \leqslant d_1} U_0^2(i) &\leqslant 3C_1 d_1 r_1 \rho(r_2)^2 \log^4(2d_1) H(t_n) \\ &\leqslant cA_n^2 \rho(r_2)^2 \log^4(2l_n) \\ &\leqslant cA_n^2 \rho(r_2)^2 \Big( \sum_{i=1}^{[\log n]} \rho(2^i)^{2/3} \Big)^4, \end{aligned}$$

其中

$$\begin{aligned} \sum_{i=1}^{[\log n]} \rho(2^i)^{2/3} &\leqslant \sum_{i=1}^{[\log r_2]} \rho(2^i)^{2/3} + \rho(r_2)^{2/3} \log(n/r_2) \\ &\leqslant \rho(r_2)^{-1/3} \sum_{i=1}^{[\log r_2]} \rho(2^i) + 2\rho(r_2)^{2/3} \sum_{i=1}^{[\log n]} \rho(2^i)^{2/3}. \end{aligned}$$

由此即得

$$\sum_{i=1}^{[\log n]} \rho(2^i)^{2/3} = O(\rho(r_2)^{-1/3}) \qquad n \to \infty.$$

所以我们得

(4.4.54) $$E\max_{1\leqslant i\leqslant d_1}U_0^2(i)\leqslant cA_n^2\rho(r_2)^{2/3}.$$

且进一步对充分大的 $n$

(4.4.55) $$I_1^{(2)}\leqslant\epsilon/\lambda^2.$$

对 $I_2$，类似于(4.4.52)和(4.4.55)，对充分大 $n$ 我们得

(4.4.56)
$$\begin{aligned}I_2&\leqslant\epsilon/\lambda^2+4(\lambda A_n)^{-2}ET_{d_2}^2(2)\\&\leqslant\epsilon/\lambda^2+4C_1(\lambda A_n)^{-2}d_2r_2EX_1^2I(t_n/l_n<|X_1|\leqslant t_n)\\&\leqslant\epsilon/\lambda^2+4C_1(\lambda A_n)^{-2}nl_n^{-1}H(t_n)\\&\leqslant\epsilon/\lambda^2+c\lambda^{-2}l_n^{-1}\leqslant 2\epsilon/\lambda^2.\end{aligned}$$

现在我们回到对 $I_1^{(1)}$ 的估计.

$$\begin{aligned}&ET^*(d_1)^2I(|T^*(d_1)|\geqslant\lambda A_n)\\&\leqslant 4ET_{d_1}^2(1)I\left(|T_{d_1}(1)|\geqslant\frac{1}{2}\lambda A_n\right)+4EU_0^2(d_1)\\&\leqslant 36\left(E(S_{n2}^{(n)})^2I\left(|S_{n2}^{(n)}|\geqslant\lambda\frac{A_n}{6}\right)+E\left(\sum_{i=d_2r}^{n}X_{i2}^{(n)}\right)^2+EU_0^2(d_1)\right)\\&\leqslant 144\left(E(S_n^{(n)})^2I\left(|S_n^{(n)}|\geqslant\lambda\frac{A_n}{12}\right)+E(S_{n1}^{(n)})^2I\left(|S_{n1}^{(n)}|\geqslant\lambda\frac{A_n}{12}\right)\right.\\&\quad\left.+ET_{d_2}^2(2)+EU_0^2(d_1)+E\left(\sum_{i=d_2r}^{n}X_{i2}^{(n)}\right)^2\right).\end{aligned}$$

由(4.4.10)、(4.4.54)和(4.4.8)的一个变形，对充分大 $n$ 有

$$\begin{aligned}&A_n^{-1}\left(ET_{d_2}^2(2)+EU_0^2(d_1)+E\left(\sum_{i=d_2r}^{n}X_{i2}^{(n)}\right)^2\right)\\&\leqslant c(n^{-1}d_2r_2+\rho(r_2)^{2/3})\leqslant c(l_n^{-1}+\rho(n^{1/2})^{2/3})\leqslant\epsilon/2\,000.\end{aligned}$$

利用 $\{(S_n^{(n)})^2/A_n^2,n\geqslant 1\}$ 的一致可积性，我们求得对每一 $n\geqslant 1$ 和充分大 $\lambda$ 有

$$A_n^{-2}E(S_n^{(n)})^2I(|S_n^{(n)}|\geqslant\lambda A_n/12)\leqslant\epsilon/2\,000.$$

此外，由(4.4.46)

$$A_n^{-2}E(S_{n_1}^{(n)})^2I(|S_{n_1}^{(n)}|\geqslant\lambda A_n/12)\leqslant 4\lambda^{-1/2}A_n^{-5/2}E|S_{n_1}^{(n)}|^{5/2}\leqslant c\lambda^{-1/2}.$$

因此我们得存在常数 $\lambda_1$ 使对任何 $\lambda>\lambda_1$ 和充分大 $n$

(4.4.57) $$I_1^{(1)}\leqslant\epsilon/\lambda^2.$$

从(4.4.55)和(4.4.57)即得

(4.4.58) $$I_1 \leqslant 2\varepsilon/\lambda^2.$$

类似于(4.4.58)的证明,对充分大的 $\lambda$ 和 $n$ 我们也有

(4.4.59) $$I_3 \leqslant 2\varepsilon/\lambda^2.$$

从(4.4.45),(4.4.47),(4.4.48),(4.4.50),(4.4.56),(4.4.58),(4.4.59)即得(4.4.38)成立.定理4.4.1证毕.

# 第五章　$\varphi$混合序列的弱收敛

$\varphi$混合序列的中心极限定理是关于相依随机变量的较早结果之一. Ibragimov(1959)给出了下列两命题.

**命题 5.0.1**　设$\{X_n, n \geqslant 1\}$是强平稳$\varphi$混合序列,$EX_1 = 0$, $EX_1^2 < \infty$. 若$\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$,那么

$$\sigma^2 = EX_1^2 + 2\sum_{j=2}^{\infty} EX_1 X_j$$

绝对收敛,且当$\sigma > 0$时,那么$S_n/\sigma\sqrt{n}$依分布收敛于$N(0,1)$.

**命题 5.0.2**　设$\{X_n, n \geqslant 1\}$是强平稳$\varphi$混合序列,$EX_1 = 0$, $E|X_1|^{2+\delta} < \infty$对某$\delta > 0$且$\sigma_n^2 = ES_n^2 \to \infty$. 那么$S_n/\sigma_n$依分布收敛于$N(0,1)$.

此后,对$\varphi$混合序列的中心极限定理及弱收敛已被许多作者所讨论. Ibragimov-Linnik 和 Iosifescu 提出了下述猜测:

**猜测** 1(Ibragimov-Linnik 1971).　设$\{X_n, n \geqslant 1\}$是强平稳$\varphi$混合序列,$EX_1 = 0$,$EX_1^2 < \infty$. 若$\sigma_n^2 \to \infty (n \to \infty)$,那么中心极限定理成立.

**猜测 2**(Iosifescu 1977).　设$\{X_n, n \geqslant 1\}$如上. 那么$W_n$弱收敛于$W$,其中$W_n(t) = S_{[nt]}/\sigma_n$.

自 70 年代以来,围绕这两猜测得到了许多有意义的结果. Herrndorf(1983b)指出存在一强平稳$\varphi$混合序列,$\sigma_n^2 \to \infty$,$\liminf_{n\to\infty} \sigma_n^2/n = 0$,使猜测 2 不成立. Peligrad(1985)证明,在附设$\liminf_{n\to\infty} \sigma_n^2/n > 0$下,这两个猜测成立. 这样,我们可把上述猜测的研究归结为部分和方差的讨论. 在这两篇论文中,他们也给出了当 2 阶矩有限时$\varphi$混合序列中心极限定理和弱收敛成立的充分条件. 我们

将在§5.1中介绍.

在§5.2中,我们将介绍上两猜测和Peligrad(1990)提出的更一般的猜测.

## §5.1　2阶矩有限时的弱不变原理

因 $\rho(n)\leqslant 2\varphi^{1/2}(n)$,所以当以 $\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$ 代替 $\sum_{n=1}^{\infty}\rho(2^n)<\infty$ 时,我们有定理4.1.1同样的结论.

Herrndorf(1983)去掉了混合速度上的条件,证明着

**定理 5.1.1**　设$\{X_n,n\geqslant 1\}$是 $\varphi$ 混合序列,$EX_n=0$,$EX_n^2<\infty$,且满足

(i) $\sigma_n^2=ES_n^2=nh(n)$,其中 $h(n)$是缓变的,

(ii) 对任给 $\varepsilon>0$,$\lim\limits_{n\to\infty}P\{\max\limits_{1\leqslant i\leqslant n}|X_i|\geqslant\varepsilon\sigma_n\}=0$,

(iii) $\{S_m^2(n)/\sigma_n^2,m\geqslant 0,n\geqslant 1\}$一致可积.

那么

$$W_n\Rightarrow W.$$

**证**　我们来验证定理4.0.4的条件.由 $\varphi$ 混合的定义

$$\left|P\{\cap_{i=1}^r E_i\}-\prod_{i=1}^r P(E_i)\right|\leqslant r\varphi([n\delta])\to 0\quad n\to\infty,$$

其中 $E_i=\{W_n(t_i)-W_n(S_i)\in B_i\}$,$i=1,\cdots,r$ 定义于定理4.0.4中,$\delta=\min\limits_{2\leqslant i\leqslant r}(s_i-t_{i-1})>0$.因此$\{W_n,n\geqslant 1\}$具有渐近独立增量.其次,条件(i)和(iii)蕴含着对每一 $t\geqslant 0$,$\{W_n^2(t),n\geqslant 1\}$的一致可积性.显然由条件(i)有 $EW_n(t)=0$ 且 $EW_n^2(t)\to t(n\to\infty)$.为证明胎紧性,我们需要下述引理.

**引理 5.1.1**　设$\{X_n,n\geqslant 1\}$是 $\varphi$ 混合序列.对任给正整数 $q$ 和 $a>0$,$m\geqslant 0$,$r\geqslant q+1$,我们有

(5.1.1)　$$(1-\varphi(q)-\max_{q\leqslant j\leqslant r}P\{|S_{m+r}-S_{m+j}|>a\})$$
$$P\{\max_{1\leqslant j\leqslant r}|S_{m+j}-S_m|>3a\}$$

$$\leqslant P\{|S_{m+r}-S_m|>a\}-P\{(q-1)\max_{1\leqslant j\leqslant r}|X_{m+j}|>a\}.$$

**证** 记 $A_1=\{|X_{m+1}|>3a\}$,

$$A_j=\{|S_{m+j}-S_m|>3a, |S_{m+i}-S_m|\leqslant 3a, 1\leqslant i\leqslant j-1\}$$

$$2\leqslant j\leqslant r,$$

$$B_j=\{|S_{m+r}-S_{m+j+q-1}|\leqslant a\} \qquad 1\leqslant j\leqslant r-q,$$

$$B_j=\Omega, \quad r-q+1\leqslant j\leqslant r \qquad C=\{|S_{m+r}-S_m|>a\}.$$

显然

$$\bigcup_{j=1}^{r}A_jB_j\subset C\cup\{(q-1)\max_{1\leqslant j\leqslant r}|X_{m+j}|>a\}.$$

因此

$$(5.1.2)\qquad P(C)+P\{(q-1)\max_{1\leqslant j\leqslant r}|X_{m+j}|>a\}$$

$$\geqslant P\{\bigcup_{j=1}^{r}A_jB_j\}$$

$$=\sum_{j=1}^{r}P(A_jB_j)$$

$$\geqslant\left\{\min_{1\leqslant j\leqslant r-q}P(B_j)-\varphi(q)\right\}\sum_{j=1}^{r}P(A_j).$$

注意到

$$\sum_{j=1}^{r}P(A_j)=p\left\{\max_{1\leqslant j\leqslant r}|S_{m+j}-S_m|>3a\right\},$$

$$\min_{1\leqslant j\leqslant r-q}P(B_j)+\max_{q\leqslant j\leqslant r}P\{|S_{m+r}-S_{m+j}|>a\}\geqslant 1,$$

把它代入(5.1.2)就得证(5.1.1).证毕.

为证明$\{W_n\}$是胎紧的,只需证明

$$(5.1.3)\qquad \lim_{\delta\downarrow 0,1/\delta\in N}\frac{1}{\delta}\max_{0\leqslant k\leqslant 1/\delta}\limsup_{n\to\infty}P\left\{\sup_{k\delta\leqslant s\leqslant(k+1)\delta}|W_n(s)-W_n(k\delta)|>\epsilon\right\}=0.$$

选正整数$q$充分大使$\varphi(q)<1$.对任给$\epsilon>0$和$\delta>0$,由条件(iii)我们有

$$\sup_{m\geqslant 0}\sup_{1\leqslant j\leqslant n\delta}P\{|S_m(j)|>\epsilon\sigma_n/3\}$$

$$\leqslant 9\epsilon^{-2}\sup_{m\geqslant 0,j\geqslant 1}ES_m^2(j)/\sigma_j^2\sup_{1\leqslant j\leqslant n\delta^2}\sigma_j^2/\sigma_n^2$$

$$= C(\varepsilon) \sup_{1\leqslant j\leqslant n\delta} \sigma_j^2/\sigma_n^2.$$

从(i)和性质 $A4$ 即得

$$\lim_{\delta\downarrow 0}\lim_{n\to\infty}\sup\sup_{1\leqslant j\leqslant n\delta}\sigma_j^2/\sigma_n^2=0.$$

所以存在 $\delta_0=\delta_0(\varepsilon)>0$ 使对任何 $0<\delta\leqslant\delta_0(\varepsilon)$

$$\text{(5.1.4)}\quad \limsup_{n\to\infty}\sup_{m\geqslant 0}\sup_{1\leqslant j\leqslant n\delta}P\{|S_m(j)|>\frac{1}{3}\varepsilon\sigma_n\}\leqslant C(\varepsilon)\delta \leqslant\frac{1}{2}(1-\varphi(q)).$$

应用引理 5.1.1 于 $m=[nk\delta]$, $r=[n(k+1)\delta]-[nk\delta]$, $3a=\varepsilon\sigma_n$, 从(5.1.4)对任给 $0<\delta\leqslant\delta_0$ 且 $1/\delta\in N$, $0\leqslant k\leqslant 1/\delta$, 我们得

$$\text{(5.1.5)}\quad \begin{aligned}&\frac{1}{2}(1-\varphi(q))\limsup_{n\to\infty}P\{\sup_{k\delta\leqslant s\leqslant(k+1)\delta}|W_n(s)-W_n(k\delta)|>\varepsilon\}\\ &\leqslant\limsup_{n\to\infty}P\{|W_n((k+1)\delta)-W_n(k\delta)|>\varepsilon/3\}\\ &\quad+\limsup_{n\to\infty}P\left\{(q-1)\max_{1\leqslant j\leqslant n}|X_j|>\frac{1}{3}\varepsilon\sigma_n\right\}\\ &=:I_1+I_2.\end{aligned}$$

对固定的 $q$, 从(ii)即得 $I_2=0$. 又从(i)有

$$\lim_{n\to\infty}\sigma_{[n(k+1)\delta]-[nk\delta]}/\sigma_n=\delta^{1/2}.$$

这样我们有

$$\begin{aligned}I_1&\leqslant\limsup_{n\to\infty}P\{|S_{[n(k+1)\delta]}-S_{[nk\delta]}|>\varepsilon\sigma_{[n(k+1)\delta]-[nk\delta]}/4\sqrt{\delta}\}\\ &\leqslant 16\varepsilon^{-2}\delta\sup_{m\geqslant 0,n\geqslant 1}ES_m^2(n)I(|S_m(n)|>\varepsilon\sigma_n/(4\sqrt{\delta}))/\sigma_n^2.\end{aligned}$$

它与(5.1.5)和条件(iii)相结合得证(5.1.3)成立. 定理 5.1.1 证毕.

**推论 5.1.1** 设$\{X_n,n\geqslant 1\}$是 $\varphi$ 混合序列, $EX_n=0$, $EX_n^2<\infty$. 若定理 5.1.1 的条件(i)和(ii)及

(iv) $\sup\{ES_m^2(n)/\sigma_n^2, m\geqslant 0, n\geqslant 1\}<\infty$

被满足, 且$\{X_n,n\geqslant 1\}$服从中心极限定理, 那么 $W_n\Rightarrow W$.

**证** 从中心极限定理和(i), 即得对每一 $t>0$, $S_{[nt]}/\sigma_n$ 依分布收敛于 $W(t)$. 现在设 $0<s<t$ 是给定的. 我们来证

$$\text{(5.1.6)}\quad (S_{[nt]}-S_{[ns]})/\sigma_n\xrightarrow{d}W(t)-W(s),\qquad n\to\infty.$$

显然$\{(S_{[ns]}/\sigma_n, S_{[nt]}/\sigma_n), n\geqslant 1\}$是胎紧的(见 Billingsley 1968, 41页). 因此,从 Helly 定理即得存在 $R^2$ 上的概率测度 $Q$ 和子列$\{n_k\}$使得

$$(S_{[n_k s]}/\sigma_{n_k}, S_{[n_k t]}/\sigma_{n_k})\xrightarrow{d} Q \qquad k\to\infty.$$

设 $\pi_i: R^2\to R, i=1,2$,是两投影. 那么

$$\left(S_{[n_k s]}\Big/\sigma_{n_k}, \left(S_{[n_k t]}-S_{[n_k s]}\right)\Big/\sigma_{n_k}\right)\xrightarrow{d} Q(\pi_1,\pi_2-\pi_1)^{-1} \quad k\to\infty.$$

取 $p=p(n)\in\{0,1,\cdots,[ns]\}$使得当 $n\to\infty$时 $p(n)\to\infty$且 $p_n/\sigma_n\to 0$. 我们有

$$(ES^2_{[ns]-p(n)}(p(n)))^{1/2}/\sigma_n\leqslant p(n)\sigma_n^{-1}\sup_j(EX_j^2)^{1/2}\to 0 \quad n\to\infty.$$

由此即得

$$\left(S_{[n_k s]-p(n_k)}\Big/\sigma_{n_k}, \left(S_{[n_k t]}-S_{[n_k s]}\right)\Big/\sigma_{n_k}\right)\xrightarrow{d} Q(\pi_1,\pi_2-\pi_1)^{-1} \quad k\to\infty.$$

对任何 Boret 集 $A, B\subset R$,当 $Q(\pi_1,\pi_2-\pi_1)^{-1}(\partial(A\times B))=0$ 时,由混合性条件我们得

$$\begin{aligned}
&|Q(\pi_1,\pi_2-\pi_1)^{-1}(A\times B)-Q\pi_1^{-1}(A)Q(\pi_2-\pi_1)^{-1}(B)|\\
&=\lim_{k\to\infty}|P\{S_{[n_k s]-p(n_k)}/\sigma_{n_k}\in A, (S_{[n_k t]}-S_{[n_k s]})/\sigma_{n_k}\in B\}\\
&\quad -P\{S_{[n_k s]-p(n_k)}/\sigma_{n_k}\in A\}P\{(S_{[n_k t]}-S_{[n_k s]})/\sigma_{n_k}\in B\}|\\
&=0.
\end{aligned}$$

因此 $\pi_1$ 和 $\pi_2-\pi_1$ 是 $Q$ 独立的. 由于 $Q\pi_1^{-1}=N(0,s)$和 $Q\pi_2^{-1}=N(0,t)$,即得 $Q(\pi_2-\pi_1)^{-1}=N(0,t-s)$. 这就证明了(5.1.6).

余下的证明仅需验证(5.1.3). 从(5.1.6)即得

$$\begin{aligned}
(5.1.7)\quad &\limsup_{n\to\infty}P\{|W_n((k+1)\delta)-W_n(k\delta)|\geqslant\varepsilon/3\}\\
&\leqslant N(0,\delta)\{x: |x|\geqslant\varepsilon/3\}.
\end{aligned}$$

进一步

$$\begin{aligned}
&\frac{1}{\delta}N(0,\delta)\{x: |x|\geqslant\varepsilon/3\}\\
&=\frac{1}{\delta}\frac{1}{\sqrt{2\pi\delta}}\int_{|x|\geqslant\varepsilon/3}\exp\left(-\frac{x^2}{2\delta}\right)dx
\end{aligned}$$

$$= \frac{2}{\delta}\left(1 - \Phi\left(\frac{\varepsilon}{3\sqrt{\delta}}\right)\right)$$

$$\leqslant \frac{2}{\delta}\,\frac{3\sqrt{\delta}}{\varepsilon}\,\frac{1}{\sqrt{2\pi}}e^{-\varepsilon^2/18\delta} \to 0 \qquad \delta \to 0.$$

现在从(5.1.5)和(5.1.7)即可推出(5.1.3). 这就完成了推论 5.1.1 的证明.

**注 5.1.1** 定理 5.1.1 的条件(i)和(ii)对弱不变原理也是必要的.

假设 $W_n$ 弱收敛于 $W$. 对 $\delta>0$ 及函数 $f$, 记 $W(f,\delta)=\sup\{|f(x)-f(y)|: 0\leqslant x,y\leqslant 1, |x-y|\leqslant\delta\}$. 对任给 $\varepsilon>0$, 集 $\{f: f\in D[0,1], w(f,\delta)\geqslant\varepsilon\}$ 关于一致拓扑是闭的. 因此

$$\limsup_{n\to\infty} P\{w(W_n,\delta)\geqslant\varepsilon\}\leqslant P\{w(W,\delta)\geqslant\varepsilon\}\to 0 \quad \delta\to 0.$$

由于对每一 $\delta>0$

$$\limsup_{n\to\infty} P\{\max_{1\leqslant i\leqslant n}|X_i|\geqslant\varepsilon\sigma_n\}\leqslant\limsup_{n\to\infty} P\{w(W_n,\delta)\geqslant\varepsilon\}.$$

这就证明了(ii)成立.

为证明了(i)的必要性, 我们来证 $h(t):=\max(1,\sigma_{[t]}^2)/t, t>0$, 是缓变的. 由于 $\sigma_n^2\to\infty(n\to\infty)$, 对充分大的 $t$, $h(t)=\sigma_{[t]}^2/t$. 当 $t\in[0,1]$时, 我们有

$$S_{[nt]}/\sigma_{[nt]}\xrightarrow{d}N(0,1), S_{[nt]}/\sigma_n\xrightarrow{d}N(0,t) \quad n\to\infty.$$

所以对每一 $t\in[0,1]$我们得

(5.1.8)
$$\sigma_{[nt]}^2/\sigma_n^2\to t \quad n\to\infty.$$

其次, (ii)蕴含着 $X_n/\sigma_n\xrightarrow{P}0$. 因此 $S_n/\sigma_n$ 和 $S_n/\sigma_{n+1}$ 都弱收敛于 $N(0,1)$ 且 $\sigma_n/\sigma_{n+1}\to 1$. 由此推得对 $t\in[0,1]$

(5.1.9)
$$\sigma_{[ts]}^2/\sigma_{[t[s]]}^2\to t \quad s\to\infty.$$

从(5.1.8)和(5.1.9)我们得对每一 $t\in[0,1]$ $\lim\limits_{s\to\infty}h(ts)/h(s)=1$. 因此 $h$ 是缓变函数.

对强平稳情形直接可写出下述推论.

**推论 5.1.2** 设 $\{X_n, n\geqslant 1\}$ 是强平稳 $\varphi$ 混合序列, $EX_1=0$, $EX_1^2<\infty$ 且 $\sigma_n^2\to\infty$. 记

$$Y_n(t)=\frac{1}{\sigma_n}(S_{[nt]}+(nt-[nt])X_{[nt]+1}).$$

那么下述断言是等价的：

(a) $W_n \Rightarrow W$，

(b) $Y_n \Rightarrow W$，

(c) $\{X_n\}$服从中心极限定理且(ii)被满足，

(d) $\{S_n^2/\sigma_n^2, n\geq 1\}$是一致可积的且(ii)被满足.

**证** 显然，(a)和(b)是等价的且(b)蕴含着(c). 从 Billingsley (1968)的定理 5.4 即得(c)蕴含着(d). 最后从定理 5.1.1 知(d)蕴含着(a).

Peligrad(1985)减弱了 Herrndorf(1983)的条件并给出了下述定理.

**定理 5.1.2** 设$\{X_n, n\geq 1\}$是$\varphi$混合序列，$EX_n=0$，$EX_n^2<\infty$，假设条件(i)，(iv)和 Lindeberg 条件

(v)对任给 $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{\sigma_n^2}\sum_{i=1}^n EX_i^2 I(X_i^2>\varepsilon\sigma_n^2)=0$$

被满足. 那么 $W_n\Rightarrow W$.

**注 5.1.2** 若 $\varphi^*=\lim\limits_{n\to\infty}\varphi(n)<1$，那么(ii)等价于对任给 $\varepsilon>0$

$$\sum_{i=1}^n P\{|X_i|>\varepsilon\sigma_n\}\to 0; \tag{5.1.10}$$

且 Lindeberg 条件(v)等价于

$$E\max_{1\leq i\leq n}X_i^2/\sigma_n^2\to 0. \tag{5.1.11}$$

事实上，显然地(5.1.10)蕴含着(ii)，另一方面，由(ii)我们可选 $n_0$ 和 $p_0$ 使得当 $n\geq n_0$ 时

$$P\{\max_{1\leq i\leq n}X_i^2<\varepsilon\sigma_n^2\}-\varphi(p_0)\geq a>0. \tag{5.1.12}$$

所以对每一 $x\geq\varepsilon>0$，$n\geq n_0$ 和 $j=0,1,\cdots,p_0-1$ 我们有

$$\begin{aligned}&P\{\max_{1\leq i\leq n}X_i^2\geq x\sigma_n^2\}\\ \geq\ &P\{X_j^2\geq x\sigma_n^2,\ \max_{p_0+j\leq i\leq n}X_i^2<x\sigma_n^2\}\end{aligned}$$

$$+ P\{X_{p_0+j}^2 \geqslant x\sigma_n^2, \max_{2p_0+j\leqslant i\leqslant n} x_i^2 < x\sigma_n^2\}$$

$$+ \cdots + P\{X_{[(n-j)/p_0]p_0+j}^2 \geqslant x\sigma_n^2\}$$

$$\geqslant \sum_{0\leqslant i\leqslant [(n-j)/p_0]} P\{X_{ip_0+j}^2 \geqslant x\sigma_n^2\}(P\{\max_{1\leqslant i\leqslant n} X_i^2 < x\sigma_n^2\} - \varphi(p_0))$$

$$\geqslant a \sum_{0\leqslant i\leqslant [(n-j)/p_0]} P\{X_{ip_0+j}^2 \geqslant x\sigma_n^2\}.$$

这就推得

$$(5.1.13) \qquad \sum_{j=1}^{n} P\{X_j^2 > x\sigma_n^2\} \leqslant \frac{p_0}{a} P\{\max_{1\leqslant j\leqslant n} X_j^2 > x\sigma_n^2\}.$$

因此从(ii)可得(5.1.10). 事实上从下式可知(v)蕴含着(5.1.11):对任给 $\varepsilon>0$ 有

$$E \max_{1\leqslant i\leqslant n} X_i^2/\sigma_n^2 \leqslant \varepsilon + E \max_{1\leqslant i\leqslant n} X_i^2 I(|X_i|^2 > \varepsilon\sigma_n^2)/\sigma_n^2.$$

注意到在(5.1.11)下(5.1.12)且进一步(5.1.13)成立. 因此,从下列熟知的关系式可知(5.1.11)蕴含着(v):对每一正的可积随机变量 $X$ 有

$$(5.1.14) \qquad EXI(X>b) = bP(X>b) + \int_b^\infty P(X>x)dx.$$

**注 5.1.3** Utev(1990)指出对 $\varphi$ 混合序列 $\{X_n\}$, Lindeberg 条件(v)蕴含着中心极限定理. 进一步, Grin(1991)指出对于平稳 $\varphi$ 混合序列 $\{X_n\}$, $\varphi(1)<1$, 存在数列 $\{A_n\}$ 使得 $B_n^{-1}S_n - A_n \xrightarrow{d} N(0,1)$, 其中 $B_n = \sup\{z\geqslant 0: \mathrm{Var}(\sum_{i=1}^{n} X_i I(|X_i|\leqslant z)) \geqslant z^2\}$, 当且仅当 $\{B_n\}$ 是具有指数为 1/2 的正则变化函数.

定理 5.1.2 的证明需要下列引理. 首先我们叙述一个与引理 2.2.7 类似的引理.

**引理 5.1.2** 设 $\{Y_n, n\geqslant 1\}$ 是随机变量序列. 记 $T_n = \sum_{i=1}^{n} Y_i$. 若对某 $b>0$, $p\in N$ 和 $a_0>0$ 有

$$(5.1.15) \qquad \varphi(p_0) + \max_{1\leqslant i\leqslant m} P\{T_m - T_i | ba_0/2\} \leqslant \eta < 1,$$

那么对每一 $a\geqslant a_0$ 和 $n>p$ 下述关系式成立:

$$(5.1.16)\quad P\{\max_{1\le i\le m}|T_i|>(1+b)a\}\leqslant\frac{1}{1-\eta}P(|T_m|>a)+$$

$$+\frac{1}{1-\eta}P\{\max_{1\le i\le m}|Y_i|>\frac{ba}{2(p-1)}\}$$

和

$$(5.1.17)\quad P\{|T_m|>(1+2b)a\}\leqslant\frac{\eta}{1-\eta}P\{|T_m|>a\}$$

$$+\frac{1}{1-\eta}P\left\{\max_{1\le i\le m}|Y_i|>\frac{ba}{2p}\right\}.$$

为简单计，记

$$E_aX=EXI(X>a).$$

**引理 5.1.3** 设随机变量序列$\{Y_n,n\geqslant1\}$满足(5.1.15)．那么对每一$A\geqslant a_0^2$我们有

$$E_{(1+2b)^2A}T_m^2\leqslant(1+2b)^2\frac{\eta}{1-\eta}E_AT_m^2$$

$$+\left(\frac{2p(1+2b)}{b}\right)^2\frac{1}{1-\eta}E_{A(b/2p)^2}\max_{1\le i\le m}Y_i^2.$$

**证** 由(5.1.14)和变量代换我们得

$$E_{(1+2b)^2A}T_m^2=(1+2b)^2AP\{T_m^2>(1+2b)^2A\}$$

$$+(1+2b)^2\int_A^\infty P\{T_m^2>(1+2b)^2y\}dy.$$

应用(5.1.17)并再次由(5.1.14)就得引理成立．

**引理 5.1.4** 设$\{X_n,n\geqslant1\}$是中心化序列，$\varphi^*<1/4$且$\{\max_{1\le i\le n}EX_i^2/\sigma_n^2,n\geqslant1\}$有界，那么

$$\{\max_{1\le i\le n}E(S_n-S_i)^2/\sigma_n^2,n\geqslant1\}$$

也是有界的．

**证** 设整数$p$使$\varphi(p)<1/4$．我们有

$$(5.1.18)\quad \max_{1\le i\le n}E(S_n-S_i)^2\leqslant\max_{1\le i<n-p}E(S_n-S_i)^2+p^2\max_{1\le i\le n}EX_i^2.$$

对每一$i<n-p$我们又有

$$|\,\|S_n\|_2-\|S_i+(S_n-S_{i+p})\|_2|\leqslant p\max_{1\le i\le n}\|X_i\|_2.$$

由引理 1.2.8，我们有

$$\| S_i+(S_n-S_{i+p}) \|_2 \geqslant (1-2\varphi^{1/2}(p))^{1/2}(\sigma_i^2+E(S_n-S_{i+p})^2)^{1/2}.$$

所以对每一 $i<n-p$

$$\sigma_i<(1-2\varphi^{1/2}(p))^{-1/2}(\sigma_n+p\max_{1\leqslant i\leqslant n}\| X_i \|_2),$$

由此，从(5.1.18)有

$$\begin{aligned}&\max_{1\leqslant i\leqslant n}E(S_n-S_i)^2/\sigma_n^2\\&\leqslant 2+4(1-2\varphi^{1/2}(p))^{-1}+p^2(1+4(1-2\varphi^{1/2}(p))^{-1})\\&\quad\max_{1\leqslant i\leqslant n}EX_i^2/\sigma_n^2.\end{aligned}$$

引理得证.

**引理 5.1.5** 设$\{X_n,n\geqslant 1\}$是中心化序列，$\varphi^*<1/4$. 那么$\{\max_{1\leqslant i\leqslant n}S_i^2/\sigma_n^2,n\geqslant 1\}$是一致可积的当且仅当$\{\max_{1\leqslant i\leqslant n}X_i^2/\sigma_n^2,n\geqslant 1\}$是一致可积的.

**证** 首先，因为对任给 $x>0$

$$P\{\max_{1\leqslant i\leqslant n}|X_i|>2x\sigma_n\}\leqslant P\{\max_{1\leqslant i\leqslant n}|S_i|>x\sigma_n\}. \tag{5.1.19}$$

由关系式(5.1.14)得证条件必要.

为证条件充分，设$\{\max_{1\leqslant i\leqslant n}X_i^2/\sigma_n^2,n\geqslant 1\}$是一致可积的. 由 Chebyshev 不等式并由引理 5.1.4 对任给 $b>0$

$$\lim_{t\to\infty}\sup_n\max_{1\leqslant i\leqslant n}P\{(S_n-S_i)^2>(b/2)^2t\sigma_n^2\}=0. \tag{5.1.20}$$

由 $\varphi^*<1/4$ 和(5.1.20)，我们可找到某常数 $b>0,\eta<1/2,p\in\mathbb{N}$ 和 $a_0\in R$ 使得

$$(1+2b)^2\eta/(1-\eta)<1 \tag{5.1.21}$$

且对每一 $n\geqslant 1$

$$\varphi(p)+\max_{1\leqslant i\leqslant n}P\{(S_n-S_i)^2>(b/2)^2a_0^2\sigma_n^2\}\leqslant\eta. \tag{5.1.22}$$

从引理 5.1.3，(5.1.21)和(5.1.22)即得对每一 $A>a_0^2$ 和 $n\geqslant 1$ 有

$$\begin{aligned}E_{(1+2b)^2A}\left(\frac{S_n^2}{\sigma_n^2}\right)\leqslant&(1+2b)^2\frac{\eta}{1-\eta}E_A\left(\frac{S_n^2}{\sigma_n^2}\right)\\&+\left(\frac{2p(1+2b)}{b}\right)^2\frac{1}{1-\eta}E_{A(b/2p)^2}\left(\max_{1\leqslant i\leqslant n}\frac{X_i^2}{\sigma_n^2}\right).\end{aligned}$$

在此关系式中对 $n$ 取上确界，并注意到$\sup_n E_A(S_n^2/\sigma_n^2)$关于 $A$ 是减

的且$\{\max_{1\leqslant i\leqslant n}X_i^2/\sigma_n^2,n\geqslant 1\}$是一致可积的，我们得

$$\limsup_{A\to\infty}\sup_n E_A\left(\frac{S_n^2}{\sigma_n^2}\right)\leqslant(1+2b)^2\frac{\eta}{1-\eta}\limsup_{A\to\infty}\sup_n E_A\left(\frac{S_n^2}{\sigma_n^2}\right).$$

因此，并由(5.1.21)，(5.1.22)得证$\{S_n^2/\sigma_n^2,n\geqslant 1\}$是一致可积的.
由(5.1.16)和(5.1.14)得证$\{\max_{1\leqslant i\leqslant n}S_i^2/\sigma_n^2,n\geqslant 1\}$是一致可积的.

**定理 5.1.2 的证明.**

从定理 5.1.1 的证明知，$\{W_n,n\geqslant 1\}$是渐近独立的。由注 5.1.2 知在 Lindeberg 条件(v)下$\{\max_{1\leqslant i\leqslant n}X_i^2/\sigma_n^2,n\geqslant 1\}$是一致可积的. 由此从(i)和引理 5.1.5，对每一 $t$，$\{W_n^2(t),n\geqslant 1\}$是一致可积的. 其次，再由(i)知对每一 $t$ $EW_n(t)=0$ 且 $EW_n^2(t)\to t(n\to\infty)$.

为证$\{W_n,n\geqslant 1\}$是胎紧的，从 Billingsley(1968)定理 8.3 的证明，只需证明

$$(5.1.23)\qquad \lim_{\delta\to 0}\lim_{n\to\infty}\sup\sum_{i=0}^{[1/\delta]-1}P\left\{\max_{i\delta\leqslant s\leqslant(i+1)\delta}|W_n(s)-W_n(i\delta)|>\varepsilon\right\}=0.$$

对每一 $0\leqslant i\leqslant 1/\delta-1$ 记

$$f_i=f_i(n,\delta,a):=\max_{in\delta\leqslant j\leqslant(i+1)n\delta}P\left\{\left|\sum_{k=j}^{(i+1)n\delta}X_k\right|>\frac{b}{2}a^{1/2}\sigma_n\right\}.$$

由 Chebyshev 不等式我们有

$$f_i\leqslant\left(\frac{2}{b}\right)^2\frac{1}{a}\max_{in\delta\leqslant j\leqslant(i+1)n\delta}E\left(\sum_{k=j}^{(i+1)n\delta}X_k\right)^2/\sigma_n^2.$$

由(i)，(iv)和缓变函数的性质(见附录)我们有

$$(5.1.24)\qquad \lim_{\delta\to 0}\lim_{n\to\infty}\sup\max_{0\leqslant i\leqslant 1/\delta-1}f_i=0.$$

选取 $p$ 和 $b$ 使得

$$\varphi(p)(1+2b)^2/(1-\varphi(p))<1;$$

并选 $\delta_0$ 和 $n_0$ 使对任何 $\delta<\delta_0$ 和 $n<n_0$ 有

$$(5.1.25)\qquad \varphi(p)+\max_{1\leqslant i\leqslant 1/\delta}f_i=\eta'(n,\delta,a):=\eta'<1.$$

从(5.1.25)和引理 5.1.3，对每一 $0\leqslant i\leqslant 1/\delta-1$ 我们得

$$(5.1.26)\qquad E_{(1+2b)^2a}\left\{\left(\sum_{ni\delta\leqslant j\leqslant n(i+1)\delta}X_j\right)^2/\sigma_n^2\right\}$$

$$\leqslant (1+2b)^2 \frac{\eta'}{1-\eta'} E_a \left( \frac{\sum_{ni\delta\leqslant j\leqslant n(i+1)\delta} X_j}{\sigma_n} \right)^2$$

$$+ \left( \frac{2p(1+2b)}{b} \right)^2 \frac{1}{1-\eta'} \sum_{ni\delta\leqslant j\leqslant n(i+1)\delta} E_{a(b/2p)^2} \left( \frac{X_j^2}{\sigma_n^2} \right).$$

注意到条件(i)和(iv),对固定的 $\delta>0$ 我们有

$$\limsup_{n\to\infty} \sum_{i=0}^{[1/\delta]-1} E\Big( \sum_{ni\delta\leqslant j\leqslant n(i+1)\delta} X_j \Big)^2 / \sigma_n^2 = O\Big( \limsup_{n\to\infty} \sum_{j\leqslant 1/\delta} \sigma_{[n\delta]}^2 / \sigma_n^2 \Big) = O(1).$$

记

$$l(a) = \limsup_{\delta\to 0} \limsup_{n\to\infty} \sum_{i=0}^{[1/\delta]-1} E_a\Big( \sum_{ni\delta\leqslant j\leqslant n(i+1)\delta} X_j \Big)^2 / \sigma_n^2.$$

从(5.1.26),(5.1.24)和条件(v),对每一 $a>0$ 我们得

$$l((1+2b)^2 a) \leqslant \frac{(1+2b)^2 \varphi(p)}{1-\varphi(p)} l(a).$$

由于 $l(a)$ 是 $a$ 的减函数,且 $[(1+2b)^2\varphi(p)]/(1-\varphi(p))<1$,那么得 $\lim_{a\to\infty} l(a)=0$. 因此对每一 $a>0$,$l(a)=0$,由此即得对任一 $\varepsilon>0$

$$\limsup_{\delta\to 0} \limsup_{n\to\infty} \sum_{i=0}^{[1/\delta]-1} P\Big( \Big| \sum_{ni\delta\leqslant j\leqslant n(i+1)\delta} X_j \Big| > \varepsilon\sigma_n \Big) = 0.$$

现在由(5.1.16),(v)和(5.1.24)即得(5.1.23). 定理 5.1.2 证毕.

特别地,由定理 5.1.2 我们有下述推论

**推论 5.1.3** 设 $\{X_n, n\geqslant 1\}$ 是平稳 $\varphi$ 混合序列,$EX_1=0$,$EX_1^2<\infty$,$\sigma_n^2\to\infty$ 且 Lindeberg 条件(v)被满足. 那么 $W_n\Rightarrow W$.

**推论 5.1.4** 设 $\{X_n, n\geqslant 1\}$ 是强平稳 $\varphi$ 混合序列,$EX_1=0$,$EX_1^2<\infty$,$\sigma_n^2\to\infty$ 且对任给 $\varepsilon>0$

$$\lim_{n\to\infty} \frac{n}{\sigma_n^2} EX_1^2 I(X_1^2>\varepsilon\sigma_n)=0, \tag{5.1.27}$$

那么 $W_n\Rightarrow W$.

作为推论 5.1.4 的特殊情形有

**推论 5.1.5** 设 $\{X_n, n\geqslant 1\}$ 是强平稳 $\varphi$ 混合序列,$EX_1=0$,$EX_1^2<\infty$ 且 $\liminf_{n\to\infty} \sigma_n^2/n>0$. 那么 $W_n\Rightarrow W$.

**注 5.1.4** Peligrad(1985)指出:在某些情形中,Lindeberg 条

件(v)也是必要的. 若$\{X_n,n\geqslant 1\}$是$\varphi$混合序列,使得$W_n\Rightarrow W,\sigma_n^2\to\infty$且$\varphi(1)<1$. 那么 Lindeberg 条件被满足.

事实上,由注 5.1.1 我们有$\sigma_i^2=ih(i)$,其中$h$是$R^+$上缓变函数,因此

$$\left\{\max_{1\leqslant i\leqslant n}E(S_n-S_i)^2/\sigma_n^2,n\geqslant 1\right\}$$

有界. 故存在$t_0>0$使对每一$n\geqslant 1$

$$\varphi(1)+\max_{1\leqslant i\leqslant n}P\{|S_n-S_i|>t_0\sigma_n\}\leqslant C<1.$$

那么从引理 2.2.7 的证明,对任给的$x>t_0^2$和每一$n\in N$,我们有

$$P\left\{\max_{1\leqslant i\leqslant n}S_i^2>4x\sigma_n^2\right\}\leqslant\frac{1}{1-c}P\left\{S_n^2>x\sigma_n^2\right\}. \tag{5.1.28}$$

另一方面,向$W$的弱收敛性蕴含着$\{S_n^2/\sigma_n^2,n\geqslant 1\}$的一致可积性. 由此及(5.1.28)和(5.1.14)可知$\{\max\limits_{1\leqslant i\leqslant n}S_i^2/\sigma_n^2,n\geqslant 1\}$是一致可积的. 从注 5.1.1,对任给$\epsilon>0$,我们有

$$\lim_{n\to\infty}P\left\{\max_{1\leqslant i\leqslant n}|X_i|>\epsilon\sigma_n\right\}=0.$$

因此从$\{\max\limits_{1\leqslant i\leqslant n}X_i^2/\sigma_n^2,n\geqslant 1\}$的一致可积性,我们得(5.1.11). 从注 5.1.2 得出,对于弱收敛于$W$,(v)是必要的.

## § 5.2 Ibragimov-Linnik-Iosifescu 猜测

在推论 5.1.5 中,我们已指出在假设$\liminf\sigma_n^2/n>0$下 Iosifescu 猜测正确. 而 Herrndof(1983b)通过一个例子指出,若$\liminf\sigma_n^2/n=0$,Iosifescu 猜测不成立.

**例 5.2.1** 假设$\{\eta_n,n\geqslant 1\}$是强平稳$\varphi$混合序列,$E\eta_n=0,E\eta_n^2<\infty,\tau_n^2=E\left(\sum\limits_{i=1}^{n}\eta_i\right)^2\to\infty,\liminf\tau_n^2/n=0$. 若$\{\eta_n,n\geqslant 1\}$不满足弱不变原理,那么我们就取$X_n=\eta_n,n\in N$. 假设$\{\eta_n,n\geqslant 1\}$满足弱不变原理. 设$\{\alpha_n,n\geqslant 0\}$是一个相互独立同分布的随机变量序列,它与$\{\eta_n\}$独立且满足

$$(5.2.1)\qquad P\{\alpha_0 = a_k T_{n_k}\} = b_k n_k^{-1},\quad k \in \mathbb{N},$$
$$P\{\alpha_0 = 0\} = 1 - \sum_k b_k n_k^{-1},$$

其中正整数 $n_1 < n_2 < n_3 < \cdots$ 且实数列 $\{a_k\}$，$\{b_k\}$，$a_k \to \infty$，$b_k \to \infty$ 使得

$$(5.2.2)\qquad \sum_k b_k n_k^{-1} \leqslant 1/2,\ \sum_k a_k^2 T_{n_k}^2\, b_k n_k^{-1} < \infty.$$

(例如从 $\tau_n^2/n \to 0$，我们可选取 $n_k$ 使得 $\tau_{n_k}^2 n_k^{-1} < 2^{-2k-1}$ 且 $n_k$ 是增加的，$a_k = k$，$b_k = 2^k$.) (5.2.1)和(5.2.2)蕴含着 $E\alpha_0^2 < \infty$，$P(\alpha_0 = 0) \geqslant 1/2$. 现在令

$$(5.2.3)\qquad X_n = \eta_n + \alpha_n - \alpha_{n-1}.$$

记 $S_n = \sum_{j=1}^{n} X_j$，$T_n = \sum_{j=1}^{n} \eta_j$. 我们有

$$(5.2.4)\qquad S_n = T_n + \alpha_n - \alpha_0.$$

显然，$\{X_n, n \geqslant 1\}$ 是一个强平稳 $\varphi$ 混合序列，$EX_1 = 0$，$EX_1^2 < \infty$ 且

$$(5.2.5)\qquad \sigma_n^2/\tau_n^2 \to 1.$$

对 $n, m \geqslant 1$，我们有

$$(5.2.6)\qquad \left(\frac{S_{n+m} - S_m}{\sigma_n}\right)^2 \leqslant 2\left(\frac{T_{n+m} - T_m}{\tau_n}\right)^2 \left(\frac{\tau_n}{\sigma_n}\right)^2 + 2\left(\frac{\alpha_{n+m} - \alpha_m}{\sigma_n}\right)^2.$$

众所周知，若 $\{\eta_n\}$ 满足弱不变原理，那么 $\{(T_{n+m} - T_m)^2/\tau_n^2, m \geqslant 0, n \geqslant 1\}$ 是一致可积的. 这样，从(5.2.6)，(5.2.5)和

$$(5.2.7)\qquad \|(\alpha_{n+m} - \alpha_m)/\sigma_n\|_2 \leqslant 2\|\alpha_0\|_2/\sigma_n \to 0,$$

即得 $\{(S_{n+m} - S_m)^2/\sigma_n^2, m \geqslant 0, n \geqslant 1\}$ 也是一致可积的，因为 $\{\eta_n\}$ 满足弱不变原理且对任何 $0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant 1$

$$\sigma_n^{-1}(S_{[nt_1]}, \cdots, S_{[nt_k]}) - \tau_n^{-1}(T_{[nt_1]}, \cdots, T_{[nt_k]}) \xrightarrow{P} 0,$$

$(W_{[nt_1]}, \cdots, W_{[nt_k]})$ 弱收敛于 $W\pi_{t_1\cdots t_k}^{-1}$，其中 $W_n(t) = S_{[nt]}/\sigma_n$. 现在我们来证明 $W_n$ 不弱收敛于 $W$，对 $t > 0$ 有

$$(5.2.8)\qquad P\{\max_{1\leqslant i\leqslant n} |S_i| \geqslant t\sigma_n\}$$
$$\geqslant P\{\max_{1\leqslant i\leqslant n} |\alpha_i - \alpha_0| \geqslant 2t\sigma_n\} - P\{\max_{1\leqslant i\leqslant n} |T_i| \geqslant t\sigma_n\},$$

$$(5.2.9)\qquad P\{\max_{1\leqslant i\leqslant n} |\alpha_i - \alpha_0| \geqslant 2t\sigma_n\}$$

$$\geqslant P\{\alpha_0=0,\max_{1\leqslant i\leqslant n}|\alpha_i|\geqslant 2t\sigma_n\}$$

$$\geqslant \frac{1}{2}P\{\max_{1\leqslant i\leqslant n}|\alpha_i|\geqslant 2t\sigma_n\}$$

$$=\frac{1}{2}(1-(P\{|\alpha_0|<2t\sigma_n\})^n)$$

$$\geqslant \frac{1}{2}-\frac{1}{2}\exp(-nP\{|\alpha_0|\geqslant 2t\sigma_n\}).$$

应用(5.2.9),(5.2.5),$a_k\to\infty$,(5.2.1)和 $b_k\to\infty$,我们得

$$\limsup_{n\to\infty}P\{\max_{1\leqslant i\leqslant n}|\alpha_i-\alpha_0|\geqslant 2t\sigma_n\}$$

$$\geqslant \frac{1}{2}-\frac{1}{2}\lim_{k\to\infty}\exp(-n_k b_k/n_k)=\frac{1}{2}.$$

由于$\{\eta_n\}$满足弱不变原理以及(5.2.5),我们可选 $t_0>0$使$\limsup\limits_{n\to\infty}P\{\max\limits_{1\leqslant i\leqslant n}|T_i|\geqslant t_0\sigma_n\}\leqslant 1/4$. 那么从(5.2.8)和(5.2.9)推出,对每一 $t\geqslant t_0$有

$$\limsup_{n\to\infty}P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant t\sigma_n\}\geqslant\frac{1}{4}.$$

因此$\{X_n,n\geqslant 1\}$不满足弱不变原理.

进一步,Peligrad(1990) 指出对于 $\varphi$混合序列,猜测1和2并不是人们所期望的最一般结果. 在全面地综合有关工作后,她提出下述猜测可能是正确的.

**猜测3**(Peligrad 1990). 设$\{X_n,n\geqslant 1\}$是强平稳中心化 $\varphi$混合序列,满足:当 $x\to\infty$时

(5.2.10) $\qquad H(x):=EX_1^2I(|X_1|\leqslant x)$是缓变的,

且 $\varphi(1)<1$. 那么 $\tilde{W}_n$ 弱收敛于 $W$,其中

$$\tilde{W}_n(t)=S_{[nt]}/((\pi/2)^{1/2}b_n)\quad 0\leqslant t\leqslant 1,$$

$b_n=E|S_n|$.

**注5.2.1** 至少有两种有兴趣的情形,那时容易验证(5.2.10). 第一种是 $EX_1^2<\infty$情形,第二种为 $P(|x_1|>x)$是指数为$-2$的正则变化函数情形,即

(5.2.11) $\qquad P(|x_1|>x)=1/(x^2h(x))$,

其中当 $x\to\infty$ 时 $h(x)$ 是缓变的.

事实上,我们可改写 $P(|x_1|>x)=h(x)/x^2$. 由分部积分,我们有

$$H(x)=-\int_0^x y^2 dP(|x_1|>y)$$
$$=-h(x)+\int_0^x h(y)y^{-1}dy.$$

由 Karamata 表示(见附表 $A$ 的定理 $A_1$),我们可选 $z_1=z_1(x)<x$ 使得

$$\limsup_{x\to\infty}\sup_{z_1\leqslant y\leqslant x} h(y)/h(x)=1,$$
$$\lim_{x\to\infty} x/z_1(x)=\infty.$$

由此

$$\int_0^x h(y)y^{-1}dy \geqslant \int_{z_1}^x h(y)y^{-1}dy > \frac{1}{2}h(x)\log(xz_1^{-1}).$$

所以

$$H(x)=(1+0(1))\int_0^x h(y)y^{-1}dy.$$

对任给 $k>0$ 我们有

$$\left|\int_x^{kx} h(y)y^{-1}dy\right| \leqslant 2(\log k)h(x)=0\left(\int_0^x h(y)y^{-1}dy\right),$$

由此即得

$$\lim_{x\to\infty} H(kx)/H(x)=1.$$

Peligrad(1990) 在条件(5.2.11)下证明了猜测了成立,这就是下述定理.

**定理5.2.1** 设 $\{X_n,n\geqslant 1\}$ 是中心化强平稳 $\varphi$ 混合序列,满足(5.2.11)且 $\varphi(1)<1$. 那么

$$\widetilde{W}_n\Rightarrow W.$$

定理5.2.1的证明不在此详述了.

# 第六章 混合相依随机场的弱收敛性

随机场的混合相依的定义有两大类，其一是由经典情形，从相依随机变量序列到相依随机场的一种自然的拓广，这方面已被 Bulinskii 和 Zurbenko(1981)，Gorodetskii(1982，1984)，Bolthausen(1982)，Nahapetian(1987)，Bradley(1992)，Doukhan 和 Guyon(1991)，Gugon(1992)及 Doukhan(1994)等所讨论. 另一种是出现在弱相依随机场的集指标部分和过程的研究中，这方面已被 Goldie 和 Greenwood(1986a，b)，陈冬青(1991)及陆传荣(1995)等所研究. 我们在 §6.1 中介绍前一种情形，在 §6.2 和 §6.3 中，讨论后一种情形.

## §6.1 混合相依随机场的中心极限定理

从 $\alpha$ 混合序列到 $\alpha$ 混合随机场的自然拓广已被许多作者所讨论.

随机场 $\{X_t, t\in \mathbb{Z}^d\}$，$d\geqslant 1$，说是 $\alpha_*$ 混合的，若

(6.1.1) $$\alpha_{m,n}(r)=\sup\{|P(AB)-P(A)P(B)|: A\in \sigma_U, B\in \sigma_V,\ U,V\subset \mathbb{Z}^d, |U|\leqslant m, |V|\leqslant n, d(U,V)\geqslant r\}\to 0, r\to\infty.$$

其中 $d(U,V)=\inf\{d(t,s): t\in U, s\in V\}$，$d(t,s)=\max\limits_{1\leqslant i\leqslant d}|t_i-s_i|$，$m$，$n\in \mathbb{N}\cup\{\infty\}$，$\sigma_A=\sigma\{X_t, t\in A\}$，$|A|$ 记 $A$ 的元素个数.

对 $d$ 维立方体 $J_n=[-n,n]^d$ 的某些子集 $\Delta_j^{(n)}$，$j=1,\cdots,k$，记

(6.1.2) $$S(n,j)=\sigma_n^{-1}S_{\Delta_j^{(n)}}, \qquad \sigma_n^2=\mathrm{Var}S_{J_n},$$

$$S_I=\sum_{t\in I}X_t, I\subset \mathbb{Z}^d, |I|<\infty,$$

$$M_k=\left|E\prod_{j=1}^{k}e^{iuS^{(n,j)}}-\prod_{j=1}^{k}Ee^{iuS^{(n,j)}}\right|,$$

$$L_k(\varepsilon,\delta) = \sum_{j=1}^{k} \int_{|S(n,j)|\geq \varepsilon} |S(n,j)|^{2+\delta} P(dw), \varepsilon,\delta \geq 0.$$

Nahapetian(1987)证明着下述结论，它是定理 3.2.1 的一个推广.

**定理 6.1.1** 设$\{X_t, t\in \mathbb{Z}^d\}$是强平稳$\alpha$-混合随机场，$EX_t=0$，$E|X_t|^{2+\delta}<\infty$某$\delta>0$. 若对某$\tau>0$,

(i) $\alpha_{m,n}(r)\leqslant f(m)n^{\tau}\alpha(r)$，其中$f(m)$是非负函数，$m\in N\cup\{\infty\}$；

(ii) $\sum_{r=1}^{\infty} r^{d-1}\alpha^{\delta/(2+\delta)}(r)<\infty, \alpha(r)=o(r^{-(2\tau+1)d}), r\to\infty$.

那么

$$\sigma^2 = \sum_{t\in \mathbb{Z}^d} EX_0X_t < \infty$$

且当$\delta\neq 0$时我们有

$$S_{J_n}/\sigma_n \xrightarrow{d} N(0,1). \tag{6.1.3}$$

定理 6.1.1 的证明需要若干引理. 显然，从引理 1.2.3 的证明可写出下述引理.

**引理 6.1.1** 设随机变量$X$和$Y$分别关于$\sigma$域$\sigma_U$和$\sigma_V$是可测的，$|U|\leqslant m$, $|V|\leqslant n$, $d(U,V)\geqslant r$, $E|X|^p<\infty$, $E|Y|^q<\infty$, $p,q>1$, $p^{-1}+q^{-1}\leqslant 1$. 那么

$$|EXY - EXEY| \leqslant c(E|X|^p)^{1/p}(E|Y|^q)^{1/q}\alpha_{m,n}^{1-p^{-1}-q^{-1}}(r).$$

特别地，若$|X|\leqslant C_1$ a.s.，$|Y|\leqslant C_2$ a.s.，我们有

$$|EXY - EXEY| \leqslant cC_1C_2\alpha_{m,n}(r).$$

**引理 6.1.2** 设$\xi_1,\cdots,\xi_n$是随机向量序列，$\left|E\prod_{j=i}^{n}\xi_j\right|<\infty$，$i=1,\cdots,n-1$，$|E\xi_i|\leqslant 1$，$i=1,\cdots,n$. 那么

$$\left|E\prod_{s=1}^{n}\xi_s - \prod_{s=1}^{n}E\xi_s\right| \tag{6.1.4}$$

$$\leqslant \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|E(\xi_i-1)(\xi_j-1)\prod_{s=j+1}^{n}\xi_s\right.$$

$$-E(\xi_i-1)E(\xi_j-1)\prod_{s=j+1}^{n}\xi_s\Big|.$$

**证** 显然地，我们有

$$(6.1.5)\qquad \Big|E\prod_{s=1}^{n}\xi_s-\prod_{s=1}^{n}E\xi_s\Big|\leqslant\sum_{i=1}^{n-1}\Big|E\xi_i\prod_{j=i+1}^{n}\xi_j-E\xi_iE\prod_{j=i+1}^{n}\xi_j\Big|.$$

此外

$$\Big|E\xi_i\prod_{j=i+1}^{n}\xi_j-E\xi_iE\prod_{j=i+1}^{n}\xi_j\Big|$$

$$\leqslant\Big|E(\xi_i-1)(\xi_{i+1}-1)\prod_{j=i+2}^{n}\xi_j-E(\xi_i-1)E(\xi_{i+1}-1)\prod_{j=i+2}^{n}\xi_j\Big|$$

$$+\Big|E(\xi_i-1)\xi_{i+2}\prod_{j=i+3}^{n}\xi_j-E(\xi_i-1)E\xi_{i+2}\prod_{j=i+3}^{n}\xi_j\Big|.$$

由这一递推过程，我们得

$$(6.1.6)\qquad \Big|E\xi_i\prod_{j=i-1}^{n}\xi_j-E\xi_iE\prod_{j=i+1}^{n}\xi_j\Big|$$

$$\leqslant\sum_{j=i+1}^{n}\Big|E(\xi_i-1)(\xi_j-1)\prod_{s=j+1}^{n}\xi_s$$

$$-E(\xi_i-1)E(\xi_j-1)\prod_{s=j+1}^{n}\xi_s\Big|,$$

把(6.1.6)代入(6.1.5)得证(5.1.4).

**定理6.1.1的证明.**

设 $p=p(n)$，$q=q(n)$ 是正整数满足：$p,q\to\infty$，$p=o(n)$，$q=o(p)$，$n\to\infty$. 记 $\hat{k}=\hat{k}(n)=[2n/(p+q)]$，

$$I_n(i)=[-n+ip+iq,-n+(i+1)p+iq]\quad i=0,1,\cdots,\hat{k}-1,$$

$$\boldsymbol{I}_n=\bigcup_{j=0}^{\hat{k}-1}I_n(j),\ \boldsymbol{I}_n^d=\overbrace{\boldsymbol{I}_n\times\boldsymbol{I}_n\times\cdots\times\boldsymbol{I}_n}^{d\text{个}}.$$

那么 $\boldsymbol{I}_n^d$ 是由 $\hat{k}^d$ 个边长为 $p$ 的 $d$ 维立方体组成的. 对给定的 $n$，记这 $k=\hat{k}^d$ 个 $d$ 维立方体为 $\Delta_j^{(n)}$，$j=1,\cdots,k$，即

$$\boldsymbol{I}_n^d=\bigcup_{j=1}^{k}\Delta_j^{(n)}.$$

令 $A_n=\boldsymbol{J}_n\backslash\boldsymbol{I}_n^d$，由 Bernstein 分段法，我们只需证明

1) $\sigma_n^{-2}\mathrm{Var}S_{A_n}\to0$,

2) $M_k \to 0$, $\sum_{j=1}^{k} \mathrm{Var} S(n,j) \to 1$且对任给 $\varepsilon > 0$

$$L_k(\varepsilon,\delta) \to 0, n \to \infty.$$

从 $E|X_t|^{2+\delta} < \infty$和 $\sum_{r=1}^{\infty} r^{d-1}\alpha(r)^{\delta/(2+\delta)} < \infty$，由引理6.1.1即得 $\sigma^2$是有限的且对任一$d$维立方体$I$，当$|I| < \infty$时，$\mathrm{Var}S_I \sim \sigma^2|I|$. 所以我们有 $\sigma_n^{-2}\mathrm{Var}S_{A_n} \to 0$且当 $n \to \infty$时

$$\sum_{j=1}^{k} \mathrm{Var} S(n,j) = k\sigma_n^{-2}\mathrm{Var} S_{\Delta_1^{(n)}}$$

$$= \frac{k}{(2n)^d \sigma^2(1+o(1))} p^d \sigma^2(1+o(1)) \to 1.$$

这也允许我们仅对有界的随机场给出2)的证明(见 Ibragimov 和 Linnik1971，定理 18.5.3和18.5.4的证明).

现在我们证明 $M_k \to 0$. 从(6.1.5)即得

(6.1.7) $$M_k \leqslant \sum_{j=1}^{k-1} \left| E e^{itS(n,j)} \prod_{m=j+1}^{k} e^{itS(n,m)} - E e^{itS(n,j)} E \prod_{m=j+1}^{k} e^{itS(n,m)} \right|$$

$$\leqslant \sum_{j=1}^{k-1} \Big| E\big(e^{itS(n,j)} - 1 - itS(n,j) + \frac{t^2}{2}S(n,j)^2\big) \prod_{m=j+1}^{k} e^{itS(n,m)}$$

$$- E\big(e^{itS(n,j)} - 1 - itS(n,j) + \frac{t^2}{2}S(n,j)^2\big) E \prod_{m=j+1}^{k} e^{itS(n,m)} \Big|$$

$$+ |t| \sum_{j=1}^{k-1} \left| ES(n,j) \prod_{m=j+1}^{k} e^{itS(n,m)} - ES(n,j) E \prod_{m=j+1}^{k} e^{itS(n,m)} \right|$$

$$+ \frac{t^2}{2} \sum_{j=1}^{k-1} \left| ES(n,j)^2 \prod_{m=j+1}^{k} e^{itS(n,m)} - ES(n,j)^2 E \prod_{m=j+1}^{k} e^{itS(n,m)} \right|$$

$$=: T_1 + |t| T_2 + \frac{t^2}{2} T_3.$$

取 $p = o(n^{1/2})$，我们有

(6.1.8) $$T_1 \leqslant 2\frac{|t|^3}{3!} \sum_{j=1}^{k-1} E|S(n,j)|^3$$

$$\leqslant c|t|^3 \left(\frac{n}{p}\right)^d \sigma_n^{-3} E|S_{\Delta_1^{(n)}}|^3$$

$$\leqslant c|t|^3 \left(\frac{n}{p}\right)^d (n^d)^{-3/2} p^d E S^2_{\Delta_1^{(n)}}$$

$$\leqslant c|t|^{3}\left(\frac{p}{n^{1/2}}\right)^{d}\to 0, n\to\infty.$$

利用不等式(6.1.4)的证明我们有

(6.1.9) $$T_2\leqslant\sigma_n^{-1}\sum_{j=1}^{k-1}\sum_{s\in\Delta_j^{(n)}}\left|EX_s\prod_{m=j+1}^{k}e^{itS(n,m)}-EX_sE\prod_{m=j+1}^{k}e^{itS(n,m)}\right|$$

$$\leqslant\sigma_n^{-1}\sum_{j=1}^{k-1}\sum_{s\in\Delta_j^{(n)}}\sum_{m=j+1}^{k}\left|EX_s(e^{itS(n,m)}-1)\prod_{r=m+1}^{k}e^{itS(n,r)}\right.$$

$$\left.-EX_sE(e^{itS(n,m)}-1)\prod_{r=m+1}^{k}e^{itS(n,r)}\right|.$$

注意到$|X_s|\leqslant C_0$，$|e^{itS(n,m)}-1|\leqslant cp^d/\sigma_n$ a. s.，应用引理6.1.1于(6.1.9)，我们有

$$T_2\leqslant cp^d\sigma_n^{-2}\sum_{j=1}^{k-1}\sum_{s\in\Delta_j^{(n)}}\sum_{m=j+1}^{k}\alpha_{1,n^d}(d(\Delta_j^{(n)},\Delta_m^{(n)}))$$

$$\leqslant cp^{2d}f(1)n^{\tau d}\sigma_n^{-2}\sum_{j=1}^{k-1}\sum_{m=j+1}^{k}\alpha(d(\Delta_j^{(n)},\Delta_m^{(n)}))$$

$$\leqslant cp^{2d}n^{\tau d}p^{-d}\sum_{j=1}^{k}\alpha(d(\Delta_1^{(n)},\Delta_j^{(n)}))$$

$$\leqslant cp^{d}n^{\tau d}\sum_{l=1}^{\infty}\sum_{j:(l-1)q\leqslant d(\Delta_1^{(n)},\Delta_j^{(n)})<lq}\alpha(d(\Delta_1^{(n)},\Delta_j^{(n)}))$$

$$\leqslant cp^{d}n^{\tau d}\sum_{l=1}^{\infty}l^{d-1}\alpha(lq).$$

从定理6.1.1的条件(ii)即得

(6.1.10) $$T_2\leqslant cp^{d}n^{\tau d}q^{-(2\tau+1)d}\sum_{l=1}^{\infty}\frac{\beta(lq)}{l^{1+2\tau d}}$$

其中$\beta(n)=\alpha(n)n^{(2\tau+1)d}$. 由于$\beta(n)\to 0(n\to\infty)$，我们可取$p=p(n)=o(n^{1/2})$，$p(n)\to\infty(n\to\infty)$，$q=q(n)\to\infty$且$q=o(p)$，使得(6.1.10)的右边趋于0.

估计$T_3$.

(6.1.11) $$T_3\leqslant\sigma_n^{-2}\sum_{j=1}^{k-1}\sum_{s,u\in\Delta_j^{(n)}}\left|EX_sX_u\prod_{m=j+1}^{k}e^{itS(n,m)}\right.$$

$$
\begin{aligned}
&-EX_sX_uE\sum_{m=j+1}^{k}e^{itS(n,m)}\Big| \\
&\leqslant \sigma_n^{-2}\sum_{j=1}^{k-1}\sum_{m=j+1}^{k}\sum_{s,u\in\Delta_j^{(n)}}\Big|EX_sX_u(e^{itS(n,j+1)}-1) \\
&\quad\cdot\prod_{m=j+2}^{k}e^{itS(n,m)}-EX_sX_uE(e^{itS(n,j+1)}-1)\prod_{m=j+2}^{k}e^{itS(n,m)}\Big| \\
&\leqslant c\frac{p^{2d}}{n^d}\frac{p^d}{n^{d/2}}\sum_{j=1}^{k-1}\sum_{m=j+1}^{k}\alpha_{2,u^d}(d(\Delta_j^{(n)}\Delta_m^{(n)})) \\
&\leqslant c\frac{p^{3d}}{n^{3d/2}}\left(\frac{n}{p}\right)^d f(2)n^{\tau d}\sum_{s=1}^{\infty}s^{d-1}\alpha(sq) \\
&\leqslant c\frac{p^{2d}n^{\tau d}\beta(q)}{n^{d/2}q^{(2\tau+1)^d}}\to 0,n\to\infty.
\end{aligned}
$$

结合(6.1.7)—(6.1.11)得 $M_k\to 0(n\to\infty)$.

最后，我们来证 $L_k(\varepsilon,\delta)\to 0$. 注意到$\{\xi\}$的平稳性，我们有

$$
L_k(\varepsilon,\delta)\leqslant c\left(\frac{n}{p}\right)^d n^{d(1-\delta/2)}\int_{|S_{\Delta_1^{(n)}}|\geqslant\varepsilon\sigma_n}|S_{\Delta_1^{(n)}}|^{2+\delta}p(dw)
$$

$$
\leqslant c(n^{\delta/2}p)^{-d}\int_{S_{\Delta_1^{(n)}}|\geqslant\varepsilon\sigma_n}|S_{\Delta_1^{(n)}}|^{2+\delta}p(dw).
$$

存在着 $p=p(n)\to\infty,p(n)=o(n)$和 $q=q(n)\to\infty,q=o(p)$使当 $n\to\infty$时，上式右边趋于0. 这就完成了定理6.1.1的证明.

**注6.1.1** 在上述 $\alpha$ 混合定义中，集 $U$ 和 $V$ 所处地位对称，故(i)中 $\alpha_{m,n}(r)\leqslant f(m)n^\tau\alpha(r)$不尽合理. 修改上述定义，称随机场$\{X_t,t\in\mathbb{Z}^d\}$为 $\alpha$ 混合的，若

$$
\begin{aligned}
&\alpha(r)=\sup\{|p(AB)-P(A)P(B)|;A\in\sigma_U,B\in\sigma_V, \\
&U,V\subset\mathbb{Z}^d,d(U,V)\geqslant r\}\to 0\qquad 当\ r\to\infty.
\end{aligned}
\tag{6.1.12}
$$

我们可给出较定理6.1.1更一般的结论，它是定理3.2.3在随机场情形的拓广.

**定理6.1.2** 设$\{X_t,t\in\mathbb{Z}^d\}$是 $\alpha$ 混合随机场，$EX_t=0$、$EX_t^2<\infty$. 若存在 $g\in G$ 使得

$$
\sup Eg(|X_t|)<\infty,\ \sum_{r=1}^{\infty}r^{d-1}f_g(\alpha(r))\infty,
\tag{6.1.13}
$$

且

(6.1.14) $\lim\limits_{n\to\infty}\mathrm{Var}S_{J_n}/n^d=\sigma^2>0.$

那么

(6.1.15) $W_n\Rightarrow W,$

其中 $W_n(t)=S_{J_{[nt]}}/\sigma_n, 0\leqslant t\leqslant 1.$

定理6.1.2的证明与定理3.2.3的证明类似，从略.

**注6.1.2** 对于非平稳的随机场，定理6.1.1的类似结果被Guyon(1992)所讨论. 具有连续参数的 $\alpha$ 混合随机场的中心极限定理已被 Gorodezkii(1984)，Zhurbenko(1984)等所讨论，对它的弱收敛结果也已被给出了.

Bradley(1992)在"无约束 $\rho$ 混合"条件下证明了强平稳随机场的中心极限定理，此时只设2阶矩有限且没有混合速度上的限制. 设 $\{X_t, t\in\mathbb{Z}^d\}$ 是强平稳随机场. 对非空不相交的集 $S, D\subset\mathbb{Z}^d$，令

$$\rho(S,D) = \rho(\sigma(X_k, k\in S), \sigma(X_k, k\in D)).$$

对每一 $r\geqslant 1$，定义

$$\rho^*(r) = \sup\rho(S,D),$$

其中 sup 是对所有非空不相交子集 $S, D\subset\mathbb{Z}^d$ 使当 $d(S,D)\geqslant r$ 来取的. 记

$$L:=L^{(n)}=(l_1^{(n)},\cdots,l_d^{(n)})\in\mathbb{N}^d, S(X,L)=\sum_{1\leqslant t\leqslant L}X_t,$$

Bradley(1992)证明着下述定理.

**定理6.1.3.** 设 $\{X_t, t\in\mathbb{Z}^d\}$ 是中心化强平稳随机场，$0<EX_0^2<\infty$，$\rho^*(r)\to 0(r\to\infty)$ 且在 $T^d$ 上有连续正的谱密度 $f(\cdot)$，满足 $f(1,\cdots,1)>0$，其中 $T$ 是复平面上的单位圆. 那么当 $\|L^{(n)}\|=l_1^{(n)}\cdots l_d^{(n)}\to\infty$ 时我们有 $\|S(X,L)\|_2\to\infty$ 且

$$S(X,L)/\|S(X,L)\|_2\xrightarrow{d}N(0,1).$$

定理6.1.3的证明不在此叙述了.

## §6.2 有限维分布的收敛性

记

$J_n=\{j=(j_1/n,j_2/n,\cdots,j_d/n):j_1,j_2,\cdots,j_d\in\{1,2,\cdots,n\}\}$,

$c_{n,j}=(j-n^{-1}\mathbf{1},j)$, $(a,b]=\{(x_1,x_2,\cdots,x_d)$:

$a_i<x_i\leqslant b_i, i=1,\cdots,d\}$,

$I^d=\{(a,b]:a,b\in[0,1]^d\}$.

设$\{\xi_{n,j},j\in J_n,n\geqslant 1\}$是随机场组列. 易见 $d$ 维整格点上随机场$\{\xi_t,t\in\mathbb{Z}^d\}$是随机场组列$\{\xi_{n,j},j\in J_n,n\geqslant 1\}$的特殊情形(写 $\xi_{n,j}=\xi_{nj,nj}=(j_1,j_2,\cdots,j_d)$). 现在从随机场组列$\{\xi_{n,j},j\in J_n,n\geqslant 1\}$构造第 $n$ 级集指标部分和过程为

(6.2.1) $Z_n(A)=n^{-d/2}\sum\limits_{j\in J_n}\dfrac{|A\cap C_n,j|}{|C_n,j|}(\xi_{n,j}-E\xi_{n,j}), A\in\beta^d, n\in\mathbb{N}$,

其中$|\cdot|$是 Lebesgue 测度,$\beta^d$ 是$[0,1]^d$ 的 Borel 集类. 对于随机场$\{\xi_t,t\in\mathbb{Z}^d\}$,对应地定义

(6.2.1′) $$Z_n(A)=n^{-d/2}\sum_{t\in\mathbb{Z}^d}|A\cap C_t|(\xi_t-E\xi_t),$$

其中 $A/n^d\in\beta^d$,$C_t=(t-\mathbf{1},t)$,$t=(t_1,\cdots,t_d)$,$t_i\in\mathbb{Z}$.

在§6.2和§6.3中,我们来证明 $Z_n$ 弱收敛于 Wiener 过程 $W$,其中假设 $Z_n$ 的定义域是满足一定度量熵条件的 $\beta^d$ 的子集,也在$\{\xi_{n,j}\}$上附设某些矩条件和混合相依条件. 令 $x>0$.

**定义6.2.1** 随机场$\{\xi_{n,j},j\in J_n\}$说是 $\alpha$ 混合的,若当 $n\to\infty$ 时,$\alpha(nx)\to 0$,其中

$$\alpha(nx)=\sup_{\substack{I,J\subset J_n\\ d(I,J)\geqslant x}}\ \sup_{\substack{A\in\sigma(\xi_{n,j},j\in I)\\ B\in\sigma(\xi_{n,j},j\in J)}}|P(AB)-P(A)P(B)|.$$

**定义6.2.2** 随机场$\{\xi_{n,j},j\in J_n\}$说是 $\rho$ 混合的,若当 $n\to\infty$ 时,$\rho(nx)\to 0$,其中

$$\rho(nx)=\sup_{\substack{I,J\subset J_n\\ d(I,J)\geqslant x}}\ \sup_{\substack{X\in L_2(\sigma(\xi_{n,j},j\in I))\\ Y\in L_2(\sigma(\xi_{n,j},j\in J))}}\frac{|\mathrm{Cov}(X,Y)|}{\sqrt{\mathrm{Var}X\ \mathrm{Var}Y}},$$

这里 $L_2(\mathscr{F})$ 表示关于 $\mathscr{F}$ 可测且平方可积的随机变量全体.

**定义6.2.3** 随机场 $\{\xi_{n,j}, j\in J_n\}$ 说是对称 $\varphi$ 混合的,若当 $n\to\infty$ 时,$\varphi(nx)\to 0$,其中

$$\varphi(nx) = \sup_{\substack{I,J\subset J_n \\ d(I,J)\geqslant x}} \sup_{\substack{A\in\sigma(\xi_{n,j},j\in I) \\ B\in\sigma(\xi_{n,j},j\in J) \\ P(A)P(B)>0}} \operatorname{man}(|P(A|B) - P(A)|,$$
$$|P(B|A) - P(B)|).$$

**定义6.2.4** 随机场 $\{\xi_{n,j}, j\in J_n\}$ 说是绝对正则的,若当 $n\to\infty$ 时,$\beta(nx)\to 0$,其中

$$\beta(nx) = \sup_{I,J\subset J_n, d(I,J)\geqslant x} \|\mathscr{L}(\xi_{n,j}, j\in I\cup J)$$
$$- \mathscr{L}(\xi_{n,j}, j\in I)\mathscr{L}(\xi_{n,j}, j\in J)\|_{\mathrm{Var}},$$

$\mathscr{L}(\xi(\cdot))$ 是 $\{\xi(\cdot)\}$ 的分布律,$\|\cdot\|_{\mathrm{Var}}$ 的全变差范数.

显然地

$$(6.2.2)\quad \alpha(nx)\leqslant\rho(nx)\leqslant 2\varphi(nx),\ \alpha(nx)\leqslant\beta(nx)\leqslant\varphi(nx).$$

其次,我们引入度量熵条件.我们称 $\beta^d$ 中 Borel 集 $A$、$B$ 是等价的,若 $|A\triangle B|=0$,用 $\mathscr{E}$ 记等价类构成的集合.令 $d_L(A,B)=|A\triangle B|$,可证 $d_L(\cdot,\cdot)$ 是 $\mathscr{E}$ 上一个度量,在 $d_L$ 下集 $\mathscr{E}$ 构成一完备度量空间.

**定义6.2.5** $\mathscr{E}$ 的子集 $\mathscr{A}$ 说是(内含)全有界的,若对每一 $\delta>0$,存在一有限集 $\mathscr{A}_\delta\subset\mathscr{E}$,使对每一 $A\in\mathscr{A}$ 存在 $A^+, A_-\in\mathscr{A}_\delta$ 满足 $A_-\subset A\subset A^+$ 且 $|A^+\backslash A_-|\leqslant\delta$.

注意到 $\mathscr{A}_\delta$ 关于 $d_L$ 是 $\mathscr{A}$ 的一个 $\delta$ 网.

设 $\mathscr{A}$ 是 $\mathscr{E}$ 的一个全有界子集.它的闭包 $\overline{\mathscr{A}}$ 是完备且全有界的,因此是紧的.设 $C(\overline{\mathscr{A}})$ 是 $\overline{\mathscr{A}}$ 上具有上确界范数的连续函数空间.因 $\overline{\mathscr{A}}$ 是紧的,所以 $C(\overline{\mathscr{A}})$ 是可分的.这样 $C(\overline{\mathscr{A}})$ 是可分完备度量空间.设 $CA(\overline{\mathscr{A}})$ 是 $C(\overline{\mathscr{A}})$ 的处处可加的元 $f$ 的全体,即对 $A, B, A\cup B, A\cap B\in\overline{\mathscr{A}}$,元 $f$ 满足 $f(A\cup B)=f(A)+f(B)-f(A\cap B)$.可以证明对固定的 $w$,$Z_n(\cdot)\in CA(\overline{\mathscr{A}})$,即 $Z_n$ 是 $CA(\overline{\mathscr{A}})$ 的随机元.$\overline{\mathscr{A}}$ 上的标准 Wiener 过程就是 $CA(\overline{\mathscr{A}})$ 的一个随机元 $W$,它的有限维分布是具有 $EW(A)=0$,$EW(A)W(B)=|A$

$\cap B|$ 的正态分布. 为证 $W$ 的存在性，要求 $\mathscr{A}$ 满足度量熵条件(见 Dudley1973).

**定义6.2.6** 设 $\mathscr{A}$ 是 $\mathscr{E}$ 的一个全有界子集，$\mathscr{A}_\delta$ 是 $\mathscr{A}$ 的最小 $\delta$ 网. 记

$$N(\delta,\mathscr{A}) = \mathrm{Card}\mathscr{A}_\delta, \quad H(\delta) = \log N(\delta,\mathscr{A}).$$

$\mathscr{A}$ 说是满足度量熵条件的，或具有收敛熵积分，若

$$\int_0^1 \left(\frac{H(\delta)}{\delta}\right)^{1/2} d\delta < \infty. \tag{6.2.3}$$

用 $r := \inf\{s, s>0, H(\delta)=O(\delta^{-s}), \delta\to 0\}$ 定义 $\mathscr{A}$ 的度量熵指数. 若 $r<1$，那么(6.2.3)成立.

**注6.2.1** 满足度量熵条件的某些集类的例子有：

若 $\mathscr{C}^d$ 记 $[0,1]^d$ 的凸子集的体，则 $r=(d-1)/2$(见 Dudley1974).

若 $\mathscr{I}^d=\{(\boldsymbol{a},\boldsymbol{b}):\boldsymbol{a},\boldsymbol{b}\in[0,1]^d\}$ 如上，则 $r=0$.

若 $\mathscr{P}^{d,m}$ 记 $[0,1]^d$ 的所有多边形区域族，其顶点不超过 $m$ 个，则 $r=0$(见 Erickson 1981).

若 $\mathscr{E}^d$ 记 $[0,1]^d$ 中所有椭球区域全体，则 $r=0$(见 Gaeussler 1983).

Vapnik-Cervonenkis 类 $\nu$ 是包含后三个例子的一种重要集类，它对某 $c,\nu>0$ 满足 $N(\delta,\nu,d_\lambda)=\mathrm{Card}\ \nu_\delta\leqslant c\delta^{-\nu}$(Dudley 1978).

当 $\{\xi_t, t\in\mathbb{Z}^d\}$ 独立时，$Z_n$ 弱收敛于 $W$ 的条件已被 Bass 和 Pyke (1984,1985), Alexander 和 Pyke(1986), 陆传荣(1992)等所研究. 对于混合相依随机场，$Z_n$ 弱收敛于 $W$ 首先被 Goldie 和 Greenwood (1986$a$,$b$)所讨论. 他们证明了下述定理.

**定理6.2.1** 假设 $E\xi_{n,j}=0$ 且

(i) 对某 $s>2$，$\{|n^{d/2}\xi_{n,j}|^s, j\in J_n, n\geqslant1\}$ 是一致可积的；

(ii) $\mathscr{A}$ 的(内含)度量熵的指数 $r<1$；

(iii) $\beta(nx)=O((nx)^b)(nx\to\infty)$，绝对正则指数 $b$ 满足

$$b\geqslant ds/(s-2) \text{ 及 } b>d(1+r)/(1-r);$$

(iv) 对称 $\varphi$ 混合系数满足

$$\sup_{n\geqslant 1}\sum_{j=1}^{\infty}\varphi^{1/2}(2^j n^{-1}) < \infty$$

(v) 对 $\mathscr{I}^d$ 中任一零族 $\{D_h, 0<h<h_0\}$（一个零族是一族满足 $D_h \subseteq D_{h'}$ $(h \leqslant h')$ 且对每一 $h$ $|D_h| = h$ 的集），

$$\lim_{h\downarrow 0}\lim_{n\to\infty}\sup\left|\frac{EZ_n^2(D_h)}{|D_h|} - 1\right| = 0.$$

那么在 $CA(\overline{\mathscr{A}})$ 中 $Z_n$ 弱收敛于 $W$.

对于随机场 $\{\xi_t, t\in \mathbb{Z}^d\}$，定义6.2.1，6.2.2，6.2.3和6.2.4对应地为当 $x\to\infty$ 时

$$\alpha(x) = \sup_{\substack{I,J\subset \mathbb{Z}^d \\ d(I,J)\geqslant x}} \sup_{\substack{A\in\sigma(\xi_i, i\in I) \\ B\in\sigma(\xi_j, j\in J)}} |P(AB) - P(A)P(B)| \to 0,$$

$$\rho(x) = \sup_{\substack{I,J\subset \mathbb{Z}^d \\ d(I,J)\geqslant x}} \sup_{\substack{X\in L_2(\sigma(\xi_i, i\in I)) \\ Y\in L_2(\sigma(\xi_j, j\in J))}} \frac{|\mathrm{Cov}(X,Y)|}{\sqrt{\mathrm{Var}X\ \mathrm{Var}Y}} \to 0,$$

$$\beta(x) = \sup_{I,J\subset \mathbb{Z}^d, d(I,J)\geqslant x} \|\mathscr{L}(\xi_j, j\in I\cup J) - \mathscr{L}(\xi_j, j\in I)\mathscr{L}(\xi_j, j\in J)\|_{\mathrm{Var}} \to 0,$$

$$\varphi(x) = \sup_{\substack{I,J\subset \mathbb{Z}^d \\ d(I,J)\geqslant x}} \sup_{\substack{A\in\sigma(\xi_i, i\in I) \\ B\in\sigma(\xi_j, j\in J), P(A)P(B)>0}} \max(|P(A|B) - P(A)|, |P(B|A) - P(B)|) \to 0.$$

**推论6.2.1** 设 $\{\xi_t, t\in \mathbb{Z}^d\}$ 是强平稳实随机场，$E\xi(\mathbf{0}) = 0$. 假设

(i) 对某 $s>2$，$E|\xi_0|^s<\infty$；

(ii) $\mathscr{A}$ 的(内含)度量熵指数 $r<1$；

(iii) $\beta(x) = O(x^{-b})$ $(x\to\infty)$ 且 $b>\max(ds/(s-2), d(1+r)/(1-r))$；

(iv) $\sum_{j=1}^{\infty}\rho^{1/2}(2^j)<\infty$；

(v) $\sum_{t\in\mathbb{Z}^d} E\xi_0\xi_t = 1$.

那么在 $CA(\overline{\mathscr{A}})$ 中，$Z_n$ 弱收敛于 $W$.

Dobrushin(1968)举例指出：对具有物理背景的 Gibbs 随机场，它不满足 $\varphi$ 混合条件. Dobrushin 和 Nahapetian(1974)引入了

非一致 $\varphi$ 混合条件.

**定义6.2.7** 随机场 $\{\xi_t, t\in \mathbb{Z}^d\}$ 说是非一致 $\varphi$ 混合的，若对 $\Lambda_i \subset \mathbb{Z}^d, |\Lambda_i|<\infty, i=1,2$，存在一个依赖于 $|\Lambda_1|$ 的非负函数 $\varphi_{|\Lambda_1|}(\cdot)$ 使得

$$\sup_{E\in\sigma(\Lambda_1),F\in\sigma(\Lambda_2),P(F)>0} |P(E|F)-P(E)| \leqslant \varphi_{|\Lambda_1|}(d(\Lambda_1,\Lambda_2)),$$

其中 $\varphi_{|\Lambda_1|}(x)\to 0(x\to\infty)$，$|\Lambda|$ 是 $\Lambda$ 的元数.

陈冬青(1991)对具有非一致 $\varphi$ 混合条件的集指标部分和过程给出了它收敛于 Brown 运动的一个充分条件，此时指标集 $\mathscr{A}=\mathscr{I}^d=\{(\boldsymbol{a},\boldsymbol{b}], \boldsymbol{a},\boldsymbol{b}\in[0,1]^d\}$.

**定理6.2.2** 设 $\{\xi_t, t\in\mathbb{Z}^d\}$ 是强平稳，非一致 $\varphi$ 混合随机场，满足

(i) 在 $R^1$ 上存在一个非负函数 $\varphi(\cdot)$，使对任一 $\Lambda\subset\mathbb{Z}^d, |\Lambda|<\infty$ 有 $\varphi_{|\Lambda|}(\cdot)\leqslant|\Lambda|\varphi(\cdot)$ 且对某 $\delta>0$

(6.2.4) $$\limsup_{r\to\infty}(\varphi(r))^{1/2}r^{3d+4d/\delta}<\infty,$$

(ii) $E\xi_0=0, E|\xi_0|^{2+\delta}<\infty$,

(6.2.5) (iii) $0<\sigma^2 := \sum_{t\in\mathbb{Z}^d}\mathrm{Cov}\ (\xi_0,\xi_t)<\infty$,

那么在 $CA(\overline{\mathscr{A}})$ 中，$Z_n/\sigma$ 弱收敛于 Brown 运动.

通过一个直接计算，陆传荣(1995)改进了定理6.2.2，对于较一般的指标集 $\mathscr{A}$（具有度量熵指标 $r$, $0<r<1$）并在较弱的非一致 $\varphi$ 混合速度下证明了定理6.2.2的结论仍成立.

**定理6.2.3** 设 $\{\xi_t, t\in\mathbb{Z}^d\}$ 是强平稳非一致 $\varphi$ 混合随机场，满足

(i) $\varphi_{|\Lambda|}(\cdot)\leqslant|\Lambda|\varphi(\cdot)$, $\varphi(x)=O(x^{-2d-1-2d/\delta})$ 某 $\delta>0$,

(ii) $E\xi_0=0, E|\xi_0|^{2+\delta}<\infty$,

(iii) $\mathscr{A}$ 具有内含度量熵指数 $r<1$.

那么在 $CA(\overline{\mathscr{A}})$ 中，$Z_n/\sigma$ 弱收敛于 Brown 运动 $W$.

**注6.2.2** 陈冬青(1991)指出从定理6.2.2即可推出对某些 Gibbs 场的一致中心极限定理成立，由定理6.2.3同样可得当 $\mathscr{A}$ 具有度量熵指数 $r<1$ 时，相应的结论也对.

定理6.2.1和6.2.2的证明在此被省略了.

定理6.2.3的证明需要下述引理.关于矩估计的某些引理有着独立的兴趣.

$R^d$ 中的一个切片是指一个集

$$S(\boldsymbol{c},a,\eta)=\{\boldsymbol{x}\in R^d;a\leqslant \boldsymbol{c}'\boldsymbol{x}\leqslant a+\eta\},$$

其中 $\boldsymbol{x}\in R^d$, $|\boldsymbol{c}|=1$, $a\in R$, $\eta>0$. 切片厚度是 $\eta$,方向是 $\boldsymbol{c}$,位移是 $a$. 一个切片可把集 $A\subseteq R^d$ 分成三部分,即 $A\cap S(\boldsymbol{c},a,\eta)$ 和集

$$A_+=A\cap\{\boldsymbol{x}\in R^d;\boldsymbol{c}'\boldsymbol{x}>a+\eta\},\ A_-=A\cap\{\boldsymbol{x}\in R^d;\boldsymbol{c}'\boldsymbol{x}<a\}.$$

若 $A$ 是可测的且 $|A_+|=|A_-|$,就称切片等分集 $A$.

**引理6.2.1**(等分引理) 存在着仅依赖于 $d$ 的正常数 $C_0$ 和 $q$,使对所有满足 $0<p<1/d$ 的 $p$ 和每一有有限测度的可测集 $A\subset R^d$,我们可求得切片 $S$ 等分 $A$,且有厚 $(|A|/2)^p$,使

$$|A\cap S|\leqslant C_0(|A|/2)^{(q+pd)/(q+1)}. \tag{6.2.6}$$

引理6.2.1的证明请参见 Goldie 和 Greewood(1986b).

设 $\{X_i,i\in \mathbb{Z}^d\}$ 是 $\rho$ 混合随机场,$EX_i=0$. 记

$$\rho=\sum_{j=0}^{\infty}\rho(2^j)<\infty,\qquad \sigma=\sup_i\|X_i\|_2,$$

$$Z(A)=\sum_{i\in \mathbb{Z}^d}|A\cap C_i|X_i,\qquad A\in\beta^d.$$

**引理6.2.2** 存在仅依赖于 $d$ 的常数 $a,b$ 使得

$$\|Z(A)\|_2\leqslant ae^{b\rho}\sigma|A|^{1/2},A\in\beta^d. \tag{6.2.7}$$

**证** 不妨设 $\sigma=1$. 记

$$\sigma(h)=\sup_{A\in\beta^d,|A|=h}\|Z(A)\|_2,\qquad \bar{\sigma}(h)=\sup_{h'\leqslant h}\sigma(h').$$

注意到

$$\|Z(A)\|_2\leqslant\sum_{j\in \mathbb{Z}^d}|A\cap C_j|\,\|X_j\|_2\leqslant\sum_{j\in \mathbb{Z}^d}|A\cap C_j|=|A|,$$

所以

$$\sigma(h)\leqslant h. \tag{6.2.8}$$

在引理6.2.1中取 $p$ 使得指数 $r=(q+pd)/(q+1)$ 的任何正整幂不等于1/2. 这是为了避免以后枝节上的麻烦. 设 $A$ 有正的有

限测度$2m$，引理6.2.1中的切片$S$是$S(c,a,m^p)$. 将$\{x\in R^d: c'x>a+m^p\}$和$\{x\in R^d: c'x<a\}$看作为包含$A_+$和$A_-$的$S$的两个侧面. 由于$|A_+|=|A_-|$，我们知

$$m\geqslant|A_+|=|A_-|\geqslant m-|A\cap S|\geqslant m-C_0m^r.$$

我们可求得$A'_+$，它在$S$的包含$A_+$的那一面与$A_-$不相交且有测度$|A'_+|=m-|A_+|$. 这样$|A'_+|\leqslant C_0m^r$. 令$A''_+=A_+\cup A'_+$，则$|A''_+|=m$. 类似地，我们在$S$的包含$A_-$一侧构造$A'_-$和$A''_-$. 现若$x\in A''_+$，$y\in A''_-$，那么

$$\|x-y\|\geqslant d^{-1/2}|x-y|\geqslant d^{-1/2}m^p.$$

因此若$C_i$和$C_j$分别与$A''_+$和$A''_-$相交，那么

$$\|i-j\|\geqslant d^{-1/2}m^p-2,$$

因此

$$\begin{aligned}&E(Z(A''_+)+Z(A''_-))^2\\ &\leqslant(1+\rho(d^{-1/2}m^p-2))(EZ^2(A''_+)+EZ^2(A''_-))\\ &\leqslant(1+\rho(d^{-1/2}m^p-2))2\sigma(m)^2.\end{aligned}$$

由于

$$Z(A)=Z(A''_+)+Z(A''_-)-Z(A'_-)-Z(A'_-)+Z(A\cap S), \tag{6.2.9}$$

从三角不等式得

$$\sigma(2m)\leqslant 2^{1/2}(1+\rho(d^{-1/2}m^p-2))^{1/2}\sigma(m)+3\bar\sigma(C_0m^r).$$

选$h>1$，那么可写$h=2^km$，其中$k\in N$，$1/2<m\leqslant 1$，我们有

$$\sigma(m)\leqslant 1,\ \sigma(2^{j+1}m)\leqslant\alpha_j\sigma(2^jm)+\beta_j,$$

其中

$$\alpha_j:=2^{1/2}(1+\rho(d^{-1/2}2^{jp}m^p-2))^{1/2},\ \beta_j:=3\bar\sigma(C_02^{jr}).$$

迭代得

$$\sigma(h)\leqslant\prod_{j=0}^{k-1}\alpha_j+\sum_{j=0}^{k-1}\beta_j\prod_{i=j+1}^{k-1}\alpha_i. \tag{6.2.10}$$

设$l\in N$满足$p>1/l$. 对$j\geqslant j_0\geqslant 1$，有$d^{1/2}2^{jp}2^{-p}-2>2^{j/l}$，所以

$$\sum_{j=j_0}^{\infty}\rho(d^{-1/2}2^{jp}m^p-2)\leqslant\sum_{j=1}^{\infty}\rho(2^{j/l})$$

$$= \sum_{j=0}^{\infty}\sum_{i=1}^{l}\rho(2^j 2^{i/l}) \leqslant \sum_{j=0}^{\infty}\sum_{i=1}^{l}\rho(2^j) = lp.$$

这样

$$(6.2.11) \qquad \prod_{j=0}^{\infty} (1+\rho(d^{-1/2}2^{j\rho}m^{\rho}-2))^{1/2} \leqslant 2^{j_0/2}e^{lp/2} =: a_1 e^{b_1\rho},$$

结合(6.2.10)得

$$(6.2.12) \qquad \sigma(h) \leqslant a_1 e^{b_1\rho}\Big(2^{k/2} + 3\sum_{j=0}^{k-1}2^{(k-1-j)/2}\bar{\sigma}(C_0 2^{jr})\Big).$$

设整数$\nu$使得$r^{\nu}<1/2<r^{\nu-1}$(回顾到$r^{\nu}=1/2$在$r$的选取时已被排除). 我们重复地应用(6.2.12)$\nu$次就可得引理的结论

$$(6.2.13) \qquad \sigma(h) \leqslant ae^{b\rho}h^{1/2}.$$

事实上,应用(6.2.8)于(6.2.12)的右边,若$\nu=1$,即$r<1/2$,有

$$\sigma(h) \leqslant a_1 e^{b_1\rho}2^{k/2}\{1 + 3C_0(1 - 2^{-(1/2-r)})^{-1}\},$$

由于$2^{k/2}\leqslant 2^{1/2}h^{1/2}$,就得(6.2.13). 在其它情形,即当$r>1/2$,我们得

$$\sigma(h) \leqslant ae^{b_1\rho}\left\{2^{k/2} + \frac{3}{2}C_0(2^{r-1/2} - 1)^{-1}2^{kr}\right\},$$

由此对某$a'_1$, $\bar{\sigma}(h)\leqslant a'_1 e^{b_1\rho}h^{r}$,代入(6.2.12),若$r^2<1/2$,我们就得(6.2.13),否则$\sigma(h)\leqslant a_2 e^{b_2\rho}h^{r^2}$. 运用(6.2.12)$\nu$次后就得证(6.2.13)成立. 证毕.

用同样方法,可证下述引理.

**引理6.2.3** 假设$\tau:=\sup_i\|X_i\|_s<\infty$, $s=2+\delta$, $0\leqslant\delta\leqslant 1$且

$$\rho'' = \sum_{j=1}^{\infty}\rho^{2/s}(2^j) < \infty.$$

那么对任给$A\in\beta^d$有

$$(6.2.14) \qquad \|Z(A)\|_s \leqslant ce^{c\rho''}\tau|A|^{1/2}.$$

**证** 首先,从引理6.2.2即得引理6.2.3对$\delta=0$成立. 当$0<\delta\leqslant 1$时,设$\tau(m)=\sup\limits_{A\in\beta^d,|A|=m}\|Z(A)\|_s$, $\bar{\tau}(m)=\sup\limits_{m'\leqslant m}\tau(m')$.

从(6.2.9)我们得

$$(6.2.15) \qquad \tau(2m) \leqslant \|Z(A''_+)+Z(A''_-)\|_{2+\delta}+3\bar{\tau}(C_0 m^r).$$

应用不等式

$$|1+x|^{2+\delta}\leqslant 1+9|x|+9|x|^{1+\delta}+|x|^{2+\delta}, \tag{6.2.16}$$

我们有

$$\begin{aligned} E|Z(A''_+)+Z(A''_-)|^{2+\delta} &\leqslant 2\tau^{2+\delta}(m)+9(E|Z(A''_+)|\,|Z(A''_-)|^{1+\delta} \\ &\quad + E|Z(A''_+)|^{1+\delta}|Z(A''_-)|). \end{aligned} \tag{6.2.17}$$

由引理1.2.7,我们有

$$\begin{aligned} E|Z(A''_+)|\,|Z(A''_-)|^{1+\delta} &\leqslant E|Z(A''_+)|E|Z(A''_-)|^{1+\delta} \\ &\quad + 4\rho^{\frac{2}{2+\delta}}\left(d^{-\frac{1}{2}}m^p-2\right)\tau^{2+\delta}(m) \\ &\leqslant \left\{1+4\rho^{\frac{2}{2+\delta}}\left(d^{-\frac{1}{2}}m^p-2\right)\right\}\tau^{2+\delta}(m). \end{aligned} \tag{6.2.18}$$

类似地,我们有

$$\begin{aligned} E|Z(A''_+)|^{1+\delta}|Z(A''_-)| &\leqslant (1 \\ &\quad + 4\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}m^p-2))\tau^{2+\delta}(m). \end{aligned} \tag{6.2.19}$$

把(6.2.18)和(6.2.19)代入(6.2.17)中,我们得

$$\|Z(A''_+)+Z(A''_-)\|_{2+\delta}\leqslant 20^{\frac{1}{2+\delta}}(1+2\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}m^p-2))^{\frac{1}{2+\delta}}\tau(m).$$

写 $\rho_0=\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}m^p-2)$,由此

$$\tau(2m)\leqslant 20^{\frac{1}{2+\delta}}(1+2\rho_0)^{\frac{1}{2+\delta}}\tau(m)+3\bar{\tau}(C_0m^r).$$

对 $h=2^km, k\in N$, $1/2<m\leqslant 1$,重复代入得

$$\tau(h)\leqslant \prod_{j=0}^{k-1}\alpha_j+\sum_{j=0}^{k-1}\beta_j\prod_{i=j+1}^{k-1}\alpha_i,$$

其中

$$\alpha_j\leqslant 20^{\frac{1}{2+\delta}}\{1+2\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}2^{jp}m^p-2)\}^{\frac{1}{2+\delta}},\ \beta_j=3\bar{\tau}(C_02^{jr}m^r).$$

由条件 $\rho''<\infty$ 得

$$\sum_{j=j_0}^{\infty}\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}2^{jp}m^p-2)<\infty.$$

其中 $j_0$ 定义如引理6.2.2中. 余下的证明与 $\delta=0$ 情形相同. 引理6.2.3证毕.

记

$$\rho' = \sum_{i=0}^{\infty}\rho^{1/2}(2^i), g(y) = \sup_i EX_i^2 I(|X_i| > y).$$

**引理6.2.4** 设$\{X_i, i\in \mathbb{Z}^d\}$是$\rho$混合随机场. 若$\{X_i^2, i\in \mathbb{Z}^d\}$一致可积且$\rho'<\infty$,那么随机变量$\{Z^2(A)/|A|, A\in \mathscr{B}^d, |A|<\infty\}$是一致可积的,即存在仅依赖于$d$的常数$C$使得

(6.2.20) $$E\left(\frac{Z^2(A)}{|A|}I\left(\frac{Z^2(A)}{|A|}>y\right)\right)<\{C(1\wedge y^{-1})+Cg(y^{1/4})\}e^{c\rho'}.$$

**引理6.2.5** 设$\{W(A), A\in \beta^d\}$是可加过程,满足

(i) 对任一$C\in \mathscr{Z}^d$, $EW(C)=0$,

(ii) 对任一$C\in \mathscr{Z}^d$, $EW^2(C)=|C|$,

(iii) 当$C_1,\cdots,C_k\in \mathscr{Z}^d$且$d(C_i,C_j)>0(i\neq j)$时, $W(C_1),\cdots,W(C_k)$是相互独立的,

(iv) 对任给$\varepsilon>0$, $\lim\limits_{m\to\infty}\sum\limits_{j\in J_m}P\{|W(C_{m,j})|\geqslant\varepsilon\}=0$.

那么在$\mathscr{R}=\mathscr{R}(\mathscr{Z}^d)$($\mathscr{Z}^d$的元的有限并所生成的环)上, $W$是一Wiener过程.

**引理6.2.6** 设$\{Z_n(A), A\in \mathscr{R}=\mathscr{R}(\mathscr{Z})^d, n\geqslant 1\}$是可加过程序列,满足

(i) 对任给$C\in \mathscr{Z}^d$, $EZ_n(C)\to 0(n\to\infty)$,

(ii) 对任给$C\in \mathscr{Z}^d$, $EZ_n^2(C)\to |C|(n\to\infty)$,

(iii) 对任何$C_1\cdots,C_k\in \mathscr{Z}^d$, $\rho(C_i,C_j)>0(i\neq j)$,对一切实数$z_1,\cdots,z_k$有

(6.2.21) $$P\{\bigcap_{i=1}^{k}(Z_n(C_i)\leqslant z_i)\}-\prod_{i=1}^{k}P\{Z_n(C_i)\leqslant z_i\}\to 0, n\to\infty,$$

(iv) 对任给$\varepsilon>0$

$$\lim_{m\to\infty}\ \limsup_{n\to\infty}\sum_{j\in J_m}P\{|Z_n(C_{m,j})|\geqslant\varepsilon\}=0,$$

(v) 对每一$C\in \mathscr{Z}^d$,集$\{Z_n^2(C), n\geqslant 1\}$是一致可积的. 那么在$R$上$Z_n$的有限维分布弱收敛于Wiener过程对应的有限维分布.

**引理6.2.7** 设$\beta^d$上可加过程列$\{Z_n\}$使得$\{Z_n^2(A)/|A|, A\in \beta^d, n\geqslant 1\}$是一致的可积的,且满足引理6.2.6的(i),(ii),(iii). 那么

在 $\beta^d$ 上,$Z_n$ 的有限维分布弱由敛于标准 Wiener 过程对应的有限维分布.

引理 6.2.4—6.2.7 的证明请参见 Goldie 和 Greenwood (1986a,b).

**定理6.2.4** 设$\{\xi_{nj},j\in J_n\}$是 $\rho$ 混合三角组列,由(6.2.1)定义的光滑化部分和过程 $Z_n$ 满足:

(i) 对任给 $n\geqslant 1,j\in J_n,E\xi_{n,j}=0$,

(ii) 集$\{n^d\xi_{n,j}^2,n\geqslant 1,j\in J_n\}$是一致可积的,

(iii) $\sup\limits_n\sum\limits_{j=1}^{\infty}\rho_n^{1/2}(n^{-1}2^j)<\infty$

和引理6.2.6 的(ii).那么在 $\beta^d$ 上,$Z_n$ 的有限维分布弱收敛于标准 Wiener 过程对应的有限维分布.

**证** 由引理6.2.4知集$\{Z_n^2(A)/|A|,A\in\beta^d,n\geqslant 1\}$是一致可积的.因 $\rho_n(x)$关于 $x$ 是不增的,条件(iii)蕴含对每一固定的 $x$,$\rho_n(x)\to 0(n\to\infty)$,这样 $\alpha_n(x)\to 0(n\to\infty)$.显然地,(6.2.21)的左边不超过$(k-1)\alpha_n(\theta)$,其中 $\theta>0$是集 $C_1,\cdots,C_k$ 间的最小距离,因此引理6.2.6的(iii)被满足.由引理6.2.7,定理6.2.4的结论成立,证毕.

## §6.3 胎紧性(Tightness)

首先我们给出一个与引理6.2.3类似的关于非一致 $\varphi$ 混合序列的引理.

**引理 6.3.1** $\{\xi_t,t\in Z^d\}$,$\{Z_n(A),A\in\beta^d\}$如定理6.2.3中所设,除去以 $\varphi(x)=O(x^{-(d+1+\theta)})$,某 $\theta>0$,代替 $\varphi(x)=O(x^{-(2d+1+2d/\delta)})$外.那么对任给的 $A\in\beta^d$

$$\|Z_n(A)\|_{2+\delta}\leqslant c\sigma_0|nA|^{1/2}, \tag{6.3.1}$$

其中 $\sigma_0^2=E\xi_0^2$.

**证** 首先,我们来证对 $\delta=0$有

$$\sigma(2m)\leqslant 2^{1/2}(1+2m^{1/2}\varphi^{1/2}(d^{-1/2}m^{\rho}-2))^{1/2}\sigma(m)+3\bar{\sigma}(cm^{r}),$$

其中 $\sigma(m)=\sup\limits_{A\in \mathscr{I}^d, |A|=m}\|Z_\pi(A)\|_2, \bar{\sigma}(m)=\sup\limits_{m'\leqslant m}\sigma(m')$. 在等分引理中取 $0<p<1/d$ 使得指数 $r=(q+pd)/(q+1)$ 的任何正整数幂不等于1/2. 对任给的 $h>1$, 我们写 $h=2^k m, k\in N, 1/2<m\leqslant 1$. 注意到 $\sigma(m)\leqslant 1$, 且对非一致 $\varphi$ 混合情形, 如引理6.2.3的证明中同样讨论, 我们有

$$\sigma(2^{j+1}m)\leqslant \alpha_j\sigma(2^j m)+\beta_j,$$

其中

$$\alpha_j=2^{1/2}(1+2\cdot 2^{j/2}m^{j/2}\varphi(d^{-1/2}2^{jp}m^p-2))^{1/2},$$

$$\beta_j=3\bar{\sigma}(c2^{jr}).$$

迭代得

$$\sigma(h)\leqslant\prod_{j=0}^{k-1}\alpha_j+\sum_{j=0}^{k-1}\beta_j\prod_{i=j+1}^{k-1}\alpha_j. \tag{6.3.2}$$

其次, 我们再取 $p$ 使 $\frac{1}{d}>p>\frac{1}{d+1}$ 且指数 $r=(q+pd)/(q+1)$ 的任何正整幂不等于1/2. 那么存在 $j_0\geqslant 1$ 使对 $j\geqslant j_0$ $d^{1/2}2^{jp}2^{-p}>2^{j/(d+1)}$, 所以

$$\begin{aligned}\varphi:&=\sum_{j=j_0}^{\infty}2^{j/2}\varphi^{j/2}(d^{-1/2}2^{jp}m^p-2)\\&\leqslant\sum_{j=1}^{\infty}2^{j/2}\varphi^{j/2}(2^{j/(d+1)})\\&\leqslant c\sum_{j=1}^{\infty}2^{j/2}2^{-\frac{1}{2}\frac{j}{d+1}(d+1+\theta)}\\&=c\sum_{j=1}^{\infty}(2^{\theta/(2(d+1))})^{-j}<\infty.\end{aligned} \tag{6.3.3}$$

这样我们就得

$$\sigma(h)\leqslant ce^{c\varphi}\left(2^{k/2}+3\sum_{j=0}^{k-1}2^{\frac{k-j-1}{2}}\bar{\sigma}(c2^{jr})\right).$$

余下的证明与引理6.2.2的证明相同. 这就证明了 $\delta=0$ 时(6.3.1)成立.

考虑 $0<\delta\leqslant 1$ 情形. 回顾

引理6.2.2的记号, 写

(6.3.4) $$Z_n(A)=Z_n(A''_+)-Z_n(A''_-)-Z_n(A'_+)-Z_n(A'_-)+Z_n(A\cap S),$$

其中$|A|=2m$，$S$是$A$的一个等分切片使$|A''_+|=|A''_-|=m$，$A''_+$和$A''_-$位于$S$的不同的两侧，其距离$d(A''_+,A''_-)\geqslant d^{-1/2}m^{\rho}$，$|A'_+|$，$|A'_-|$和$|A\cap S|$均不超过$cm^{\gamma}$．记

$$\tau(h)=\sup_{A\in\mathscr{A}^d,|A|=h}\|Z_n(A)\|_{2+\delta},\ \bar{\tau}(h)=\sup_{h'\leqslant h}\tau(h').$$

从(6.3.4)得

(6.3.5) $$\tau(2m)\leqslant\|Z_n(A''_+)+Z_n(A''_-)\|_{2+\delta}+3\bar{\tau}(cm^{\gamma}).$$

由(6.2.16)我们有

(6.3.6) $$\begin{aligned}&E|Z_n(A''_+)+Z_n(A''_-)|^{2+\delta}\\&\leqslant 2\tau^{2+\delta}(m)+9(E|Z_n(A''_+)|\cdot|Z_n(A''_-)|^{1+\delta}\\&\quad+E|Z_n(A''_+)|^{1+\delta}|Z_n(A''_-)|).\end{aligned}$$

由非一致$\varphi$混合的性质，我们有

(6.3.7) $$\begin{aligned}&E|Z_n(A''_+)|\,|Z_n(A''_-)|^{1+\delta}\\&\leqslant E|Z_n(A''_+)|E|Z_n(A''_-)|^{1+\delta}\\&\quad+2\varphi^{1/(2+\delta)}_{A''_+}(d^{-1/2}m^{\rho}-2)\tau^{2+\delta}(m)\\&\leqslant(1+2m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}m^{\rho}-2))\tau^{2+\delta}(m).\end{aligned}$$

类似地我们有

(6.3.8) $$\begin{aligned}&E|Z_n(A''_+)|^{1+\delta}|Z_n(A''_-)|\\&\leqslant(1+2m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}m^{\rho}-2))\tau^{2+\delta}(m).\end{aligned}$$

把(6.3.7)、(6.3.8)代入(6.3.6)我们得

$$\begin{aligned}\|Z_n(A''_+)+Z_n(A''_-)\|_{2+\delta}\leqslant c(1+m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}\\(d^{-\frac{1}{2}}m^{\rho}-2))^{2+\delta}\tau(m).\end{aligned}$$

由此我们有

$$\tau(2m)\leqslant c(1+m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}m^{\rho}-2))^{2+\delta}\tau(m)+3c\bar{\tau}(cm^{\gamma}).$$

对$h=2^k m$，$k\in\mathbb{N}$，$1/2<m\leqslant 1$，重复代入$k$次得

$$\tau(h)\leqslant\prod_{j=0}^{k-1}\alpha_j+\sum_{j=0}^{k-1}\beta_j\prod_{i=j+1}^{k-1}\alpha_i,$$

其中

$$\alpha_j \leqslant c\{1 + 2^{\frac{j}{2+\delta}} m^{\frac{j}{2+\delta}} \varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}2^{jp}m^p - 2)\}^{\frac{1}{2+\delta}}.$$

注意到

$$\varphi = \sum_{j=j_0}^{\infty} 2^{\frac{j}{2+\delta}} \varphi^{\frac{1}{2+\delta}}(d^{-\frac{1}{2}}2^{jp}m^p - 2)$$

$$\leqslant c\sum_{j=1}^{\infty}(2^{r/(2(2+\delta)(d+1))})^{-j} < \infty,$$

余下的证明与 $\delta=0$情形的证明相同. 引理6.3.1证毕.

定理6.2.3中胎紧性的证明. 对 $f\in C(\overline{\mathscr{A}})$我们记

$$w_\delta(f) = \sup_{A,B\in\mathscr{A},|A\Delta B|\leqslant\delta}|f(A) - f(B)| \tag{6.3.9}$$

作为 $f$ 的连续模. 那么, 由于 $\overline{\mathscr{A}}$ 是紧的, 由 Arzela-Ascoli 定理: $C(\overline{\mathscr{A}})$的子集 $U$ 有紧闭包当且仅当它是等度有界(即 $\sup\limits_{f\in U}\sup\limits_{A\in\mathscr{A}}|f(A)|<\infty$)且等度连续(即$\lim\limits_{\delta\downarrow 0}\sup\limits_{f\in U}w_\delta(f)=0$). 由此, 从 Billingsley (1968)定理8.2即得 $C(\overline{\mathscr{A}})$的随机元序列$\{Z_n\}$是相对紧的, 即$\{Z_n\}$的每一子列包含一个弱收敛的子列当且仅当

(a) 对 $\mathscr{A}$ 中某可列稠密集的每一元 $A$, $\{Z_n(A), n\geqslant 1\}$是胎紧的, 且

(b) 对每一 $\lambda>0$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}P\{w(Z_n) > \lambda\} = 0.$$

从定理6.2.4即得(a). 我们仅需证明(b). 不失一般性可设 $\sigma^2=1, E\xi_t=0$. 对$0\leqslant u\leqslant v<\infty$, 定义

$$\eta_{n,j}(u,v) = n^{-d/2}\xi_{nj}I(u\leqslant n^{d\delta/(2(1+\delta))}n^{-d/2}|\xi_{nj}|<v) \quad j\in J_n,$$

$$Z_n(A,u,v) = \sum_{j\in J_n}\frac{|A\cap C_{n,j}|}{|C_{n,j}|}(\eta_{n,j}(u,v) - E\eta_{n,j}(u,v)).$$

**引理6.3.2** 假设$\{\xi_t, t\in\mathbb{Z}^d\}$满足定理6.2.3的条件, 那么当 $n\to\infty$时 $U_n(I^d,a,\infty)\to 0$ a.s., $EU_n(I^d,a,\infty)\to 0$, 其中 $I^d=[0,1]^d$, $a>0$且

$$U_n(A,a,\infty) = \sum_{j\in J_n}\frac{|A\cap C_{n,j}|}{|C_{n,j}|}|\eta_{n,j}(a,\infty)|.$$

**证** 我们运用 Bass 的方法(1985). 对 $\boldsymbol{k}=(k_1,\cdots,k_d)\in\mathbb{Z}_+^d$，设 $\mu(\boldsymbol{k})=\max(k_1,\cdots,k_d)$，$\Psi(i,a)=\sup\{k\in\mathbb{Z}_+ : ak^{d/(2(1+\delta))}<i+1\}$. 不难看出 $\mathrm{Card}\{\boldsymbol{k}\in\mathbb{Z}_+^d : \mu(\boldsymbol{k})=r\}\leqslant Cr^{d-1}$，其中 $C$ 是依赖于 $d$ 的正常数.

$$
\begin{aligned}
&\sum_{\boldsymbol{k}\in\mathbb{Z}_+^d} I(i+1>a\mu(\boldsymbol{k})^{d/(2(1+\delta))})\mu(\boldsymbol{k})^{-d/2} \\
&=\sum_{r=1}^{\infty}\sum_{j:\mu(j)=(r)} I(i+1>ar^{d/(2(1+\delta))})r^{-d/2} \\
&=c\sum_{r=1}^{\infty} I(i+1>ar^{d/(2(1+\delta))})r^{(d-2)/2} \\
&\leqslant c(\Psi(i,a))^{d/2}\leqslant ca^{-1}(i+1)^{1+\delta}.
\end{aligned}
$$

这样

$$
\begin{aligned}
&\sum_{\boldsymbol{k}\in\mathbb{Z}_+^d} E|\xi_{\boldsymbol{k}}|I(a(\mu(\boldsymbol{k}))^{d/(2(1+\delta))}\leqslant|\xi_{\boldsymbol{k}}|<\infty)(\mu(\boldsymbol{k}))^{-d/2} \\
\leqslant&\sum_{\boldsymbol{k}}\sum_{i+1>a(\mu(\boldsymbol{k}))^{d/(2(1+\delta))}}(i+1)P\{i<|\xi_{\boldsymbol{k}}|\leqslant i+1\}(\mu(\boldsymbol{k}))^{-d/2} \\
\leqslant&\sum_{i=0}^{\infty}\Big[\sum_{\boldsymbol{k}} I(i+1>a(\mu(\boldsymbol{k}))^{d/(2(1+\delta))})(\mu(\boldsymbol{k}))^{-d/2}\Big] \\
&\cdot(i+1)P\{i<|\xi_0|\leqslant i+1\} \\
\leqslant&\, ca^{-1}\sum_{i=0}^{\infty}(i+1)^{2+\delta}P\{i<|\xi_0|\leqslant i+1\} \\
\leqslant&\, ca^{-1}E(|\xi_0|+1)^{2+\delta}<\infty.
\end{aligned}
$$

那么对任给 $\varepsilon>0$，存在 $n_1(w)$，使得

$$
\sum_{\boldsymbol{k}:\mu(\boldsymbol{k})>n_1}|\xi_{\boldsymbol{k}}I(a\mu(\boldsymbol{k})^{d/(2(1+\delta))}\leqslant|\xi_{\boldsymbol{k}}|<\infty)|/(\mu(\boldsymbol{k}))^{d/2}<\varepsilon.
$$

由此

$$
U_n(I^d,a,\infty)\leqslant\sum_{\mu(\boldsymbol{k})\leqslant n_1}|\xi_{\boldsymbol{k}}|I(an^{d/(2(1+\delta))},\leqslant|\xi_{\boldsymbol{k}}|<\infty)n^{-d/2}.
$$

这样，当 $n\to\infty$时，$U_n(I^d,a,\infty)\to0$ a. s. 类似地，我们可得

$$
EU_n(I^d,a,\infty)\to0.
$$

引理6. 3. 2证毕.

为证明(b)，我们只需证明

(6.3.10) $$\lim_{\nu\downarrow 0}\ \limsup_{n\to\infty} P\{\|Z_n\|_{\mathscr{A}_\nu}>\lambda\}=0,$$

其中 $\mathscr{A}_\nu=\{A\backslash B: A,B\in\mathscr{A},|A\backslash B|\leqslant\nu\}$. 由于 $Z_n(A)=Z_n(A,0,a)+Z_n(A,a,\infty)$,且

$$|Z_n(A,a,\infty)|\leqslant U_n(I^d,a,\infty)+EU_n(I^d,a,\infty),$$

由引理6.3.2,为证(6.3.10)式,我们仅需证明

(6.3.11) $$\lim_{\nu\downarrow 0}\ \limsup_{n\to\infty} P\{\|Z_n(A,0,a)\|_{\mathscr{A}_\nu}>\lambda\}=0.$$

设 $p_n=[n^{(2+\delta)/2(1+\delta)}]$, $m_n=n/(2p_n)$. 我们把 $I^d$ 按下述方法分划成:$C_{p_n,l}$, $l\in J_{p_n}$ 和 $C_{2p_n,l}$, $l\in J_{2p_n}$. 这样在每一 $C_{p_n,l}$ 中有 $2^d$ 个 $C_{2p_n,l}$. 记 $C_{p_n,l}$ 中的第 $i$ 个 $C_{2p_n,j}$ 为 $I_{n,l,i}$, $l\in J_{p_n}$, $j\in J_{2p_n}$. 设

$$I_{n,i}=\bigcup_{l\in J_{p_n}} I_{n,l,i},\ i=1,2,\cdots,2^d.$$

那么

$$Z_n(\cdot,0,a)=\sum_{i=1}^{2^d}Z_n(\cdot\cap I_{n,i},0,a).$$

现在为证(6.3.11),我们仅需证明

(6.3.12) $$\lim_{\nu\downarrow 0}\ \limsup_{n\to\infty} P\{\|Z_n(A\cap I_{n,i},0,a)\|_{\mathscr{A}_\nu}>\lambda\}=0.$$

写

$$Z_n(A\cap I_{n,i},0,a)=\sum_{l\in J_{p_n}}\sum_{j\in S(n,l,i)}\frac{|A\cap I_{n,l,i}\cap C_{n,j}|}{|C_{n,j}|}$$

$$(\eta_{nj}(0,a)-E\eta_{nj}(0,a))=:\sum_{l\in J_{p_n}}V_{nl}(A\cap I_{n,i},0,a),$$

$$\eta_{nj}=n^{-d/2}\xi_{nj}I(n^{(2+\delta)/2(1+\delta)}n^{-d/2}|\xi_{nj}|<a)\quad j\in J_n,$$

其中 $s(n,l,i)=\{j\in J_n: C_{n,j}\cap I_{n,l,i}\neq\phi\}$.

记 $V_{nl}=V_{nl}(A\cap I_{n,i},0,a)$. 由非一致 $\varphi$ 混合的性质,我们有

(6.3.13) $$Ee^{\alpha\sum_{k\in J_{p_n}}V_{nk}}\leqslant Ee^{\alpha V_{nl}}Ee^{\alpha\sum_{k\neq l,k\in J_{p_n}}V_{nk}}$$

$$+2\varphi_{S(n,l,i)}\left(\frac{n}{2p_n}\right)Ee^{\alpha\sum_{k\neq l}V_{nk}}\|e^{\alpha V_{nl}}\|_\infty.$$

注意到 $|V_{nl}|\leqslant 2a$. 且当 $\alpha a\leqslant 1/4$ 时 $e^{\alpha V_{nl}}\leqslant 1+\alpha V_{nl}+\alpha^2V_{nl}^2$. 从引理6.3.1即得

$$EV_{nl}^2 \leqslant C|A \cap I_{n,l,i}| \quad i = 1,2,\cdots,2^d.$$

所以对 $\alpha a \leqslant 1/4$ 有

$$Ee^{\alpha V_{nl}} \leqslant e^{E\alpha^2 V_{nl}^2} \leqslant e^{C\alpha^2|A\cap I_{n,l,i}|}. \tag{6.3.14}$$

把(6.3.14)代入(6.3.13)得

$$Ee^{\alpha \sum\limits_{l\in J_{p_n}} V_{nl}} \leqslant wEe^{\alpha \sum\limits_{k\neq l,\ k\in J_{p_n}} V_{nk}},$$

其中

$$\begin{aligned} w &\leqslant e^{c\alpha^2|A\cap I_{n,l,i}|}\left(1 - 2e^{1/2}|S(n,l,i)|\varphi\left(\frac{n}{2p_n}\right)\right) \\ &\leqslant e^{c\alpha^2|A\cap I_{n,l,i}|}\left(1 + 2e^{1/2}\frac{n^d}{(2p_n)^d}\varphi\left(\frac{n}{2p_n}\right)\right). \end{aligned}$$

迭代得对充分大 $n$

$$\begin{aligned} Ee^{\alpha \sum\limits_{l\in J_{p_n}} V_{nl}} &\leqslant e^{c\alpha^2|A\cap I_{n,l}|}\left(1 + 2e^{1/2}\frac{n^d}{(2p_n)^d}\varphi\left(\frac{n}{p_n}\right)\right)^{p_n^d} \\ &\leqslant \exp\left\{c\alpha^2|A| + cn^d\left(\frac{n}{p_n}\right)^{2d-2d/\delta-1}\right\} \\ &= \exp\left\{c\alpha^2|A| + cn^{-\frac{\delta}{2(1+\delta)}}\right\} \\ &\leqslant c\exp(\alpha^2|A|). \end{aligned} \tag{6.3.15}$$

现在我们来估计(6.3.12)式中左边的概率. 因 $0\leqslant r<1$, 故可选 $s>0$ 使 $r<1/(1+s)$. 令

$$\begin{aligned} &\delta_j = \nu/2^j && j = 0,1,\cdots, \\ &\lambda_j = \lambda_0 e^{-j(1+r-r(2+s))/(2+s)} && j = 1,2,\cdots, \\ &\lambda_0 = \lambda(1 - 2^{-(1+r-r(2+s))/(2+s)}), && \\ &\alpha_j = e^{-j(1+r)/(2+s)}a && j = 1,2,\cdots, \\ &a = C\nu^{1/(1+s)}, && c_0 = (6E|\xi_0|^{2+s}/\lambda_0)^{1/(1+s)}. \end{aligned}$$

对任给 $A\in\mathscr{A}_0$ 存在 $A_j, A_j^+\in\mathscr{A}_0(\delta_j)$ 使得 $A_j\subsetneqq A\subseteq A_j^+$ 且 $|A_j^+\setminus A_j|\leqslant\delta_j$.

那么

$$Z_n(A\cap I_{n,i},0,a)$$

$$= Z_n(A_0 \cap I_{n,i}, 0, a) + \sum_{j=0}^{\infty}\{Z_n(A_{j+1} \cap I_{n,i}, 0, a_j)$$

$$- Z_n(A_j \cap I_{n,i}, 0, a)\} + \sum_{j=0}^{\infty}\{Z_n(A \cap I_{n,i}, a_j, a_{j-1})$$

$$- Z_n(A_j \cap I_{n,i}, a_j, a_{j-1})\}.$$

故若$\|Z_n(\cdot \cap I_{n,i}, 0, a)\|_{\mathscr{A}_0}$超过$\lambda$，下述事实至少有一成立：

(a) 对某 $A_0 \in \mathscr{A}_0(\delta_0)$，$|Z_n(A_0 \cap I_{n,i}, 0, a)| > \lambda_0$；

(b) 对某 $j$，某 $A_j \in \mathscr{A}_0(\delta_j)$，$A_{j+1} \in \mathscr{A}_0(\delta_{j+1})$，$|A_j \Delta A_{j+1}| \leqslant 2\delta_j$，

$$|Z_n(A_{j+1} \cap I_{n,i}, 0, a_j) - Z_n(A_j \cap I_{n,i}, 0, a_j)| > 2\lambda_j;$$

(c) 对某 $j$，某 $A_j, A_j^+ \in \mathscr{A}_0(\delta_j)$，$A_j \subseteq A \subseteq A_j^+$，$|A_j^+ \backslash A_j| \leqslant \delta_j$，

$$|Z_n(A \cap I_{n,i}, a_j, a_{j-1}) - Z_n(A_j \cap I_{n,i}, a_j, a_{j-1})| > \lambda_j.$$

在 $\mathscr{A}_0(\delta_j)$中，集对 $A_j, A_j^+$ 的个数$\leqslant \exp(4H(\delta_j/2))$，而集对 $A_j \in \mathscr{A}_0(\delta_j)$，$A_{j-1} \in \mathscr{A}_0(\delta_{j-1})$的个数$\leqslant \exp(4H(\delta_{j+1}/2))$.

我们有

$$P\{\|Z_n(\cdot \cap I_{n,i}, 0, a)\|_{\mathscr{A}_0} > \lambda\} \leqslant p_0 + \sum_{j=0}^{\infty} r_j + \sum_{j=1}^{\infty} s_j,$$

其中

$$p_0 \leqslant 2\exp\{2H(\delta_0/2)\} \max_{|A_0| \leqslant 2\delta_0} P\{|Z_n(A_0 \cap I_{n,i}, 0, a)| > \lambda_0\},$$

$$r_j \leqslant 4\exp\{4H(\delta_{j-1}/2)\} \max_{|A_{j+1}\Delta A_j| \leqslant 2\delta_j}$$

$$\cdot (P\{|Z_n((A_{j+1} \backslash A_j) \cap I_{n,i}, 0, a)| > \lambda_j\}$$

$$+ P\{|Z_n((A_j \backslash A_{j+1}) \cap I_{n,i}, 0, a)| > \lambda_j\}),$$

$$s_j \leqslant \exp\{4H(\delta_j/2)\} \max_{A_j \subseteq A_j^+, |A_j^+ \backslash A_j| \leqslant 2\delta_j}$$

$$\cdot P\{\sup_{A_j \subseteq A \subseteq A_j^+} |Z_n(A \cap I_{n,i}, a_j, a_{j-1})$$

$$- Z_n(A_j \cap I_{n,i}, a_j, a_{j-1})| > \lambda_j\}.$$

由(6.3.15)，取 $\alpha = 1/4a_0$，我们有

$$p_0 \leqslant 2\exp\{2H(\delta_0/2)\}\exp\left\{-\frac{\lambda_0}{4a_0} + c_0\frac{\delta_0}{a_0^2}\right\}$$

$$\leqslant 2\exp\left\{c2^{r+1}\delta_0^{-r} - \frac{\lambda_0}{4a_0} + c_0\frac{\delta_0}{a_0}\right\}.$$

类似地

$$r_j \leqslant 4\exp\left\{c2^{r+1}\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c_0\frac{\delta_j}{a_j}\right\},$$

$$s_j \leqslant \exp\left\{c2^{r+1}\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c_0\frac{\delta_j}{a_j}\right\}.$$

这样

$$\begin{aligned} & p_0 + \sum_{j=0}^{\infty} r_j + \sum_{j=0}^{\infty} s_j \\ & \leqslant 6\sum_{j=0}^{\infty}\exp\left\{c\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c_0\frac{\delta_j}{a_j}\right\} \\ & \leqslant 6\sum_{j=0}^{\infty}\exp\{(c\nu^{-r} - c'\nu^{-\frac{1}{1+s}} + c\nu^{-\frac{1-s}{1+s}}2^{-\frac{jsr}{2-s}})2^{jr}\} \end{aligned}$$

因为 $r<1/(1+s)$，可选 $\nu$ 充分小使 $2^{jr}$ 的系数是负的，由此得(6.3.12)成立. 定理6.2.3证毕.

# 第七章 Berry-Esseen 不等式和弱收敛的速度

众所周知，独立随机变量序列$\{X_n,n\geqslant 1\}$前 $n$ 项的正则化和的分布函数 $F_n(x)$与正态分布函数 $\Phi(x)$之差的一致估计是由 Esseen 和 Berry-Esseen 不等式给出的．进一步，对$\{X_n\}$为 $i.i.d.$ 随机变量序列．当 $E|X_1|^3<\infty$、$EX_1=0$ 时存在一个简明的非一致估计

$$|F_n(x)-\Phi(x)|\leqslant A\rho_0/(\sqrt{n}(1+|x|^3)),$$

其中 $\rho_0=E|X_1|^3/\sigma^3$，$\sigma^2=EX_1^2$，$F_n(x)=P\{S_n/\sigma\sqrt{n}<x\}$．

在本章中，我们将对 $\alpha$ 混合序列和 $\rho$ 混合序列给出一致估计于 § 7.1 中．

在 § 7.2 中，我们将讨论由部分和过程$\{W_n(t),0\leqslant t\leqslant 1,n\geqslant 1\}$所生成的测度 $W_n$ 与 Wiener 测度 $W$ 间的 Prohorov 距离 $L(P_0W_n^{-1},W)$．对 $\varphi$ 混合序列，我们将给出 $L(p_0W_n^{-1},W)$的估计．

## § 7.1 $\alpha$ 混合和 $\rho$ 混合序列依分布收敛的速度

对独立随机变量序列，Esseen 和 Berry-Esseen 不等式的证明是基于下述命题（见 Petrov 1975）

**命题 7.1.1** 设 $F_n(x)$是分布函数，其对应特征函数为 $f_n(t)$．那么对任给 $T>0$ 和 $b>1/(2\pi)$，存在常数 $r(b)>0$，我们有

$$\sup_x|F_n(x)-\Phi(x)|\leqslant b\int_{-T}^{T}\left|\frac{f_n(t)-e^{-t^2/2}}{t}\right|dt+r(b)\frac{1}{\sqrt{2\pi}T}. \tag{7.1.1}$$

给出$|f_n(t)-e^{-t^2/2}|$的估计是证明的关键．在独立随机变量情形中，还比较简单．对于混合相依变量情形，如何去获得$|f_n(t)-e^{-t^2/2}|$的估计呢？Tikhomirov（1980）给出了一个方法，此

方法被 Sunklodas(1984)所改进.该方法的概略如下:

设$\{X_n, n\geqslant 1\}$是$\alpha$混合序列,$EX_n=0$.记

$$\sigma_n^2 = ES_n^2,\quad Z_n = S_n/\sigma_n,\quad F_n(x) = P(Z_n < x).$$

假设$1\leqslant h\leqslant n-1, 2\leqslant k\leqslant n-1$使$2kh+1<n$.令

$$Y_j = X_j/\sigma_n,$$

$$Z_j^{(0)} = X_j/\sigma_n = Y_j,\quad Z_j^{(l)} = \sum_{|p-j|\leqslant lh} Y_p,$$

(7.1.2) $$z_j^{(0)}=Z_n,\ z_j^{(l)}=Z_n-Z_j^{(l)},$$

$$\xi_j^{(l)}=\exp\{it(z_j^{(l-1)}-z_j^{(l)})\}-1, a_j^{(r-1)}=E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}\Big),$$

$$\eta_j^{(r)}=e^{-itZ_j^{(r)}}-1,\ f_n(t)=Ee^{itZ_n},$$

其中$j=1,2,\cdots,n;l=1,2,\cdots,r;r=2,\cdots,k$.由于

$$f_n'(t) = i\sum_{j=1}^n EY_je^{itZ_n} = i\sum_{j=1}^n EY_je^{itz_j^{(0)}}$$

且对$j=1,2,\cdots,n;r=2,\cdots,k$有

$$Ee^{itz_j^{(r)}} = E(\eta_j^{(r)}+1)f_n(t) + E[(\eta_j^{(r)} - E\eta_j^{(r)})e^{itZ_n}],$$

这样特征函数$f_n(t)$的导数有表示:

(7.1.3) $$\begin{aligned} f_n'(t) &= i\Big\{\sum_{j=1}^n a_j^{(1)}E(\eta_j^{(2)}+1) \\ &\quad + \sum_{r=3}^k\sum_{j=1}^n a_j^{(r-1)}E(\eta_j^{(r)}+1)\Big\}f_n(t) \\ &\quad + i\sum_{r=2}^k\sum_{j=1}^n\Big\{E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}e^{itz_j^{(r)}}\Big) - E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}\Big)Ee^{itz_j^{(r)}}\Big\} \\ &\quad + i\sum_{r=2}^k\sum_{j=1}^n a_j^{(r-1)}E\{(\eta_j^{(r)} - E\eta_j^{(r)})e^{itZ_n}\} \\ &\quad + i\sum_{j=1}^n EY_je^{itz_j^{(1)}} + i\sum_{j=1}^n EY_j\prod_{l=1}^k\xi_j^{(l)}e^{itz_j^{(k)}}. \end{aligned}$$

若在某些适当条件下,我们可给出(7.1.3)右边每一项的估计,那么我们可得微分方程

(7.1.4) $$f'_n(t) = (-t + a(t))f_n(t) + b(t).$$

解微分方程(7.1.4)，我们可得 $|f_n(t)-e^{-t^2/2}|$ 的估计. 最后，从(7.1.1)就得 $\Delta_n=\sup_x|F_n(x)-\Phi(x)|$ 的估计.

记 $s=2+\delta$. 假设

(7.1.5) $$d:=\max_{1\leqslant j\leqslant n}E|X_j|^{2+\delta}\quad 0<\delta\leqslant 1,$$

(7.1.6) $$\sigma_n^2:=ES_n^2\geqslant c_0 n\quad 0<c_0<\infty.$$

Sunklodas(1984)给出了下述定理.

**定理 7.1.1** 设 $\{X_n,n\geqslant 1\}$ 是 $\alpha$ 混合序列，$EX_n=0$ 且满足(7.1.5)，(7.1.6)和

$$\alpha(n)\leqslant Ke^{-\lambda n}\quad \lambda>0,K>0,$$

那么存在 $c_1=c_1(K,\delta)$，$c_2=c_2(K,\delta)$ 使对 $\lambda$，$\lambda_1\leqslant\lambda\leqslant\lambda_2$ 有

(7.1.7) $$\Delta_n\leqslant c_1\frac{d}{c_1\sigma_n^2}\left(\frac{\log(\sigma_n/c_0^{1/2})}{\lambda}\right)^{1+\delta}\quad n\geqslant 1,$$

其中

$$\lambda_1=c_2(\log(\sigma_n/c_0^{1/2}))^b/n,\quad b>2(1+\delta)/\delta;$$

$$\lambda_2=4(2+\delta)\delta^{-1}\log(\sigma_n/c_0^{1/2}).$$

**定理 7.1.2** 设 $\{X_n\}$ 如定理 7.1.1，但

$$\alpha(n)\leqslant Kn^{-\mu},\mu=2\beta(1+\delta)(2+\delta)/\delta^2,\beta>1,k>0.$$

那么存在 $C=C(k,\beta,\delta)$ 使对每一 $n\geqslant 1$ 有

(7.1.8) $$\Delta_n\leqslant Cdc_0^{-(1+\beta+\delta)/(1+\beta)}\sigma_n^{-(\beta-1)\delta/(1+\beta)}.$$

**注 7.1.1** 设 $\{X_n,n\geqslant 1\}$ 是强平稳 $\alpha$ 混合序列时，定理 7.1.1 就是 Tikhomirov(1980)的定理 2. 但 Tikhomirov(1980)的定理还指出若 $E|X_1|^{4+\gamma}<\infty$，$\gamma>0$，那么

(7.1.9) $$\Delta_n\leqslant cn^{-1/2}\log n.$$

定理的证明需要下述引理.

**引理 7.1.1** 我们有

(7.1.10) $$i\sum_{j=1}^n a_j^{(1)}=-t+\Theta 20(1+12\alpha)^{1/2}d^{2/s}\sigma_n(\alpha(h+1))^{(s-2)/2s}|t|/c_0^{3/2}+\Theta(2^{3-s}/(s-1))d(2h+1)^{s-1}|t|^{s-1}/(c_0\sigma_n^{s-2}),$$

其中 $\alpha = \sum_{j=1}^{n}(\alpha(j))^{(s-2)/s}, s = 2+\delta, 0<\delta\leqslant 1$，且常数 $\Theta, |\Theta| \leqslant 1$.

**证** 设 $\tau$ 是在集 $\{1,2,\cdots,n\}$ 上均匀分布的随机变量，与 $\{X_1, X_2,\cdots,X_n\}$ 独立. 容易看出对 $i=0,1,\cdots,k$ 和任何 $b\geqslant 1$ 有

$$(7.1.11)\qquad E|Z_\tau^{(l)}|^b \leqslant (2hl+1)^b\sum_{j=1}^{n}E|Y_j|^b/n.$$

由 Hölder 不等式和(7.1.11)，我们得

$$(7.1.12)\qquad \sum_{j=1}^{n}E(|Y_j||Z_j^{(1)}|^{s-1})\leqslant n\|Y_\tau\|_s\|Z_\tau^{(1)}\|_s^{s-1}$$

$$\leqslant (2h+1)^{s-1}\sum_{j=1}^{n}E|Y_j|^s.$$

记 $\hat{z}_j^{(l)}$ 是 $z_j^{(l)}$ 中 $p<j-lh$ 的那些 $Y_p$ 的和，$\tilde{z}_j^{(l)}$ 是 $z_j^{(l)}$ 中 $p>j+lh$ 的那些 $Y_p$ 的和. 这样 $z_j^{(l)}=\hat{z}_j^{(l)}+\tilde{z}_j^{(l)}$. 从引理 1.2.4 即得

$$\begin{aligned}|EY_j\hat{z}_j^{(l)}| &\leqslant 10(\alpha(h+1))^{\frac{1}{2}-\frac{1}{s}}\|Y_j\|_s\|\hat{z}_j^{(l)}\|_2\\ &\leqslant 10(1+12\alpha)^{1/2}d^{2/s}(\alpha(h+1))^{(s-2)/2s}/(c_0^{1/2}\sigma_n).\end{aligned}$$

对 $|EY_j\hat{z}_j^{(l)}|, j=1,2,\cdots,n$ 有同样的界. 由于 $\sum_{j=1}^{n}EY_jZ_n=1$，应用(7.1.6)我们有

$$\begin{aligned}(7.1.13)\qquad \sum_{j=1}^{n}EY_jz_j^{(1)} &= 1-\sum_{j=1}^{n}(EY_j\hat{z}_j^{(1)}+EY_j\tilde{z}_j^{(1)})\\ &= 1+\Theta 20(1+12\alpha)^{1/2}d^{2/s}\\ &\quad (\alpha(h+1))^{(s-2)/2s}\sigma_n/c_0^{3/2}.\end{aligned}$$

按 Taylor 公式有

$$i\sum_{j=1}^{n}a_j^{(1)} = -\sum_{j=1}^{n}EY_jZ_j^{(1)}t+\Theta(2^{3-s}/(s-1))$$

$$\sum_{j=1}^{n}E(|Y_j||Z_j^{(1)}|^{s-1})t^{s-1}.$$

把(7.1.12)和(7.1.13)代入并利用(7.1.6)式就得(7.1.10).

**引理 7.1.2** 对 $|t|\leqslant\sigma_n/(32d^{1/s}(h+1))=:T_1, j=1,2,\cdots,n$ 和 $r=3,4,\cdots,k$，我们有

(7.1.14) $$|a_j^{r-1}|\leqslant 4^r d^{3/s}(h+1)^2 t^2(1/2)^{4r}/\sigma_n^3 + c\frac{d^{1/s}}{\sigma_n}(\alpha(h+1))^{(s-2)/s}\left[r\left(\frac{1}{2}\right)^r + r4^r(\alpha(h+1))^{1/s}\right] =: a^{(r-1)}$$

和

(7.1.15) $$|a_j^{(1)}|\leqslant 2d^{2/s}(h+1)|t|/\sigma_n^2.$$

**证** 注意到 $a_j^{(r-1)}=E\{Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}\}$. 从

$$|a_j^{(1)}| = |EY_j\xi_j^{(1)}| \leqslant |t|E|X_j|\sum_{|p-j|\leqslant h}|X_p|/\sigma_n^2 \leqslant 2d^{2/s}(h+1)|t|/\sigma_n^2,$$

即得(7.1.15).

现在来证(7.1.14). 记

$$\xi_{j_1}^{(l)} = \exp\{it\sum_{p=j-lh+1}^{j-(l-1)h} Y_p\} - 1,$$

$$\xi_{j_2}^{(l)} = \exp\{it\sum_{p=j+(l-1)h+1}^{j+lh} Y_p\} - 1.$$

显然地

$$|\xi_j^{(l)}| \leqslant |\xi_{j_1}^{(l)}| + |\xi_{j_2}^{(l)}|.$$

由此即得

(7.1.16) $$|a_j^{(r-1)}|\leqslant E\left|Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}\right| \leqslant \sum_{k=1}^{r-1}\sum{}^{*} E\left|Y_j\prod_{\nu=1}^{k}\xi_{j_1}^{(l_\nu)}\prod_{\mu=k+1}^{r-1}\xi_{j_2}^{(l_\mu)}\right|,$$

其中和 $\sum^{*}$ 是对 $1\leqslant l_1<\cdots<l_k\leqslant r-1, 1\leqslant l_{k+1}<\cdots<l_{r-1}\leqslant r-1$, $l_\mu\neq l_\nu(\nu\neq\mu)$来求和的. 对(7.1.16)右边的每一项有

(7.1.17) $$E\left|Y_j\prod_{\nu=1}^{k}\xi_{j_1}^{(l_\nu)}\prod_{\mu=k+1}^{r-1}\xi_{j_2}^{(l_\mu)}\right| \leqslant \left(E\left|Y_j\prod_{\nu}\xi_{j_1}^{(l_\nu)}\prod_{\mu}\xi_{j_2}^{(l_\mu)}\right|^{\frac{2+\delta}{1+\delta}}\right)^{\frac{1+\delta}{2+\delta}}.$$

$$\cdot \left(E\left|\prod_{\nu}{}'\xi_{j_1}^{(l_\nu)}\prod_{\mu}{}''\xi_{j_2}^{(l_\mu)}\right|^{2+\delta}\right)^{\frac{1}{2+\delta}},$$

其中 $\prod'$ 是对所有偶数 $l$ 的乘积，$\prod''$ 是对所有奇数 $l$ 的乘积. 容易看出

(7.1.18) $$E\left|Y_j\prod_{\nu}{}'\xi_{j_1}^{(l_\nu)}\prod_{\mu}{}''\xi_{j_2}^{(l_\mu)}\right|^{\frac{2+\delta}{1+\delta}}$$

$$\leqslant (E|Y_j|^{2+\delta})^{\frac{1}{1+\delta}}\prod_{\nu}{}'E|\xi_{j_1}^{(l_\nu)}|^{\frac{2+\delta}{1+\delta}}\prod_{\mu}{}''E|\xi_{j_2}^{(l_\mu)}|^{\frac{2+\delta}{1+\delta}}$$

$$+ cr2^{r-1}(\alpha(h-1))^{\frac{\delta}{1+\delta}}(E|Y_j|^{2+\delta})^{\frac{1}{1+\delta}},$$

(7.1.19) $$E\left|\prod_{\nu}{}'\xi_{j_1}^{(l_\nu)}\prod_{\mu}{}''\xi_{j_2}^{(l_\mu)}\right|^{2+\delta}$$

$$\leqslant cr2^{r-1}\alpha(h+1)+\prod_{\nu}{}'E|\xi_{j_1}^{(l_\nu)}|^{2+\delta}\prod_{\mu}{}''E|\xi_{j_2}^{(l_\mu)}|^{2+\delta}$$

且

$$\max(E|\xi_{j_1}^{(l)}|^{2+\delta},E|\xi_{j_2}^{(l_\mu)}|^{2+\delta})\leqslant\left(\frac{|t|h_h}{\sigma_n}\right)^{2+\delta}\wedge 2^{2+\delta},$$

其中 $h_n=\max\limits_{1\leqslant i\leqslant h+1}\left\|\sum\limits_{j=1}^{i}X_j\right\|_{2+\delta}$. 从(7.1.17)—(7.1.19)即得

(7.1.20) $$E\left|Y_j\prod_{\nu}{}'\xi_{j_1}^{(l_\nu)}\prod_{\mu}{}''\xi_{j_2}^{(l_\mu)}\right|$$

$$\leqslant 2^r d^{2+r}(h+1)^4t^2(1/2)^{4r}/\sigma_n^4$$

$$+c\frac{d^{1+\delta}}{\sigma_n}(\alpha(h+1))^{\frac{s-2}{s}}\left[r\left(\frac{1}{2}\right)^{2r}+r2^r(\alpha(h+1))^{1/s}\right].$$

注意到(7.1.16)右边的项数不超过 $2^r$. 得证(7.1.14)成立.

在下面我们将运用下述不等式：对任何有限的 $p\geqslant 1$

(7.1.21) $$\sum_{r=1}^{\infty}r^p/e^r<\infty,$$

且对 $j=1,\cdots,n;r=2,\cdots,k$

(7.1.22) $$(E|\eta_j^{(r)}|^s)^{1/s}\leqslant|t|d^{1/s}(2rh+1)/\sigma_n.$$

假设 $1\leqslant h\leqslant n-1,2\leqslant k\leqslant n-1$ 且

(7.1.23) $$k^{3/2}4^k(\alpha(h+1))^{1/s}\leqslant 1.$$

**引理 7.1.3** 若(7.1.23)被满足且 $|t|\leqslant T_1$，那么有

$$(7.1.24)\quad \left|\sum_{j=1}^{n}a_j^{(1)}E\eta_j^{(2)}+\sum_{r=3}^{k}\sum_{j=1}^{n}a_k^{(r-1)}E(\eta_j^{(r)}+1)\right|$$

$$\leqslant\frac{c}{c_0}[d^{3/s}(h+1)^2t^2/\sigma_n+d^{1/s}\sigma_n(\alpha(h+1))^{(s-2)/s}],$$

$$(7.1.25)\quad \sum_{r=2}^{k}\left|\sum_{j=1}^{n}a_j^{(r-1)}E[(\eta_j^{(r)}-E\eta_j^{(r)})e^{itZ_n}]\right|$$

$$\leqslant\frac{c}{c_0^{1/2}}[(h+1)^{1/2}+\alpha^{1/2}][d^{3/s}(h+1)^2t^2/\sigma_n^2$$

$$+d^{1/s}(\alpha(h+1))^{(s-2)/s}].$$

又若附设 $k\geqslant(\log n)/8\log 2$，那么

$$(7.1.26)\quad \sum_{j=1}^{n}\left|E\left(Y_j\prod_{l=1}^{k}\xi_j^{(l)}e^{it\tau_j^{(k)}}\right)\right|$$

$$\leqslant cc_0^{1/2}d^{3/s}(h+1)^2t^2/\sigma_n^2$$

$$+cc_0^{-1}d^{1/s}\sigma_n(\alpha(h+1))^{(s-2)/s}.$$

**证** 由于

$$\left|\sum_{j=1}^{n}a_j^{(1)}E\eta_j^{(2)}+\sum_{r=3}^{k}\sum_{j=1}^{n}a_j^{(r-1)}E(\eta_j^{(r)}+1)\right|$$

$$\leqslant\sum_{j=1}^{n}|a_j^{(1)}|E|\eta_j^{(2)}|+\sum_{r=3}^{k}\sum_{j=1}^{n}|a_j^{(r-1)}|,$$

利用引理 7.1.2，(7.1.22)，(7.1.23)和(7.1.6)，即得(7.1.24).

由 $\mathrm{Cov}(\xi,\eta)=E(\xi-E\xi)\overline{(\eta-E\eta)}$，应用 Hölder 不等式和引理 1.2.4，我们得对 $r=2,3,\cdots,k$ 有

$$(7.1.27)\quad \left|\sum_{j=1}^{n}a_j^{(r-1)}E[(\eta_j^{(r)}-E\eta_j^{(r)})e^{itZ_n}]\right|$$

$$\leqslant\left\{\left(\sum_{j=1}^{n}\sum_{|p-j|\leqslant 2rh}+\sum_{j=1}^{n}\sum_{|p-j|>2rh}\right)a_j^{(r-1)}\overline{a_j^{(r-1)}}\mathrm{Cov}(\eta_j^{(r)},\eta_p^{(r)})\right\}^{1/2}$$

$$\leqslant\left\{\sum_{j=1}^{n}\sum_{|p-j|\leqslant 2rh}|a_j^{(r-1)}||a_p^{(r-1)}|\cdot\|\eta_j^{(r)}\|_2\|\eta_p^{(r)}\|_2\right\}^{1/2}$$

$$+\left\{24\sum_{j=1}^{n}\sum_{|p-j|>2rh}|a_j^{(r-1)}||a_p^{(r-1)}|\right.$$

$$\left.(\alpha(|p-j|-2rh))^{(s-2)/s}\|\eta_j^{(r)}\|_s\|\eta_p^{(r)}\|_s\right\}^{1/2}.$$

设 $r=3,4,\cdots,k$. 由于对 $j=1,2,\cdots,n$，$|\eta_j^{(r)}|\leqslant 2$ 且对 $|t|\leqslant T_1$，

$|a_j^{(r-1)}| \leqslant a^{(r-1)}$，又 $a^{(r-1)}$ 与 $j$ 无关，注意到(7.1.6)，我们得(7.1.27)的右边不超过

(7.1.28) $$2a^{(r-1)}\Big\{\sum_{j=1}^{n}\sum_{|p-j|\leqslant 2rh}1\Big\}^{1/2}+ca^{r-1}$$

$$\cdot\Big\{\sum_{j=1}^{n}\sum_{|p-j|>2rh}\Big\}(\alpha(|p-j|-2rh))^{(s-2)/s}\Big\}^{1/2}$$

$$\leqslant cc_0^{-1/2}\sigma_n[r^{1/2}(h+1)^{1/2}+\alpha^{1/2}]a^{(r-1)}.$$

对 $r=2$，我们同法进行之，只是按(7.1.22)估计 $\eta_j^{(2)}, j=1,2,\cdots,n$. 我们得(7.1.27)右边在 $r=2$ 时不超过

(7.1.29) $$cc_0^{-1/2}d^{3/s}(h+1)^{5/2}t^2/\sigma_n^2+cc_0^{-1/2}\alpha^{1/2}d^{3/s}(h+1)^2t^2/\sigma_n^2.$$

把(7.1.27)对 $r$ 从 2 至 $k$ 求和，注意到(7.1.28)，(7.1.29)和(7.1.23)，我们得证(7.1.25)成立.

最后，由于当 $k\geqslant(\log n)/(8\log 2)$ 时有 $(1/2)^{4k}\leqslant n^{-1/2}$，由引理 7.1.2 的证明，(7.1.6)和(7.1.23)即得(7.1.26). 引理 7.1.3 证毕.

**引理 7.1.4** 我们有

(7.1.30) $$\sum_{j=1}^{n}|EY_je^{itz_j^{(1)}}|\leqslant 32c_0^{-1}d^{1/s}\sigma_n(\alpha(h+1))^{(s-1)/s}.$$

若(7.1.23)被满足，那么

$$\sum_{r=2}^{k}\sum_{j=1}^{n}\Big|E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}e^{itz_j^{(r)}}\Big)-E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}\Big)Ee^{itz_j^{(r)}}\Big|$$

$$\leqslant 48c_0^{-1}d^{1/s}\sigma_n(\alpha(h+1))^{(s-1)/s}.$$

**证** 我们仅给出第二个不等式的证明，第一个不等式的证明是类似的. 从 $z_j^{(r)}$ 的定义，对所有 $r=2,3,\cdots,k$，$\hat{z}_j^{(r)}$ 和 $\check{z}_j^{(r)}$ 不同时为零. 不失一般性，假设 $\hat{z}_j^{(r)}\neq 0$ 且 $\check{z}_j^{(r)}\neq 0$. 那么按引理 1.2.1-1.2.5

$$\Big|E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}e^{itz_j^{(r)}}\Big)-E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}\Big)Ee^{itz_j^{(r)}}\Big|$$

$$\leqslant\Big|E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}e^{itz_j^{(r)}}\Big)-E\Big(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}e^{it\hat{z}_j^{(r)}}\Big)Ee^{it\check{z}_j^{(r)}}\Big|$$

$$+\left|E\left(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}e^{itz_j^{(r)}}\right)-E\left(Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}\right)Ee^{itz_j^{(r)}}\right|$$

$$+\left|Ee^{itz_j^{(r)}}Ee^{itz_j^{(r)}}-Ee^{itz_j^{(r)}}\right|\cdot\left|EY_j\prod_{l=1}^{r-1}\xi_j^{(l)}\right|$$

$$\leqslant 48d^{1/s}(\alpha(h+1))^{(s-1)/s}2^{r-1}/\sigma_n.$$

把上式对 $j=1,2,\cdots,n,r=2,3,\cdots,k$ 求和，并运用(7.1.6)和(7.1.23)得证引理 7.1.4 成立.

**定理 7.1.1 的证明.**

按照证明的基本方法，我们首先建立微分方程(7.1.4). 记

$$\alpha=\sum_{\tau=1}^{n}\alpha(\tau)^{\delta/(2+\delta)},$$

$$a_0=cd^{1/s}\sigma_n(\alpha(h+1))^{(s-2)/s}/c_0,$$

$$a_1=cd^{2/s}(1+\alpha)^{1/2}\sigma_n(\alpha(h+1))^{(s-2)/2s}/c_0^{3/2},$$

$$a_2=cd^{3/s}(h+1)^2/c_0\sigma_n,\quad a_3=cd(h+1)^{s-1}/c_0\sigma_n^{s-2},$$

$$b_0=cd^{1/s}((h+1)^{1/2}+\alpha^{1/2})(\alpha(h+1))^{(s-2)/s}/c_0^{1/2}+a_0,$$

$$b_2=cd^{3/s}((h+1)^{1/2}+\alpha^{1/2})(h+1)^2/c_0^{1/2}\sigma_n^2,$$

$$T_2=\min\{1/a_0,1/(6a_2),(1/6a_3)^{1/(s-2)}\}.$$

假设 $1\leqslant h\leqslant n-1, 2\leqslant k\leqslant n-1$ 使得

$$k\geqslant\frac{\log n}{8\log 2},\ k^{3/2}4^k(\alpha(h+1))^{1/s}\leqslant 1,\quad 2kh+1<n. \tag{7.1.31}$$

从(7.1.3)，引理 7.1.1，7.1.3 和 7.1.4 即得：当 $|t|\leqslant T_1$ 时

$$f'_n(t)=(-t+\Theta a(t))f_n(t)+\Theta b(t) \tag{7.1.32}$$

其中 $|\Theta|\leqslant 1$,

$$a(t)=a_0+a_1|t|+a_2t^2+a_3|t|^{s-1},\quad b(t)=b_0+b_2t^2.$$

其次，我们解线性微分方程(7.1.32)得

$$|f_n(t)-e^{-t^2/2}|\leqslant|x_0|e^{-t^2/2+|x_0|}+e^{-t^2/2}\int_0^{|t|}b(u)\exp\left\{\frac{u^2}{2}+\int_{|u|}^{|t|}a(v)dv\right\}du, \tag{7.1.33}$$

其中 $x_0=\int_0^t\Theta a(u)du$. 设 $0\leqslant u\leqslant t$. 那么对 $a_1\leqslant 1/6$ 和 $|t|\leqslant T_2$ 我们有

(7.1.34) $$\int_u^t a(v)dv \leqslant 1+\frac{t^2-u^2}{4},$$

且容易看到

(7.1.35) $$\int_0^{|t|} u^2\exp(u^2/4)du \leqslant 2|t|\exp(t^2/4),$$

(7.1.36) $$\int_0^{|t|} \exp(u^2/4)du \leqslant \min(4/|t|,|t|)\exp(t^2/4).$$

从(7.1.33)—(7.1.36)，对$|t|\leqslant\min(T_1,T_2)$和$a_1\leqslant 1/6$即得

(7.1.37) $$|f_n(t)-e^{-t^2/2}|\leqslant\left\{a_0|t|+\frac{a_1}{2}t^2+\frac{a_2}{3}|t|^3+\frac{a_3}{s}|t|^s\right\}e^{-\frac{t^2}{4}+1} + eb_0\{4/|t|\wedge t\}+2eb_2|t|.$$

由引理1.2.4我们有

$$\sigma_n^2\leqslant(1+12\alpha)d^{2/s}n,$$

因此$1\leqslant(1-12\alpha)^{1/2}K$，$K=d^{1/s}/c_0^{1/2}$。这样从(7.1.37)和(7.1.1)我们得

(7.1.38) $$\begin{aligned}\Delta_n\leqslant c\Big\{&K^s\frac{(h+1)^{s-1}}{\tilde\sigma_n^{s-2}}+K^3\frac{(h+1)^2}{\tilde\sigma_n}\\ &+K^2((h+1)^{1/2}+\alpha^{1/2})\frac{h+1}{\tilde\sigma_n}\\ &+K^2(1+\alpha)^{1/2}\tilde\sigma_n(\alpha(h+1))^{(s-2)/2s}\\ &+K((h+1)^{1/2}+\alpha^{1/2})(\alpha(h+1))^{(s-2)/s}\Big\},\end{aligned}$$

其中$\tilde\sigma_n^2=\sigma_n^2/c_0$。

最后，我们来证(7.1.7)。设$0<\lambda\leqslant\lambda_2=\dfrac{4(2+\delta)}{\delta}\log\tilde\sigma_n$且$h=\left[\dfrac{4(2+\delta)}{\lambda\delta}\log\tilde\sigma_n\right]$，$n\geqslant 1$，那么$\dfrac{4(2+\delta)}{\lambda\delta}\leqslant\dfrac{h+1}{\log\tilde\sigma_n}$。因此，由条件$\alpha(n)\leqslant ke^{-\lambda n}$，有

(7.1.39) $$\begin{aligned}\alpha^{1/2}&=\Big(\sum_{\tau=1}^{n}(\alpha(\tau))^{\delta/(2+\delta)}\Big)^{1/2}\\ &\leqslant\frac{1}{2}K^{\delta/2(2+\delta)}(h+1)^{1/2}/(\log\tilde\sigma_n)^{1/2}\\ &\leqslant C(K,\delta)(h+1)^{1/2}.\end{aligned}$$

这就是说

(7.1.40) $$1 \leqslant C(K,\delta)(h+1)^{1/2}K.$$

对所选的 $h$ 有

(7.1.41) $$(\alpha(h+1))^{\delta/2(2+\delta)} \leqslant K^{\delta/2(2+\delta)} e^{-\lambda\delta(h+1)/2(2+\delta)} \leqslant K^{\delta/2(2+\delta)} \tilde{\sigma}_n^{-2},$$

(7.1.42) $$(\alpha(h+1))^{\delta/(2+\delta)} \leqslant K^{\delta/(2+\delta)} \tilde{\sigma}_n^{-4}.$$

从(7.1.39)—(7.1.42)和(7.1.37),我们得

(7.1.43) $$\Delta_n \leqslant C(K,\delta)\left[K^{2+\delta}\frac{(h+1)^{1+\delta}}{\tilde{\sigma}_n^{\delta}} + K^3\frac{(h+1)^2}{\tilde{\sigma}_n}\right] \leqslant C(K,\delta)K^{2+\delta}\frac{(h+1)^{1+\delta}}{\tilde{\sigma}_n^{\delta}}.$$

余下来证是否存在一个 $k\in\{2,\cdots,n-1\}$ 使对于 $h=\left[\frac{4(2+\delta)}{\lambda\delta}\log\tilde{\sigma}_n\right]$, $\lambda_1\leqslant\lambda\leqslant\lambda_2$ 有

(7.1.44) $$k \geqslant \frac{\log n}{8\log 2},\ k^{3/2}4^k(\alpha(h+1))^{1/(2+\delta)} \leqslant 1,\ 2kh+1<n.$$

容易验证,对所有充分大的 $n(>n_0)$ 及对

$$k=\left[\frac{(4/\delta)\log\tilde{\sigma}_n-(\log K)/(2+\delta)}{(3/2)+\log 4}\right],$$

前两不等式成立,而第三个不等式不压缩参数 $\lambda$ 的变化区间.

观察(7.1.40),我们有

$$n^{-1} \leqslant C(K,\delta)K^{2+\delta}(h+1)^{1+\delta}/\tilde{\sigma}_n^{\delta}.$$

这就证明了定理 7.1.1 对所有 $n\leqslant n_0$ 也成立. 定理 7.1.1 证毕.

定理 7.1.2 的证明是类似的. 我们仅指出,此时只需在(7.1.31)—(7.1.33)中令 $h=[\tilde{\sigma}_n^{\alpha}]$,其中 $\alpha=2\delta/(\beta+1)(\delta+1)$,且

$$k=\left[\frac{(\alpha\mu/(2+\delta))\log\tilde{\sigma}_n-(\log K)/(2+\delta)}{(3/2)-\log 4}\right].$$

**注 7.1.2** Tikhomirov(1980)对具有指数速度衰减的 $\rho$ 混合序列给出了 Berry-Esseen 不等式. Zuparov(1991)改进他的方法,证明了下述定理,当 $E|X_1|^s<\infty$, $2<s<s_0(<1-\sqrt{3})$ 且 $\rho(\cdot)$ 的衰减速度条件被减弱时,对 $\Delta_n$ 有理想的阶.

**定理 7.1.3** 设 $\{X_n,n\geqslant 1\}$ 是强平稳 $\rho$ 混合序列, $EX_1=0$ 且

$$\rho(n) \leqslant Kn^{-\theta},\ \theta>0,\ K>0,$$

$$E|X_1|^s < \infty \quad 当\ 2 < s < s_0(\theta) = \frac{\theta - 1}{\theta} + \sqrt{\left(\frac{\theta-1}{\theta}\right)^2 + \frac{4+2\theta}{\theta}}.$$

若 $\sigma_n^2 = ES_n^2 \geqslant \tau n EX_1^2$. 那么存在常数 $C = C(s,\theta,K,\tau)$ 使得

$$\Delta_n \leqslant C(s,\theta,K,\tau)\beta_s/n^{(s-2)/2},$$

其中 $\beta_s = E|X_1|^s/(EX_1^2)^{s/2}$, $s_0 < 1 + \sqrt{3}$.

定理 7.1.3 的证明不在此叙述了.

## §7.2 $\varphi$混合序列弱收敛的速度

设 $\{X_n, n \geqslant 1\}$ 是随机变量序列, $EX_n = 0$, $\sigma_n^2 = ES_n^2$. 定义 $C[0,1]$的随机元的部分和过程如下:

$$W_n(t) = \frac{1}{\sigma_n}\left(S_{[nt]} + \frac{t\sigma_n^2 - \sigma_{[nt]}^2}{\sigma_{[nt]+1}^2 - \sigma_{[nt]}^2} X_{[nt]+1}\right), \tag{7.2.1}$$

$$\sigma_{[nt]}^2 \leqslant t\sigma_n^2 \leqslant \sigma_{[nt]+1}^2.$$

记 $P_n$ 是 $W_n$ 在 $C[0,1]$中的分布, 即 $P_n = P \cdot W_n^{-1}$. 弱不变原理成立, 即 $P_n \Rightarrow W$, 它等价于

$$L(P_n, W) \to 0, \tag{7.2.2}$$

其中 $L(P,Q)$是 Lévy-Prohorov 距离:

$$L(P,Q) = \inf\{\varepsilon : \forall B \in \mathscr{F}, P(B) \leqslant Q(B^\varepsilon) + \varepsilon, Q(B) \leqslant P(B^\varepsilon) + \varepsilon\}, \tag{7.2.3}$$

其中 $\mathscr{G}$ 是 $C[0,1]$的 Borel $\sigma$-域, $B^\varepsilon$ 是 Borel 集 $B$ 的 $\varepsilon$ 领域,

$$B^\varepsilon = \{y : y \in C[0,1], \exists\, z \in B, \|y - z\| < \varepsilon\},$$

$$\|y - z\| = \sup_{0 \leqslant t \leqslant 1} |y(t) - z(t)|.$$

记

$$\wedge_n(\varepsilon) = \sup_{B \in \mathscr{F}} |P(W_n \in B) - P(W \in B^\varepsilon)|,$$

显然

$$L(P_n, W) = \inf_\varepsilon(\varepsilon \vee \wedge_n(\varepsilon)). \tag{7.2.4}$$

那么我们可在一较大的概率空间上，在其上有一标准 Wiener 过程 $\{W(t),t\geq 0\}$，不改变 $\{W_n,n\geq 1\}$ 的分布重新定义过程 $\{W_n,n\geq 1\}$ 使得

$$\Lambda_n(\varepsilon)\leqslant P\{\|W_n-W\|\geqslant\varepsilon\}.$$

所以为估计弱收敛的速度，我们只需考虑下述不等式：

$$L(P_n,W)\leqslant\inf_{\varepsilon}(\varepsilon\vee P\{\|W_n-W\|\geqslant\varepsilon\}).$$

对于独立随机变量 $\{X_n,n\geq 1\}$，Prohorov(1965)给出了一个精确的估计：

$$L(P_n,W)=0(L_3^{1/4}\log^2 L_3),$$

其中 $L_3=\sum_{k=1}^{n}E|X_k|^3/\sigma_n^3$. 对于 $i.i.d$ 情形，Borovkov(1973)证明着：若 $E|X_1|^{2+\delta}<\infty,0<\delta\leqslant 1$，那么

(7.2.5) $$L(P_n,W)=0(L_{2+\delta}^{1/(3+\delta)})=0(n^{-\frac{\delta}{2(3+\delta)}}).$$

Borovkov(1973)又指出，若仅运用 skorohod 嵌入法，弱收敛速度(7.2.5)是不能被改进的. Vtev(1981)拓广这一估计于 $0<\delta<3$ 情形，证得当 $E|X_1|^{2+\delta}<\infty,0<\delta<3$ 时

(7.2.6) $$L(P_n,W)=O(n^{-\frac{\delta}{2(3+\delta)}}).$$

Utev(1984)对 $\varphi$ 混合序列在较强的条件下给出了如(7.2.6)同样的估计.

**定理 7.2.1** 设 $\{X_n,n\geq 1\}$ 是强平稳 $\varphi$ 混合序列，$EX_1=0$，$E|X_1|^{2+\delta}<\infty$，某 $\delta,0<\delta<3$. 假设

$$\varphi(n)\leqslant cn^{-g},$$

其中 $g>j(u)(j(u)-1),u=(12+5\delta)/(2(3-\delta)),j(u)=\min\{2k:2k\geq u,k\in\mathbb{N}\}$. 那么

$$L(P_n,W)\leqslant cn^{-\frac{\delta}{2(3+\delta)}}.$$

利用引理 2.2.10，陆传荣(1993)减弱了上述条件证明着下述定理.

**定理 7.2.2** 设 $\{X_n,n\geq 1\}$ 是 $\varphi$ 混合序列，$EX_n=0$，$A_0=\sup_n EX_n^2<\infty$，$A_\delta=\sup_n E|X_n|^{2+\delta}<\infty$，$0<\delta<3$. 假设

(i) 当 $n\to\infty$ 时 $\sigma_n^2\to\infty$,

(ii) $\varphi(n)\leqslant cn^{-g}, g\geqslant(3\delta+\varepsilon)/(2(3-\delta))\vee(2+\varepsilon)$,任给 $\varepsilon>0$.

那么我们有

$$L(P_n,W)=O(n^{-\frac{\delta}{2(3+\delta)}}). \tag{7.2.7}$$

**证** 由(i),$A_\delta<\infty$,引理 2.2.2 及引理 2.2.10,我们有

$$\begin{aligned}E\max_{1\leqslant i\leqslant n}|S_k(i)|^{2+\delta}&\leqslant c\{(c_3n)^{\frac{2+\delta}{2}}+E\max_{k\leqslant i\leqslant k+n}|X_i|^{2+\delta}\}\\&\leqslant cn^{1+\delta/2}.\end{aligned} \tag{7.2.8}$$

不失一般性,我们可设 $\sigma_n^2=c_0n, c_0=1$. 那么 $W_n$ 是以 $(k/n, S_k/\sqrt{n}), k=0,1,\cdots,n$ 为结点的随机折线. 记

$$X_i^{(1)}=X_iI(|X_i|\leqslant y\sqrt{n})-EX_iI(|X_i|\leqslant y\sqrt{n})$$
$$i=1,\cdots,n,$$

其中 $y=n^{-1/2}\left(\sum_{i=1}^{n}E|X_i|^{2+\delta}\right)^{1/(2+\delta)}\leqslant n^{-\frac{\delta}{2(2+\delta)}}A_\delta^{\frac{1}{2+\delta}}$. 随机变量 $X_i^{(1)}$ 具有一切有限阶矩.

设 $S_k^{(1)}=\sum_{i=1}^{k}X_i^{(1)}$. $W_n^{(1)}$ 是具有结点 $(k/n, S_k^{(1)}/\sqrt{n}), k=0,1,\cdots,n$,的随机折线. 对它应用引理 2.2.10 有

$$\begin{aligned}&P\{\|W_n^{(1)}-W_n\|\geqslant cn^{-\delta/(2(3+\delta))}\}\\&\leqslant P\left\{\max_{1\leqslant k\leqslant n}\left|\frac{1}{\sqrt{n}}(S_k-S_k^{(1)})\right|\geqslant cn^{-\delta/(2(3+\delta))}\right\}\\&\leqslant cn^{-3(2+\delta)/(2(3+\delta))}E\max_{1\leqslant k\leqslant n}|S_k-S_k^{(1)}|^{2+\delta}\\&\leqslant cn^{-3(2+\delta)/(2(3+\delta))}\Big[\big(\max_{1\leqslant k\leqslant n}E(|S_k-S_k^{(1)}|^2)\big)^{\frac{2+\delta}{2}}\\&\qquad+\sum_{i=1}^{n}E|X_i-X_i^{(1)}|^{2+\delta}\Big]\\&\leqslant cn^{-\delta/(2(3+\delta))}.\end{aligned} \tag{7.2.9}$$

令 $l=[n^{3/(3+\delta)-\eta}], m=[n^{\delta/(3+\delta)+\eta}], \eta>0$(下面取定),写 $n=lm+r$, $0\leqslant r<l$. 设 $W_n^{(2)}$ 是具有结点 $(kl/n, S_{kl}^{(1)}/\sqrt{n}), k=0,1,\cdots,m$ 和 $(1, S_n^{(1)}/\sqrt{n})$ 的随机折线. 我们有

$$
\begin{aligned}
&P\{\|W_n^{(2)}-W_n^{(1)}\|\geqslant b\}\\
&\leqslant \sum_{k=0}^{m} P\left\{\max_{1\leqslant i\leqslant l}\left|S_{kl+i}^{(1)}-S_{kl}^{(1)}\right|\geqslant b\sqrt{n}\right\}\\
&\leqslant c\sum_{k=0}^{m} b^{-u}n^{-u/2}E\max_{1\leqslant i\leqslant l}\left|S_{kl}^{(1)}(i)\right|^{u}\\
&\leqslant cmb^{-u}n^{-u/2}l^{u/2}\\
&= cb^{-u}\left(\frac{n}{l}\right)^{1-u/2}
\end{aligned}
$$

其中应用了引理 2.2.10. 若我们取 $b=cn^{-\delta/2(3+\delta)}$, $u\geqslant 2+3\delta/[(3+\delta)\eta]$, 那么

$$
(n/l)^{1-u/2}\leqslant n^{-\delta(u+1)/(2(3+\delta))}.
$$

所以我们有

$$
(7.2.10)\qquad P\left\{\|W_n^{(2)}-W_n^{(1)}\|\geqslant cn^{-\frac{\delta}{2(3+\delta)}}\right\}\leqslant cn^{-\frac{\delta}{2(3+\delta)}}.
$$

令 $h=[n^{\theta}]$, $\theta>0$(下面取定). 记 $d=l-h$,

$$
\xi_k^{(1)}=\sum_{i=1}^{d}X_{(k-1)l+i}^{(1)},
$$

$W_n^{(3)}$ 是具有结点$\left(kl/n,\sum_{j=1}^{k}\xi_j^{(1)}/\sqrt{n}\right)$, $k=1,\cdots,m+1$, 的随机折线, 我们有

$$
\begin{aligned}
&P\{\|W_n^{(3)}-W_n^{(2)}\|\geqslant b\}\\
&=P\left\{\max_{1\leqslant k\leqslant m+1}\left|S_{kl}^{(1)}-\sum_{j=1}^{k}\sum_{i=1}^{d}X_{(j-1)l+i}^{(1)}\right|\geqslant b\sqrt{n}\right\}\\
&=P\left\{\max_{1\leqslant k\leqslant m+1}\left|\sum_{j=1}^{k}\sum_{i=1}^{h}X_{(j-1)l+d+i}^{(1)}\right|\geqslant b\sqrt{n}\right\}\\
&\leqslant cb^{-u}n^{-u/2}\left\{\left[\max_{1\leqslant k\leqslant m+1}E\left(\sum_{j=1}^{k}\sum_{i=1}^{h}X_{(j-1)l+d+i}^{(1)}\right)^{2}\right]^{u/2}\right.\\
&\qquad\left.+\sum_{j=1}^{m+1}E\left|\sum_{i=1}^{h}X_{(j-1)l+d+i}^{(1)}\right|^{u}\right\}
\end{aligned}
$$

此时由引理 2.2.4, 括号中前一项不超过 $c(mh)^{u/2}$, 后一和式不超过 $mh^{u/2}=0((mh)^{u/2})$, 所以

$$P\{\|W_n^{(3)}-W_n^{(2)}\|\geqslant b\}\leqslant cb^{-u}(h/l)^{u/2}.$$

若我们取 $b=cn^{-\delta/(2(3+\delta))}$和

(7.2.11) $$0<\theta\leqslant\frac{3-\delta}{3+\delta}-\frac{4\eta}{3},$$

那么$(h/l)^{u/2}\leqslant cn^{-\delta(u+1)/(2(3+\delta))}$,所以有

(7.2.12) $$P\{\|W_n^{(3)}-W_n^{(2)}\|\geqslant cn^{-\delta/(2(3+\delta))}\}\leqslant cn^{-\delta/(2(3+\delta))}.$$

余下的证明仿照 Utev(1984)中证明,即由 Berkes 和 Philipp (1979)的定理 2,我们有

(7.2.13) $$P\{\|W_n^{(4)}-W_n^{(3)}\|\geqslant c\varphi(h)n/l\}\leqslant c\varphi(h)n/l,$$

其中$W_n^{(4)}$是具有结点$\left(kl/n,\sum_{j=1}^{k}\xi_j^{(2)}/\sqrt{n}\right),k=0,1,\cdots,m+1$的随机折线,$\{\xi_j^{(2)}\}$是相互独立与$\xi_j^{(1)}$有相同分布的随机变量. 又由 Sakhanenko(1981),我们有

(7.2.14) $$P\{\|W_n^{(5)}-W_n^{(4)}\|\geqslant q_u^{\frac{1}{u+1}}\}\leqslant q_u^{\frac{1}{u+1}}\quad u>2,$$

其中$W_n^{(5)}$是具有结点$\left(kl/n,\sum_{j=1}^{k}Y_j/\sqrt{n}\right),k=0,1,\cdots,m+1$的随机折线,$\{Y_j\}$是相互独立的正态随机变量,$EY_j=0,\mathrm{Var}Y_j=\mathrm{Var}\xi_j^{(i)},j=1,2,\cdots,m+1$ 且

$$q_u=c\left(\sum_{i=1}^{m+1}E|\xi_i^{(1)}|^u\right)\Big/\left(\sum_{i=1}^{m+1}\mathrm{Var}\xi_i^{(1)}\right)^{u/2}.$$

易见由(7.2.8)我们有

(7.2.15) $$q_u=cn^{-u/2}\sum_{i=1}^{m}E|\xi_i^{(1)}|^u\leqslant cn^{-u/2}ml^{u/2}$$
$$\leqslant c(n/l)^{1-u/2}.$$

所以当 $u\geqslant2+3\delta/((3+\delta)\eta)$时

$$q_u^{\frac{1}{u+1}}\leqslant cn^{-\delta/(2(3+\delta))}.$$

最后,如我们取

$$\theta\geqslant\frac{2(3-\delta)}{(3+\epsilon)\delta}\left(\frac{3\delta}{2(3+\delta)}+\eta\right),$$

那么

$$(7.2.16)\qquad \varphi(h)n/l = O(n^{\varepsilon\theta-\frac{\delta}{3+\delta}-\eta}) \leqslant cn^{-\frac{\delta}{2(3+\delta)}}.$$

结合(7.2.11)我们有

$$\frac{2(3-\delta)}{(3+\varepsilon)\delta}\left(\frac{3\delta}{2(3-\delta)}+\eta\right) \leqslant \theta < \frac{3-\delta}{3+\delta}-\frac{4\eta}{3},$$

这就是说，我们需令 $\eta$ 满足

$$0 < \eta < \frac{3-\delta}{3+\delta}\cdot\frac{3\varepsilon\delta}{2(9-3\delta+2\varepsilon\delta)}.$$

结合(7.2.9)—(7.2.16)我们得

$$P\{\|W_n - W\| \geqslant cn^{-\delta/(2(3+\delta))}\} \leqslant cn^{-\delta/(2(3+\delta))}.$$

由于 $L(W_n, W) \leqslant \inf(\varepsilon + P\{\|W_n - W\| \geqslant \varepsilon\})$，得证(7.2.7)成立。定理 7.2.2 证毕.

**注 7.2.1** Utev(1984)讨论了 $\alpha$ 混合序列弱收敛的速度，得到下述结果.

设 $\{X_n, n\geqslant 1\}$ 是强平稳 $\alpha$ 混合序列，$EX_1=0$，$EX_1^2=1$ 且 $E|X_1|^{2+\delta}<\infty$，$0<\delta<3$. 假设

$$0 < \sigma^2 := EX_1^2 + 2\sum_{n=1}^{\infty} EX_1X_{n+1} < \infty$$

且

$$\alpha(n) \leqslant cn^{-g} \quad n \geqslant 1,$$

其中

$$g > \max(\delta_1^{-1} j(u)(j(u)+\delta_1),$$

$$\delta_1^{-1} j(2(2+\delta))(j(2(2+\delta))+\delta_1)),$$

$$u = \max\left\{2+(2+\delta), \frac{12+5\delta}{2(3-\delta)}\right\},$$

$$\delta_1 = \frac{\delta}{(2+\delta)(3+\delta)},$$

$j(u)$ 如定理 7.2.1 中定义. 那么

$$L(P_n, W) \leqslant cA_\delta n^{-\delta/(2(3+\delta))},$$

其中 $c=c(A, g, \delta)$，$A_\delta = E|X_1|^{2+\delta}$.

特别，若 $\delta=1$ 即假设 $E|X_1|^3<\infty$，$\alpha(n)=O(n^{-(438+\varepsilon)})$，那么 $L(P_n, W)=O(n^{-1/8})$.

Gorodezkii(1983)也讨论过类似的结果.

**注 7.2.2** Yoshihara(1979)讨论了强平稳绝对正则序列弱收敛的速度,在 $EX_1=0, E|X_1|^{4+\delta}<\infty$(某 $\delta>0$), $\sum_{n=1}^{\infty} n\beta(n)^{\delta/(4+\delta)}<\infty$ 下. 证得 $L(P_n, W)=O(n^{-1/5}(\log n)^{1/2})$.

# 第Ⅲ部分　几乎必然收敛和强逼近

在这一部分中，我们研究混合相依序列的部分和的几乎必然(a. s.)收敛性和强逼近. 60年代以来，不少作者讨论了这类序列部分和的 a. s. 收敛性. 例如，Iosifescu 和 Theodorescu(1969)建立了 $\varphi^-$ 混合序列的 0-1 律，强大数律和随机级数的收敛性等结果. 近 10 年来，各种混合序列的完全收敛性已经被邵启满等人深入地研究，从而可导出有关强大数律的深刻的结果. 我们将在第八章中讨论这些内容.

混合序列 $\{X_n, n\geq 1\}$ 的部分和 $S_n=\sum_{k=1}^{n} X_k$ 用一个 Wiener 过程强逼近已经被 Philipp 和 Stout(1975)等研究过，邵启满和陆传荣全面地改进了这些结果. 混合序列的部分和的增量的极限性质是由林正炎等人得到的. 这些深刻的定理将在第九章和第十章中分别叙述和论证.

在第十一章中，我们将介绍带有集指标的混合序列的强逼近理论.

# 第八章　大数律和完全收敛性

在本章的前两节中，我们将介绍 $\varphi$-混合序列的 Borel-Cantelli 引理、弱大数律和强大数律. 自从许宝騄和 Robbins(1947)提出完全收敛性的概念以来，许多人研究过这一课题. 弱相依序列的完全收敛性也曾被一些作者讨论过. 在 8.3-8.5 节，我们将分别介绍 $\varphi$-混合序列、$\rho$-混合序列和 $\alpha$-混合序列的完全收敛性. 最后，在 8.6 节，我们将讨论 Prohorov 提出的三个问题在 $\rho$-混合情形时的有关结论.

## §8.1　弱大数律

Chow 和 Teicher(1978)对于独立随机变量组列 $\{X_{nj}, 1\leqslant j\leqslant k_n\}$ 给出了服从弱大数律的充要条件. 杜午初将这一结果推广到了 $\varphi$-混合情形，证明了下列定理. 记 $S_n=\sum_{j=1}^{k_n} X_{nj}$.

**定理 8.1.1**　设 $\{X_{nj}, 1\leqslant j\leqslant k_n\to\infty\}$ 是行内 $\varphi$-混合的随机变量组列，存在实数 $A_n$ 使得

(8.1.1) $$S_n-A_n\xrightarrow{p}0,\ n\to\infty$$

且对任意的 $\epsilon>0$

(8.1.2) $$P\{\max_{1\leqslant j\leqslant k_n}|X_{nj}|\geqslant\epsilon\}\to 0,\ n\to\infty$$

当且仅当对任意的 $\epsilon>0$

(8.1.3) $$\sum_{j=1}^{k_n}P\{|X_{nj}|\geqslant\epsilon\}\to 0,\ n\to\infty$$

和

(8.1.4) $$\mathrm{Var}\Big(\sum_{j=1}^{k_n}X_{nj}I(|X_{nj}|<1)\Big)\to 0,\ n\to\infty.$$

此时可令

$$A_n = \sum_{j=1}^{k_n} EX_{nj} I(|X_{nj}| < 1) + 0(1).$$

定理 8.1.1 的证明需要下述引理.

**引理 8.1.1** 设$\{X_n, n \geqslant 1\}$是 $\varphi$-混合序列. 用 $m(Y)$表示随机变量 $Y$ 的中位数，那么存在 $c$，$0<c<1/2$，对一切充分大的 $k$，

$$(8.1.5) \quad P\left\{\max_{1\leqslant j\leqslant n} |S_j - m(S_j - S_{n+k-1} + S_j(k-1))| \geqslant \varepsilon\right\}$$
$$\leqslant \frac{2}{1-2c} P\left\{(k-1)\max_{1\leqslant j\leqslant n+k-1} |X_j| + |S_{n+k-1}| \geqslant \varepsilon\right\},$$

其中 $S_j(k) = \sum_{i=j+1}^{j+k} X_i$.

**证** 取 $k$ 如此大，使得 $C := \varphi(k) < 1/2$. 令

$$T = \begin{cases} \min\limits_{1\leqslant j\leqslant n} \{j : S_j - m(S_j - S_{n+k-1} + S_j(k-1)) \geqslant \varepsilon\}, \\ n+1, \text{当上面的集合是空集时}. \end{cases}$$

记 $B_j = \{S_j - S_{n+k-1} + S_j(k-1) \leqslant m(S_j - S_{n+k-1} + S_j(k-1))\}$，$1 \leqslant j \leqslant n$. 显然 $P(B_j) \geqslant 1/2$. 因为 $\{T = j\} \in \mathscr{F}_1^j = \sigma\{X_1, \cdots, X_j\}$，$B_j \in \mathscr{F}_{j+k}^{n+k-1}$，故有

$$\bigcup_{j=1}^{n} \{B_j \cap \{T = j\}\} \subset \{S_{n+k-1} - S_j(k-1) \geqslant \varepsilon\}$$
$$\subset \left\{(k-1)\max_{1\leqslant j\leqslant n+k-1} |X_j| + S_{n+k-1} \geqslant \varepsilon\right\}.$$

由 $\varphi$-混合性可得

$$(8.1.6) \quad P\left\{(k-1)\max_{1\leqslant j\leqslant n+k-1} |X_j| + S_{n+j-1} \geqslant \varepsilon\right\}$$
$$\geqslant \sum_{j=1}^{n} P\{B_j, T = j\}$$
$$\geqslant \sum_{j=1}^{n} (P(B_j) - \varphi(k)) P\{T = j\}$$
$$\geqslant \left(\frac{1}{2} - C\right) P\{1 \leqslant T \leqslant n\}$$
$$= \left(\frac{1}{2} - C\right) P\left\{\max_{1\leqslant j\leqslant n} (S_j - m(S_j - S_{n+k-1}\right.$$

$$+ S_i(k-1)) \geqslant \varepsilon \Big\}.$$

如用 $-X_j$ 代替 $X_j$ 可得另一个不等式,从而得证(8.1.5)

记

$$B = \Big\{ \max_{0 \leqslant l \leqslant n} |S_l(n-l)| > \varepsilon \Big\},$$

$$B_j = \Big\{ |S_j(n-j)| > \varepsilon, \max_{j < k \leqslant n} |S_k(n-k)| \leqslant \varepsilon \Big\}, 0 \leqslant j < n,$$

$$B_n = \{ |X_n| > \varepsilon \}, F_j = \bigcup_{l=j}^{n} B_l, 0 \leqslant j \leqslant n, F_{n+1} = \varphi.$$

**引理 8.1.2** 设 $\{X_n, n \geqslant 1\}$ 是一 $\varphi$-混合序列, $|X_j| \leqslant r < \infty$, $j=1,2,\cdots,n$. 且 $M>0$ 使得 $\varphi(M)<1/6$. 则有

$$P(B) \geqslant \frac{(1-6\varphi(M))E\Big\{\max\limits_{1 \leqslant l \leqslant n} S_l^2\Big\} - 4\varepsilon^2}{3E\Big\{\max\limits_{1 \leqslant l \leqslant n} S_l^2\Big\} + 5M^2r^2 + 12(\varepsilon+r)^2 - 2\varepsilon^2}. \tag{8.1.7}$$

**证** 对 $1 \leqslant i < k \leqslant n$, 我们有

$$\max_{1 \leqslant i \leqslant k} |S_i| \leqslant |S_k| + \max_{0 \leqslant i \leqslant k} |S_i(k-i)| \leqslant 2 \max_{0 \leqslant i \leqslant k} |S_i(k-i)|. \tag{8.1.8}$$

如果 $0<M<n, 0<i<n-M$, 则有

$$\max_{i < l \leqslant j} |S_l| \leqslant \max_{1 \leqslant l \leqslant i} |S_l| + Mr, i < j \leqslant i+M, \tag{8.1.9}$$

$$\max_{i+M < l \leqslant j} |S_l| \leqslant |S_i| + Mr + \max_{i+M < l \leqslant j} |S_{i+M}(l-i-M)|. \tag{8.1.10}$$

从(8.1.8)-(8.1.10)推得:对 $j=1,2,\cdots,n$,

$$\begin{aligned}
\max_{1 \leqslant l \leqslant j} |S_l| &= \max\Big\{ \max_{1 \leqslant k \leqslant i+M} |S_k|, \max_{i+M < l \leqslant j} |S_l| \Big\} \\
&\leqslant \max_{1 \leqslant k \leqslant i} |S_k| + Mr + \max_{i+M < l \leqslant j} |S_{i+M}(l-i-M)| \\
&\leqslant \max_{1 \leqslant k \leqslant i} |S_k| + Mr + 2 \max_{i+M \leqslant l \leqslant j} |S_l(j-l)| \\
&\leqslant \max_{1 \leqslant k \leqslant i} |S_k| + Mr + 4 \max_{i+M \leqslant l \leqslant n} |S_l(n-l)|.
\end{aligned}$$

因此对给定的 $i, 1 \leqslant i \leqslant n-M$, 我们有

$$\max_{1 \leqslant l \leqslant n} |S_l| \leqslant \max_{1 \leqslant k \leqslant i} |S_k| + Mr + 4 \max_{i+M \leqslant l \leqslant n} |S_l(n-l)|. \tag{8.1.11}$$

注意到对 $j=1,2,\cdots,n$,

$$\max_{j\leqslant l\leqslant n}|S_l(n-l)|I_{B_j}\leqslant |X_{j+1}|I_{B_j}+\max_{j+1\leqslant l\leqslant n}|S_l(n-l)|I_{B_j}$$
$$\leqslant (r+\varepsilon)I_{B_j}$$

以及$(a+b+c)^2\leqslant 3(a^2+b^2+c^2)$，由(8.1.11)可得

(8.1.12)
$$E\max_{1\leqslant l\leqslant n}S_l^2I_B=\sum_{j=0}^{n}E\max_{1\leqslant l\leqslant n}S_l^2I_{B_j}$$
$$\leqslant \sum_{j=0}^{M}E\max_{1\leqslant l\leqslant n}S_l^2I_{B_j}+\sum_{j=M+1}^{n}E\Big\{\max_{1\leqslant k\leqslant j-M}|S_k|$$
$$+Mr+4\max_{j\leqslant l\leqslant n}|S_l(n-l)|\Big\}^2I_{B_j}$$
$$\leqslant \sum_{j=0}^{M}E\max_{1\leqslant l\leqslant n}S_l^2I_{B_j}+3\sum_{j=M+1}^{n}E\max_{1\leqslant k\leqslant j-M}S_k^2I_{B_j}$$
$$+(3M^2r^2+12(r+\varepsilon)^2)P(B).$$

记 $Y_j=\max_{1\leqslant k\leqslant j-M}|S_k|$，$M<j\leqslant n$. 由引理 1.2.10 并注意到 $P(F_i)$对 $i$ 是不增的，我们有

$$\sum_{j=M+1}^{n}E\max_{1\leqslant k\leqslant j-M}S_k^2I_{B_j}=\sum_{j=M+1}^{n}EY_j^2I_{B_j}$$
$$=\sum_{j=M+1}^{n}(EY_j^2I_{F_j}-EY_j^2I_{F_{j+1}})$$
$$=EY_{M+1}^2I_{F_{M-1}}+\sum_{j=M+2}^{n}E(Y_j^2-Y_{j-1}^2)I_{F_j}$$
$$\leqslant EY_{M+1}^2\{P(F_{M+1})+2\varphi(M)\}$$
$$+\sum_{j=M+2}^{n}E(Y_j^2-Y_{j-1}^2)\{P(F_j)+2\varphi(M)\}$$
$$\leqslant EY_n^2\{P(F_{M+1})+2\varphi(M)\}.$$

此外

$$\sum_{j=0}^{M}E\max_{1\leqslant l\leqslant n}S_l^2I_{B_j}\leqslant (Mr+\varepsilon)^2P(B)$$
$$\leqslant 2(M^2r^2+\varepsilon^2)P(B).$$

将这些结果代入(8.1.12)得到

(8.1.13)
$$E\max_{1\leqslant l\leqslant n}S_l^2I_B$$

$$\leqslant 3E\max_{1\leqslant l\leqslant n} S_l^2(P(B)+2\varphi(M))$$

$$+(5M^2r^2+12(r+\varepsilon)^2+2\varepsilon^2)P(B).$$

另一方面

$$E\max_{1\leqslant l\leqslant n} S_l^2 I_B = E\max_{1\leqslant l\leqslant n} S_l^2 - E\max_{1\leqslant l\leqslant n} S_l^2 I_{B^C} \tag{8.1.14}$$

$$\geqslant E\max_{1\leqslant l\leqslant n} S_l^2 - 4\varepsilon^2(1-P(B)).$$

将它与(8.1.13)相结合即得(8.1.7).

**引理 8.1.3** $\{X_n, n\geqslant 1\}$如引理 8.1.2 所设. 如果 $S_n$ a.s. 收敛于一随机变量,那么 $E\max\limits_{1\leqslant k\leqslant n} S_k^2$ 收敛.

**证** 记 $D_{n,n+m}=\bigcup\limits_{i=n}^{n+m}\{|S_i(n+m-i)|>\varepsilon\}$. 如果 $\varphi(M)<1/6$,由引理 8.1.2,

$$P(D_{n,n+m})\geqslant \frac{(1-6\varphi(M))E\left\{\max\limits_{n\leqslant l\leqslant n+m} S_l^2\right\}-4\varepsilon^2}{3E\left\{\max\limits_{n\leqslant l\leqslant n+m} S_l^2\right\}+5M^2r^2+12(\varepsilon+r)^2-2\varepsilon^2}.$$

利用反证法. 如果 $E\max\limits_{1\leqslant k\leqslant n} S_k^2$ 发散,则对任意的 $n\geqslant 1$,当 $m$ 充分大时

$$E\max_{n\leqslant l\leqslant n+m} S_l^2 \geqslant E\max_{1\leqslant l\leqslant n+m} S_l^2 - E\max_{1\leqslant l\leqslant n} S_l^2 \geqslant \delta_0>0. \tag{8.1.15}$$

另一方面,由 $S_n \xrightarrow{\text{a.s.}} S$,当 $n,m\to\infty$时

$$P(D_{n,n+m})=P\left\{\max_{n\leqslant i\leqslant n+m}|S_i(n-m-i)|>\varepsilon\right\}$$

$$\leqslant P\left\{\max_{0\leqslant l\leqslant m}|S_n(l)|>\varepsilon/2\right\}+P\{|S_n(m)|>\varepsilon/2\}\to 0.$$

与(8.1.15)矛盾,从而得证引理.

**引理 8.1.4** 记 $B=\{\max\limits_{1\leqslant j\leqslant n}|X_j|\geqslant\varepsilon\}$, $T_n=\sum\limits_{i=1}^{n} I(|X_i|\geqslant\varepsilon)$. 如果 $\varphi(M)<1/2$,则有

$$P(B)\geqslant\frac{(1-2\varphi(M))ET_n}{ET_n+M+1}. \tag{8.1.16}$$

**证** 记

$T_0=0, T_n(m)=T_{n+m}-T_n, B_n=\{|X_n|\geqslant\varepsilon\}$,

$$B_j = \left\{\max_{j<i\leqslant n}|X_i|<\varepsilon,\ |X_j|\geqslant\varepsilon\right\},\ F_j=\bigcup_{i=j}^{n}B_i,\ j=1,\cdots,n,\ F_{n+1}=\phi.$$

我们有

$$\begin{aligned}
ET_nI_B &= \sum_{j=1}^{M}ET_nI(B_j) + \sum_{j=M+1}^{n}ET_nI(B_j)\\
&= \sum_{j=1}^{M}E(T_{j-1} + T_{j-1}(n-j+1))I(B_j)\\
&\quad + \sum_{j=M+1}^{n}E(T_{j-M-1} + T_{j-M-1}(M)\\
&\quad + T_{j-1}(n-j-1))I(B_j)\\
&\leqslant \sum_{j=1}^{M}MP(B_j) + \sum_{j=1}^{M}P(B_j)\\
&\quad + \sum_{j=M+1}^{n}(ET_{j-M-1}I(B_j) + MP(B_j) + P(B_j))\\
&\leqslant \sum_{j=M+1}^{n}ET_{j-M-1}I(B_j) + (M+1)P(B).
\end{aligned}$$

由引理 1.2.10,

$$\begin{aligned}
&\sum_{j=M+1}^{n}ET_{j-M-1}I(B_j) = \sum_{j=M+1}^{n}E(T_{j-M-1}(I(F_j) - I(F_{j+1})))\\
&= ET_1I(F_{M+2}) + \sum_{j=M+2}^{n}EI(|X_{j-M-1}|\geqslant\varepsilon)I(F_j)\\
&\leqslant \left\{\sum_{i=1}^{n-M}EI(|X_i|\geqslant\varepsilon)\right\}(P(F_{M+1}) + 2\varphi(M))\\
&\leqslant ET_n(P(B) + 2\varphi(M)).
\end{aligned}$$

因此

$$ET_n = ET_nI(B) \leqslant ET_n(P(B) + 2\varphi(M)) + (M+1)P(B).$$

由此即得(8.1.16).

**定理 8.1.1 的证明.**

先证充分性. 显然(8.1.3)推出(8.1.2). 为证(8.1.1),记

$$Y_{nj} = X_{nj}I(|X_{nj}|<1),\ V_n = \sum_{j=1}^{k_n}Y_{nj}.$$

由(8.1.4), $V_n - EV_n \xrightarrow{p} 0$. 又由(8.1.3)可得

$$(8.1.17) \quad P(V_n \neq S_n) \leqslant \sum_{j=1}^{k_n} P\{|X_{nj}| \geqslant 1\} = o(1),$$

也即 $S_n - EV_n \xrightarrow{p} 0$. 因此(8.1.1)对于 $A_n = EV_n + o(1)$成立.

再证必要性. 从引理 8.1.4,我们有

$$P\left\{\max_{1\leqslant j\leqslant k_n} |X_{nj}| \geqslant \varepsilon\right\} \geqslant \frac{(1-2\varphi(M))E\sum_{j=1}^{k_n} I(|X_{nj}| \geqslant \varepsilon)}{E\sum_{j=1}^{k_n} I(|X_{nj}| \geqslant \varepsilon) + M + 1}.$$

因此由(8.1.2)可推得(8.1.3). 为了证明(8.1.4),令 $Y'_{nj}$是与 $Y_{nj}$ 独立同分布的随机变量,记

$$Y^*_{nj} = Y_{nj} - Y'_{nj}, \quad V^*_n = \sum_{j=1}^{k_n} Y^*_{nj}.$$

从(8.1.17)和(8.1.1)可得 $V_n - A_n \xrightarrow{p} 0$. 因此 $V^*_n \xrightarrow{p} 0$. 记 $V^*_{nk} = \sum_{j=1}^{k} Y^*_{nj}$. 由引理 8.1.1,

$$(8.1.18) \quad P\left\{\max_{1\leqslant k\leqslant k_n - M} |V^*_{nk}| \geqslant \varepsilon\right\} \leqslant \frac{2}{1-2c} P\left\{(M-1)\max_{1\leqslant j\leqslant k_n} |Y^*_{nj}| + |V^*_n| \geqslant \varepsilon\right\} = o(1).$$

因此利用(8.1.2)和(8.1.18)我们有

$$\begin{aligned} &P\left\{\max_{1\leqslant k\leqslant k_n} |V^*_{nk}| \geqslant \varepsilon\right\} \\ &\leqslant P\left\{\max_{1\leqslant k\leqslant k_n - M} |V^*_{nk}| \geqslant \varepsilon\right\} + \sum_{j=k_n-M+1}^{k_n} P\{|Y^*_{nj}| \geqslant \varepsilon/M\} = o(1). \end{aligned}$$

于是,仿照引理 8.1.3 的证明,可得 Var $V^*_n$ 趋于 0. 因此 Var $V_n$ = Var $V^*_n$/2 也收敛于 0. 定理 8.1.1 证毕.

## §8.2 强大数律

首先我们证明关于 $\varphi$-混合序列的 Borel-Cantelli 引理.

**定理 8.2.1** 设$\{X_n,n\geqslant 1\}$是一$\varphi$-混合序列，$\{\mathscr{F}_n=\sigma(X_n),n\geqslant 1\}$是一$\sigma$-域序列. 那么对任意的$A_n\in\mathscr{F}_n$，$P\{A_n,i.o.\}=0$当且仅当

$$\sum_{n=1}^{\infty}P(A_n)<\infty; \tag{8.2.1}$$

此外，从$\sum\limits_{n=1}^{\infty}P(A_n)=\infty$可推出$P\{A_n,i.o.\}=1$.

**证** 对第一部分，我们只需证明：从$P\{A_n,i.o.\}=0$可推出$\sum\limits_{n=1}^{\infty}P(A_n)<\infty$. 不然的话，假设$\sum\limits_{n=1}^{\infty}P(A_n)=\infty$. 由此，存在整数$l>0$使得$\delta:=\varphi(l)<1$，且存在$j$，$0\leqslant j\leqslant l-1$，使得$\sum\limits_{n=1}^{\infty}P(A_{nl+j})=\infty$. 我们有

$$\begin{aligned}P\Big\{\bigcup_{i=m}^{n}A_{il+j}\Big\}=&P\{A_{nl+j}\}+P\{A_{nl+j}^c\cap A_{(n-1)l+j}\}+\cdots\\&+P\{A_{nl+j}^c\cap\cdots\cap A_{(m+1)l+j}^c\cap A_{ml+j}\}.\end{aligned}$$

由$\varphi$-混合性得

$$\begin{aligned}P\Big\{\bigcup_{i=m}^{n}A_{il+j}\Big\}\geqslant&P\{A_{nl+j}\}+P\{A_{(n-1)l+j}\}(P\{A_{nl+j}^c\}-\delta)+\cdots\\&+P\{A_{ml+j}\}\{P\{A_{(m+1)l+j}^c\cap\cdots\cap A_{nl+j}^c\}-\delta\}\\\geqslant&\sum_{i=m}^{n}P\{A_{il+j}\}\{P\{(\bigcup_{i=m+1}^{n}A_{il+j})^c\}-\delta\}.\end{aligned}$$

因此从$\sum\limits_{n=1}^{\infty}P\{A_{nl+j}\}=\infty$必有$P\{(\bigcup\limits_{i=m}^{\infty}A_{il+j})^c\}\leqslant\delta$，也就是说$P\{\bigcup\limits_{i=m}^{\infty}A_{il+j}\}\geqslant 1-\delta$. 因此$P\{A_n,i.o.\}\geqslant 1-\delta$，与$P\{A_n,i.o.\}=0$矛盾.

第二部分，假设$\sum\limits_{n=1}^{\infty}P(A_n)=\infty$. 从上面的讨论可知

$$P\{\limsup_n A_n\}=P\{A_n,i.o.\}\neq 0.$$

显然$\limsup\limits_n A_n\in\bigcap\limits_{n=1}^{\infty}\bigvee\limits_{i=n}^{\infty}\mathscr{F}_i$. 我们来证明$\bigcap\limits_{n=1}^{\infty}\bigvee\limits_{i=n}^{\infty}\mathscr{F}_i$是一个平凡$\sigma$-

域.不然的话,存在一集合 $B\in\bigcap_{n=1}^{\infty}\bigvee_{i=n}^{\infty}\mathscr{F}_i$,$0<P(B)<1$. 由 $\varphi$ 混合性,对任意的 $A\in\bigvee_{i=1}^{n}\mathscr{F}_i$ 有

$$|P(AB)-P(A)P(B)|<\eta P(A), \tag{8.2.2}$$

其中 $\eta<1/2$. 容易验证,满足(8.2.2)的集类包含 $\sigma$-域 $\bigvee_{i=1}^{\infty}\mathscr{F}_i$ 所以若取 $A=B$ 就得

$$P(B)-P(B)^2\leqslant\eta P(B),$$

从而推得 $P(B)>1/2$. 另一方面,我们又有 $B\in\bigcap_{n=1}^{\infty}\bigvee_{i=n}^{\infty}\mathscr{F}_i$. 引出矛盾. 这就证明了 $P\{A_n,i.o.\}=1$.

**推论8.2.1** 设 $\{X_n,n\geqslant1\}$ 是同分布的 $\varphi$-混合序列. 记 $S_n=\sum_{k=1}^{n}X_k$. 如果 $S_n/n\to b$ a.s. ,其中 $b$ 是一有限常数,那么 $E|X_1|<\infty$.

**证** 从 $S_n/n\to b$ a.s. 我们有

$$\frac{X_n}{n}=\frac{S_n}{n}-\frac{S_{n-1}}{n-1}\cdot\frac{n-1}{n}\to0\text{ a.s.}$$

因此对任意的 $\varepsilon>0$, $P\{|X_n/n|\geqslant\varepsilon,i.o.\}=0$. 由定理8.2.1,

$$E|X_1|\leqslant\sum_{n=0}^{\infty}P\{|X_1|\geqslant n\}<\infty.$$

对于同分布随机变量的 $\varphi$-混合序列,我们有下列 Marcinkiewicz 强大数律.

**定理8.2.2** 设 $\{X_n,n\geqslant1\}$ 是同分布的 $\varphi$-混合($\rho$-混合)序列. 对于某 $1\leqslant r<2$, $E|X_1|^r<\infty$ 且满足 $\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty\left(\sum_{n=1}^{\infty}\rho(2^n)<\infty\right)$. 那么

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-EX_i)=0(n^{-(1-1/r)})\quad\text{a.s.} \tag{8.2.3}$$

这个定理可从后面的推论8.3.4(推论8.4.2)直接推出,因此证明从略.

**推论8.2.2** 设$\{X_n,n\geq 1\}$是一同分布的$\varphi$-混合序列，满足$\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$. 那么$S_n/n\to b$ a.s. 当且仅当$E|X_1|<\infty$且$b=EX_1$.

**注8.2.1** 对于满足条件$\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$的$\varphi$-混合序列(未必同分布)，如果存在一个随机变量$X$使得对任意的$x>0$，$P\{|X_n|\geq x\}\leq P\{|X|\geq x\}$且对某个$1\leq r<2$，$E|X|^r<\infty$，薛留根(1994)证明定理8.2.2的结论仍然成立.

**注8.2.2** Iosifescu 和 Theodorescu(1969)讨论了$\varphi$-混合序列级数的收敛性. 例如，他们证明了：在某些条件下$\sum X_n$的a.s.收敛性和依概率收敛性是等价的. 他们还证明了下列强大数律：如果$a_n\uparrow\infty$，$\sum_{n=1}^{\infty}\mathrm{Var}\ X_n/a_n^2<\infty$且$\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$，那么

$$\frac{1}{a_n}(S_n-ES_n)\to 0\quad \text{a.s.}$$

**注8.2.3** 陈希孺和吴月华(1989)对$\alpha$-混合序列证明了一个强大数律：假设$\sup_n E|X_n|^p<\infty$(某个$p>1$)且

$$\alpha(n)=\begin{cases}O(n^{-\frac{p}{2p-2}-\varepsilon}) & 若\ 1<p<2,\\ O(n^{-\frac{2}{p}-\varepsilon}) & 若\ p\geq 2.\end{cases}$$

那么$(S_n-ES_n)/n=0(1)$ a.s.

## §8.3 $\varphi$混合序列的完全收敛性

完全收敛性的概念是由许宝騄和 Robbins(1947)提出的. 他们证明：如果$\{X_n,n\geq 1\}$是一$i.i.d.$随机变量序列，$EX_1=0$，$EX_1^2<\infty$，那么对任意的$\varepsilon>0$

$$\sum_{n=1}^{\infty}P\{|S_n|\geq\varepsilon n\}<\infty.$$

Baum 和 Katz(1965)进一步证明：$EX_1=0$且对$r>1,1\leq t<2$，$E$

$|X_1|^{rt}<\infty$当且仅当

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| \geqslant \varepsilon n^{1/t}\} < \infty.$$

白志东和苏淳(1985)得到如下结果：

**定理8.3.1** 设$\{X_n, n\geqslant 1\}$是一个$i.i.d.$随机变量序列，$r>1$，$0<t<2$，$h(x)$当$x\to\infty$时是缓变的，那么下列结论等价：

(i) $E|X_1|^{rt}h(|X_1|^t)<\infty$，

(ii) 对任意的$\varepsilon>0$，$\sum\limits_{n=1}^{\infty} n^{r-2}h(n)P\left\{|S_n-nb|\geqslant \varepsilon n^{1/t}\right\}<\infty$，

(iii) 对任意的$\varepsilon>0$，$\sum\limits_{n=1}^{\infty} n^{r-2}h(n)P\left\{\sup\limits_{k\geqslant n}|S_k-kb|/k^{1/t}\geqslant\varepsilon\right\}<\infty$，其中$b=EX_1$(若$1\leqslant t<2$)或0(若$0<t<1$).

70年代以来，一些学者研究了混合序列的完全收敛性.相应于$i.i.d.$序列的最佳结果是邵启满(1988a)给出的.

设$l(x)$和$\beta(x)$是两个正偶函数，满足条件：

(8.3.1) 对$x>0$，$l(x)$，$\beta(x)/x^\theta$(某$\theta>0$)和$x^2/\beta(x)$都是单调不减的.

邵启满(1988a)证明了下列定理.记$\alpha(x)=inv\beta(x)$.

**定理8.3.2** 设$\{X_n, n\geqslant 1\}$是同分布的$\varphi$-混合序列，$EX_1=0$且

$$E\beta(X_1)l(\beta(X_1)) < \infty. \tag{8.3.2}$$

如果满足下列条件之一：

a) $\beta(x)l(\beta(x))/x\uparrow$，$\beta(x)l(\beta(x))/x^2\downarrow$且存在$r\geqslant 2$使得

$$\sum_{n=1}^{\infty}\frac{1}{nl^r(n)}<\infty \tag{8.3.3}$$

和

$$\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\leqslant\frac{13}{42}\log l(n); \tag{8.3.4}$$

b) $\beta(x)l(\beta(x))/x\uparrow$，$\beta(x)l(\beta(x))/x^2\downarrow$且当$n\to\infty$时

$$\sum_{j=n}^{\infty}\frac{l(j)}{\alpha^2(j)}=0\left(\frac{nl(n)}{\alpha^2(n)}\right) \tag{8.3.5}$$

(8.3.6) $$\sum_{i=1}^{\infty}\varphi^{1/2}(2^i)<\infty;$$

c) $\beta(x)l(\beta(x))/x^2\uparrow$ 且存在 $q_1>q_2>2$ 使得 $\beta(x)l(\beta(x))/xq_2$ $\downarrow$,

(8.3.7) $$\sum_{n=1}^{\infty}\frac{l(n)}{n}\left(\frac{n^{1/2}}{a(n)}\right)^{q_1}\exp\left\{3q_1\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\right\}<\infty.$$

那么,对任意 $\varepsilon>0$,

(8.3.8) $$\sum_{n=1}^{\infty}\frac{l(n)}{n}P\left\{\max_{1\le i\le n}\beta(S_i)\geqslant\varepsilon n\right\}<\infty.$$

**定理8.3.3** 假设 $l(x)$ 是严格单调的. 设 $\{X_n,n\geqslant1\}$ 是同分布的 $\varphi$-混合序列. 如果(8.3.8)被满足,那么(8.3.2)成立且对任意的 $\varepsilon>0$

(8.3.9) $$\sum_{n=1}^{\infty}\frac{l(n)-l([n/2])}{n}P\left\{\sup_{i\geqslant n}\frac{\beta(S_i)}{i}>\varepsilon\right\}<\infty.$$

**注8.3.1** 如果 $l(n)=O(l(n)-l([n/2]))$,那么(8.3.8)等价于(8.3.9).

在定理8.3.2和8.3.3中令 $l(n)=n^{r-1}$, $r>1$, $\beta(n)=n^t$, $1\leqslant t<2$,我们有

**推论8.3.1** 设 $\{X_n,n\geqslant1\}$ 是同分布的 $\varphi$-混合序列. 下列结论是等价的:

(i) $E|X_1|^{rt}<\infty$, $EX_1=0$;

(ii) 对任意的 $\varepsilon>0$, $\sum_{n=1}^{\infty}n^{r-2}P\left\{\max_{1\le i\le n}|S_i|>\varepsilon n^{1/t}\right\}<\infty$;

(iii) 对任意的 $\varepsilon>0$, $\sum_{n=1}^{\infty}n^{r-2}P\left\{\sup_{k\geqslant n}|S_k|/k^{1/t}>\varepsilon\right\}<\infty$.

如果选取 $l(n)=\log n$, $\beta(n)=n^t$, $1\leqslant t<2$,则有

**推论8.3.2** 设 $\{X_n,n\geqslant1\}$ 是同分布的 $\varphi$-混合序列且对充分大的 $n$, $\varphi(n)\leqslant\frac{1}{49}(\log n)^{-2}$. 那么下列结论等价:

(i) $E|X_1|^t\log(1+|X_1|)<\infty$, $EX_1=0$;

(ii) 对任意的 $\varepsilon>0$, $\sum_{n=1}^{\infty}\frac{\log n}{n}P\left\{\max_{1\le i\le n}|S_i|\geqslant\varepsilon n^{1/t}\right\}<\infty$.

**推论8.3.3** 设$\{X_n,n\geqslant1\}$是同分布的$\varphi$-混合序列，$EX_1=0$，$E|X_1|^t<\infty$，$1\leqslant t<2$. 假设 $\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$，则对任意的$\varepsilon>0$，

$$\sum_{n=1}^{\infty}\frac{1}{n}P\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant\varepsilon n^{1/t}\}<\infty.$$

**注8.3.2** 推论8.3.1的结论可以加强如下：定理8.3.1的结论对于同分布的$\varphi$混合序列也成立，也就是说，如果$h(x)>0$是一缓变函数，那么下列结论等价：

(i)′ $E|X_1|^{rt}h(|X_1|^t)<\infty$，$EX_1=0$；

(ii)′ 对任意的$\varepsilon>0$，$\sum_{n=1}^{\infty}n^{r-2}h(n)P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant\varepsilon n^{1/t}\right\}<\infty$；

(iii)′ 对任意的$\varepsilon>0$，$\sum_{n=1}^{\infty}n^{r-2}h(n)P\left\{\max_{k\geqslant n}|S_k|/k^{1/t}\geqslant\varepsilon\right\}<\infty$.

其中$r>1$，$1<t<2$.

推论8.3.2也能作类似的加强.

**定理8.3.2的证明.**

容易看出从(8.3.1)可推出

$$\alpha(x)/x^{1/2}\uparrow,\alpha(x)/x^{1/\theta}\downarrow. \tag{8.3.10}$$

记

$$X_{nj}=X_jI\{|X_j|\leqslant\alpha(n)\},S_{ni}=\sum_{j=1}^{i}X_{nj}.$$

我们有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{l(n)}{n}P\{\max_{i\leqslant n}\beta(S_i)>\varepsilon n\}&\leqslant\sum_{n=1}^{\infty}\frac{l(n)}{n}P\{\max_{i\leqslant n}|X_i|>\alpha(n)\}\\&\quad+\sum_{n=1}^{\infty}\frac{l(n)}{n}P\{\max_{i\leqslant n}|S_{ni}|>\alpha(\varepsilon n)\}\\&=:I_1+I_2.\end{aligned}\tag{8.3.11}$$

易知

$$I_1\leqslant\sum_{n=1}^{\infty}l(n)P\{|X_i|\geqslant\alpha(n)\}\tag{8.3.12}$$

$$= \sum_{n=1}^{\infty} l(n) \sum_{j=n}^{\infty} P\{j \leqslant \beta(X_1) < j+1\}$$

$$\leqslant \sum_{j=1}^{\infty} j l(j) P\{j \leqslant \beta(X_1) < j+1\}$$

$$\leqslant E\beta(X_1) l(\beta(X_1)) < \infty.$$

现在来估计 $I_2$. 不失一般性可设$0<\varepsilon<1/2$. 由(8.3.10)

$$n|EX_1 I\{|X_1| \leqslant \alpha(n)\}| \leqslant nE|X_1| I\{|X_1| > \alpha(n)\}$$

$$\leqslant \frac{\alpha(n)}{l(n)} E\beta(X_1) l(\beta(X_1)) I\{|X_1| > \alpha(n)\}$$

$$\leqslant \frac{\alpha(\varepsilon n)}{\varepsilon^{1/\theta} l(1)} E\beta(X_1) l(\beta(X_1)) I\{|X_1| > \alpha(n)\}.$$

注意到 $l(1)>0$和$\lim\limits_{n\to\infty} E\beta(X_1) l(\beta(X_1)) I\{|X_1|>\alpha(n)\}=0$,所以当 $n$ 充分大时有

$$n|EX_1 I\{|X_1| \leqslant \alpha(n)\}| \leqslant \frac{1}{2}\alpha(\varepsilon n).$$

因此

$$I_2 \leqslant c \sum_{n=1}^{\infty} \frac{l(n)}{n} P\left\{\max_{i\leqslant n} |S_{ni} - ES_{ni}| > \frac{1}{2}\alpha(\varepsilon n)\right\}. \tag{8.3.13}$$

由引理2.2.2和引理2.2.10,对给定的 $q\geqslant 2$(将在后面指定). 存在仅依赖于 $q$ 的常数 $C_q=C(q)$使得

$$\begin{aligned} &P\left\{\max_{1\leqslant i\leqslant n} |S_{ni} - ES_{ni}| > \frac{1}{2}\alpha(\varepsilon n)\right\} \\ &\leqslant C_q(\alpha(\varepsilon n))^{-q}\Big((n \exp\Big\{3\sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i)\Big\} \\ &\quad EX_1^2 I\{|X_1| \leqslant \alpha(n)\})^{q/2} \\ &\quad + nE|X_1|^q I\{|X_1| \leqslant \alpha(n)\}\Big). \end{aligned} \tag{8.3.14}$$

如果条件 a)被满足,令 $q=28(r+1)$. 由(8.3.10),(8.3.13)和(8.3.14)我们有

$$I_2 \leqslant c \sum_{n=1}^{\infty} \frac{l(n)}{n} \left\{ \frac{n}{\alpha^2(n)} \exp\Big\{3\sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i)\Big\} \right. \tag{8.3.15}$$

$$EX_1^2 I\{|X_1| \leqslant \alpha(n)\}\Big)^{q/2}$$

$$+ c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^q(n)} E|X_1|^q I\{|X_1| \leqslant \alpha(n)\}$$

$$=: I_3 + I_4.$$

由(8.3.4)

$$I_3 \leqslant c\sum_{n=1}^{\infty} \frac{\exp\left\{(3q/2)\sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i)\right\}}{nl(n)^{(q-2)/2}} (E\beta(X_1)l(\beta(X_1)))^{q/2} \tag{8.3.16}$$

$$\leqslant c\sum_{n=1}^{\infty} \frac{1}{nl(n)^{q/2\delta-1}}$$

$$\leqslant c\sum_{n=1}^{\infty} \frac{1}{nl(n)^r} < \infty.$$

因为从 $\beta(x)l(\beta(x))/x^2 \downarrow$ 可推出 $xl(x)/\alpha^2(x) \downarrow$，故由(8.3.10)得

$$\sum_{j=n}^{\infty} \frac{l(j)}{\alpha^q(j)} = \sum_{j=n}^{\infty} \frac{jl(j)}{\alpha^2(j)} \cdot \frac{j^{(q-2)/2}}{\alpha^{q-2}(j)j^{q/2}} \tag{8.3.17}$$

$$\leqslant \frac{n^{q/2}l(n)}{\alpha^q(n)} \sum_{j=n}^{\infty} \frac{1}{j^{q/2}}$$

$$= O\left(\frac{nl(n)}{\alpha^q(n)}\right).$$

因此

$$I_4 \leqslant c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^q(n)} \sum_{j=0}^{n} E|X_1|^q I\{j \leqslant \beta(X_1) < j+1\} \tag{8.3.18}$$

$$\leqslant c\sum_{j=0}^{\infty}\left(\sum_{n=j}^{\infty} \frac{l(n)}{\alpha^q(n)}\right) E|X_1|^q I\{j \leqslant \beta(X_1) < j+1\}$$

$$\leqslant c\sum_{j=1}^{\infty} \frac{jl(j)}{\alpha^q(j)} E|X_1|^q I\{j \leqslant \beta(X_1) < j+1\}$$

$$\leqslant cE\beta(X_1)l(\beta(X_1)) < \infty.$$

从(8.3.15)，(8.3.16)和(8.3.18)得证 $I_2 < \infty$，也就是说，定理8.3.2在条件 a)下成立.

当条件 b)满足时令 $q=2$. 由类似的讨论也有 $I_2<\infty$. 如果条件 c)被满足，令 $q=q_1$. 因此从(8.3.7)易知 $I_3<\infty$. 由于从 $\beta(x)l(\beta(x))/x^{q_1}\downarrow$ 可推出 $xl(x)/\alpha(x)^{q_1}\downarrow$，故由(8.3.10)得

$$\sum_{j=n}^{\infty}\frac{l(j)}{\alpha^{q_1}(j)}\leqslant\frac{n^{1+(q_1-q_2)/2}l(n)}{\alpha^{q_1}(n)}\sum_{j=n}^{\infty}j^{-1-(q_1-q_2)/2}=O\left(\frac{nl(n)}{\alpha^{q_1}(n)}\right).$$

因此 $I_4<\infty$. 从而 $I_2<\infty$. 这就完成了定理8.3.2的证明.

**定理8.3.3的证明.**

首先我们证明从(8.3.8)可推出(8.3.9). 记 $d_n=\sum_{j=2^n}^{2^{n+1}-1}l(j)$. 我们有

(8.3.19)
$$\sum_{n=2}^{\infty}\frac{l(n)-l([n/2])}{n}P\left\{\sup_{i\geqslant n}\frac{\beta(S_i)}{i}\geqslant\epsilon\right\}$$
$$\leqslant\sum_{j=1}^{\infty}\sum_{n=2^j}^{2^{j+1}-1}\frac{l(n)-l([n/2])}{2^j}P\left\{\sup_{i\geqslant 2^j}\frac{\beta(S_i)}{i}\geqslant\epsilon\right\}$$
$$=\sum_{j=1}^{\infty}2^{-j}\left(\sum_{n=2^j}^{2^{j+1}-1}l(n)-2\sum_{n=2^{j-1}}^{2^j-1}l(n)\right)P\left\{\sup_{i\geqslant 2^j}\frac{\beta(S_i)}{i}\geqslant\epsilon\right\}$$
$$\leqslant\sum_{j=1}^{\infty}2^{-j}(d_j-2d_{j-1})\sum_{k=j}^{\infty}P\left\{\max_{2^k\leqslant i<2^{k+1}}\beta(S_i)>\epsilon 2^k\right\}$$
$$\leqslant\sum_{k=1}^{\infty}2^{-k}d_kP\left\{\max_{2^k\leqslant i<2^{k+1}}\beta(S_i)\geqslant\epsilon 2^k\right\}$$
$$\leqslant\sum_{k=1}^{\infty}4\sum_{2^{k+1}\leqslant n<2^{k+2}-1}\frac{l(n)}{n}P\left\{\max_{i<2^{k+1}}\beta(S_i)\geqslant\epsilon 2^k\right\}$$
$$\leqslant 4\sum_{n=1}^{\infty}\frac{l(n)}{n}P\left\{\max_{i\leqslant n}\beta(S_i)>\frac{\epsilon n}{2}\right\}<\infty.$$

这就证明了(8.3.9)

现在我们来证明(8.3.2)成立. 由(8.3.19)，对任意 $\epsilon>0$ 成立着

(8.3.20)
$$\sum_{k=1}^{\infty}l(2^k)P\left\{\max_{i\leqslant 2^k}|X_i|\geqslant\alpha(\epsilon 2^k)\right\}<\infty.$$

因此，当 $k\to\infty$ 时

(8.3.21)
$$P\left\{\max_{i\leqslant 2^k}|X_i|\geqslant\alpha(\epsilon 2^k)\right\}\to 0.$$

这样，再注意到当 $p\to\infty$ 时 $\varphi(p)\to 0$，就存在 $k_0$ 和 $p_0$，使对任意的 $k\geqslant k_0$，

$$\varphi(p_0)+P\left\{\max_{i\leqslant 2^k}|X_i|\geqslant \alpha(\varepsilon 2^k)\right\}<1/2.$$

于是，如同 Lai(1977)中的(3.2)，我们有

$$P\left\{\max_{1\leqslant i\leqslant 2^k}|X_i|\geqslant \alpha(\varepsilon 2^k)\right\}\geqslant \frac{2^k}{2p_0}P\{|X_1|\geqslant \alpha(\varepsilon 2^k)\}.$$

因此

$$(8.3.22)\qquad \sum_{k=1}^{\infty}l(2^k)2^kP\{|X_1|\geqslant \alpha(\varepsilon 2^k)\}$$

$$\leqslant c\sum_{k=1}^{\infty}l(2^k)P\left\{\max_{1\leqslant i\leqslant 2^k}|X_i|\geqslant \alpha(\varepsilon 2^k)\right\}<\infty.$$

另一方面

$$E\beta(X_1)l(\beta(X_1))\leqslant l(1)+\sum_{j=1}^{\infty}2^jl(2^j)P\{2^{j-1}<\beta(X_1)\leqslant 2^j\}$$

$$\leqslant l(1)+\sum_{j=1}^{\infty}2^jl(2^j)P\{|X_1|\geqslant \alpha(2^{j-1})\}.$$

从(8.3.22)即得 $E\beta(X_1)l(\beta(X_1))<\infty$. 因此(8.3.2)成立. 定理8.3.3证毕.

## §8.4 ρ混合序列的完全收敛性

对于 ρ-混合序列的完全收敛性，并不能完全达到 φ-混合情形时的结论. 但是某些理想的充分条件已被给出.

**定义8.4.1** 函数 $l(x)>0(x>0)$ 称为是拟单调不减的，如果

$$\lim_{x\to\infty}\sup\sup_{0<t\leqslant x}l(t)/l(x)<\infty;$$

类似地，称它是拟单调不增的，如果

$$\lim_{t\to\infty}\sup\ \sup_{t\geqslant x}l(t)/l(x)<\infty.$$

本节中，总设 $l(x)$ 是一个正的偶的拟单调不减的函数，$\beta(x)$ 是一个正的偶函数，使得对于某个 $\theta>0$，$\beta(x)/x^\theta$ 和 $x^2/\beta(x)$ 都是单调不减的. 记 $\alpha(x)=\mathrm{inv}\beta(x)$.

邵启满(1989c)证明了下列定理.

**定理8.4.1** 设 $\beta(x)l(\beta(x))$ 和 $x^{2-\varepsilon_0}/\beta(x)l(\beta(x))$(某个 $0<\varepsilon_0<1$)都是拟单调不减函数，$\{X_n,n\geq 1\}$ 是同分布的 $\rho$-混合序列，$EX_1=0$，$E\beta(X_1)l(\beta(X_1))<\infty$，满足

(8.4.1) $$\sum_{n=1}^{\infty}\rho(2^n)<\infty,$$

那么对任意的 $\varepsilon>0$

(8.4.2) $$\sum_{n=1}^{\infty}\frac{l(n)}{n}P\left\{\max_{1\leq i\leq n}\beta(S_i)\geq \varepsilon n\right\}<\infty.$$

**定理8.4.2** 假设存在 $q\geq 2$，$q_0\geq 2$ 和 $\delta>0$ 使得 $x^q/\beta(x)l(\beta(x))$，$\beta(x)l(\beta(x))/x^2$ 是拟单调不减函数且 $l(x)\geq x^\delta(x>0)$，

(8.4.3) $$\sum_{n=1}^{\infty}\frac{l(n)}{n}\left(\frac{n^{1/2}}{\alpha(n)}\right)^{q_0}<\infty.$$

又设 $\{X_n,n\geq 1\}$ 是同分布的 $\rho$-混合序列，$EX_1=0$，$E\beta(X_1)l(\beta(X_1))<\infty$，且对某 $r>q$

(8.4.4) $$\sum_{n=1}^{\infty}\rho^{2/r}(2^n)<\infty,$$

那么(8.4.2)成立.

**推论8.4.1** 设 $\{X_n,n\geq 1\}$ 是同分布的 $\rho$-混合序列，$EX_1=0$，$E|X_1|^p h(|X_1|^{1/\alpha})<\infty$，其中 $p\geq 1$，$p\alpha>1$，$\alpha>1/2$，$h(x)>0$ 是一缓变函数. 那么，如果

(8.4.5) $$\sum_{n=1}^{\infty}\rho^{2/r}(n)<\infty$$

其中 $r=2$，若 $1\leq p<2$；$r>p$，若 $p\geq 2$，则对任意的 $\varepsilon>0$

(8.4.6) $$\sum_{n=1}^{\infty}n^{p\alpha-2}h(n)P\{\max_{1\leq i\leq n}|S_i|\geq \varepsilon n^\alpha\}<\infty.$$

由此即可推得下列 Marcinkiewicz-Zygmund 大数律：

**推论8.4.2** 设 $\{X_n,n\geq 1\}$ 是同分布的 $\rho$-混合序列，$EX_1=0$，$E|X_1|^p<\infty(1\leq p<2)$，且满足

$$\sum_{n=1}^{\infty}\rho(2^n)<\infty,$$

则有

$$\lim_{n\to\infty} S_n/n^{1/p} = 0 \quad \text{a.s.}$$

**推论8.4.3** 设$\{X_n, n\geqslant 1\}$是同分布的$\rho$-混合序列,$EX_1=0$,$E|X_1|^p h(|X_1|^p)<\infty(1\leqslant p<2)$,其中$h(x)$是单调不减的缓变函数.又设

$$\sum_{n=1}^{\infty}\rho(2^n)<\infty.$$

那么对任意的$\varepsilon>0$,

$$\sum_{n=1}^{\infty}\frac{h(n)}{n}P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant \varepsilon n^{1/p}\right\}<\infty.$$

如果推论8.4.1中的条件$E|X_1|^p h(|X_1|^{1/\alpha})<\infty$代之以对某个$\delta>0$,$E|X_1|^{p+\delta}<\infty$,那么条件(8.4.5)可以略去.此外,孔繁超和张明俊(1994)指出:同分布的条件可被放宽.

定理8.4.1和8.4.2的证明需要下列引理.

**引理8.4.1** 设$\{\xi_n, n\geqslant 1\}$是$\rho$-混合序列,$E\xi_n=0$,$Eg(|\xi_n|)<\infty$,其中$g(x)$是一个这样的函数:存在常数$0<c<\infty$,使得对任意的$t>0$,$\sup_{x\geqslant t}x/g(x)\leqslant ct/g(t)$.记$T_k(i)=\sum_{j=k+1}^{k+i}\xi_j$.那么对任意的$q_1,q_2\geqslant 2$,存在常数$K=K(q_1,q_2,\rho(\cdot))$使得

$$\begin{aligned}
P\left\{\max_{i\leqslant n}|T_0(i)|\geqslant x\right\}\leqslant &\sum_{i=1}^{n}P\{|\xi_i|\geqslant A\}\\
&+K\Big\{x^{-q_1}\Big(n^{q_1/2}\exp\Big(K\sum_{i=1}^{[\log n]}\rho(2^i)\Big)\max_{i\leqslant n}\|\xi_i I(|\xi_i|\leqslant B)\|_2^{q_1}\\
&+n\exp\Big(K\sum_{i=1}^{[\log n]}\rho^{2/q_1}(2^i)\Big)\log^{q_1}(2n)\max_{i\leqslant n}E|\xi_i|^{q_1}I(|\xi_i|\leqslant B)\Big)\\
&+x^{-q_2}\Big(n^{q_2/2}\exp\Big(K\sum_{i=1}^{[\log n]}\rho(2^i)\Big)\max_{i\leqslant n}\|\xi_i I(B<|\xi_i|<A)\|_2^{q_2}\\
&+\Big(\frac{n}{k}\Big)^{q_2/2}k\exp\Big(K\sum_{i=1}^{[\log n]}\rho^{2/q_2}(2^i)\Big)\max_{i\leqslant n}E|\xi_i|^{q_2}I(B<|\xi_i|<A)\Big)\\
&+x^{-2}\cdot n\rho^2(k)\cdot\exp\Big(K\sum_{i=1}^{[\log n]}\rho(2^i)\Big)\log^4\Big[\frac{n}{k}\Big]
\end{aligned}$$

$$\cdot \max_{i\leqslant n} E\xi_i^2 I(B < |\xi_i| < A)\},$$

其中 $k(\leqslant n)$, $x>0$ 和 $A\geqslant B>0$ 满足下列条件：

(8.4.7) $$\frac{4nCA}{g(A)}\max_{1\leqslant i\leqslant n} Eg(|\xi_i|)I(|\xi_i|\geqslant A)\leqslant x,$$

(8.4.8) $$\frac{48kBC}{g(B)}\max_{1\leqslant i\leqslant n} Eg(|\xi_i|)I(|\xi_i|\geqslant B)\leqslant x.$$

**证** 为简计，假设 $\{\xi,\xi_i,i\geqslant 1\}$ 有相同的分布. 记

$$\xi_{i1}=\xi_i I(|\xi_i|\leqslant B)-E\xi_i I(|\xi_i|\leqslant B);$$

$$\xi_{i2}=\xi_i I(B<|\xi_i|<A)-E\xi_i I(B<|\xi_i|<A);$$

$$\xi_{i3}=\xi_i I(|\xi_i|\geqslant A)-E\xi_i I(|\xi_i|\geqslant A);$$

$$T_k=\sum_{i=1}^{n}\xi_{ik},\quad k=1,2,3.$$

易知

(8.4.9) $$P\{\max_{i\leqslant n}|T_n(i)|\geqslant x\}$$

$$\leqslant P\left\{\max_{i\leqslant n}|T_{i1}|\geqslant\frac{x}{4}\right\}+P\left\{\max_{i\leqslant n}|T_{i2}|\geqslant\frac{x}{4}\right\}$$

$$+P\left\{\max_{i\leqslant n}|T_{i3}|\geqslant\frac{x}{2}\right\}=:I_1+I_2+I_3.$$

从(8.4.7)

(8.4.10) $$I_1\leqslant P\left\{\sum_{i=1}^{n}|\xi_i|I(|\xi_i|\geqslant A)\geqslant\frac{x}{2}\right.$$

$$\left.-\sum_{i=1}^{n}E|\xi_i|I(|\xi_i|\geqslant A)\right\}$$

$$\leqslant P\left\{\sum_{i=1}^{n}|\xi_i|I(|\xi_i|\geqslant A)\geqslant\frac{x}{2}\right.$$

$$\left.-\sum_{i=1}^{n}\frac{CA}{g(A)}Eg(|\xi_i|)I(|\xi_i|\geqslant A)\right\}$$

$$\leqslant P\left\{\sum_{i=1}^{n}|\xi_i|I(|\xi_i|\geqslant A)\geqslant\frac{x}{4}\right\}$$

$$\leqslant\sum_{i=1}^{n}P(|\xi_i|\geqslant A).$$

由引理2.2.5和引理4.1.2，存在常数 $K_2=K_2(q_0,\rho(\cdot))$ 使得

$$(8.4.11)\qquad I_1\leqslant K_1x^{-q_1}\Big\{n^{q_1/2}\|\xi I(|\xi|\leqslant B)\|_2^{q_1}\exp\Big(K_1\sum_{i=0}^{[\log n]}\rho(2^i)\Big)$$

$$+n\log^{q_1}(2n)\|\xi I(|\xi|\leqslant B)\|_{q_1}^{q_1}$$

$$\cdot\exp\Big(K_1\sum_{i=0}^{[\log n]}\rho^{2/q_1}(2^i)\Big)\Big\}.$$

为了估计 $I_2$，记

$$(8.4.12)\qquad Y_i=\sum_{j=2ik+1}^{(2i+1)k}\xi_{j2},\quad i=0,1,\cdots,p_1,$$

$$Z_i=\sum_{j=(2i+1)k+1}^{2(i+1)k}\xi_{j2},\quad i=0,1,\cdots,p_2,$$

其中 $p_1=\Big[\Big(\dfrac{n}{k}-1\Big)/2\Big]$，$p_2=\Big[\Big(\dfrac{n}{k}-2\Big)/2\Big]$. 记

$$W_i=\sum_{j=0}^{i}Y_j,\quad W_i^*=\sum_{j=0}^{i}Z_j.$$

于是

$$(8.4.13)\quad I_2\leqslant P\Big\{\max_{0\leqslant i\leqslant p_1}|W_i|\geqslant\frac{x}{12}\Big\}+P\Big\{\max_{0\leqslant i\leqslant p_2}|W_i^*|\geqslant\frac{x}{12}\Big\}$$

$$+P\Big\{\max_{0\leqslant i\leqslant[n/k]}\ \max_{ik+1\leqslant j\leqslant(i+1)k}\Big|\sum_{l=ik+1}^{j}\xi_{l2}\Big|\geqslant\frac{x}{12}\Big\}$$

$$=:I_4+I_5+I_6.$$

由引理 2.2.5 和(8.4.8)，我们有

$$(8.4.14)\quad I_6\leqslant 2\Big[\frac{n}{k}\Big]\max_{0\leqslant i\leqslant[n/k]}P\Big\{\sum_{j=ik+1}^{(i+1)k}|\xi_j|I(B<|\xi_j|<A)$$

$$-E|\xi_j|I(B<|\xi_j|<A)$$

$$\geqslant\frac{x}{12}-2\sum_{j=ik+1}^{(i+1)k}E|\xi_j|I(B<|\xi_j|<A)\Big\}$$

$$\leqslant 2\cdot\Big[\frac{n}{k}\Big]\max_{0\leqslant i\leqslant[n/k]}P\Big\{\sum_{j=ik+1}^{(i+1)k}|\xi_j|I(B<|\xi_j|<A)$$

$$-E|\xi_j|I(B<|\xi_j|<A)\geqslant\frac{x}{24}\Big\}$$

$$\leqslant K_2\Big[\frac{n}{k}\Big]x^{-q_2}\Big\{k^{q_2/2}\|\xi I(B<|\xi|<A)\|_2^{q_2}\qquad\Big\}$$

$$\exp\Big(K\sum_{i=0}^{[\log n]}\rho(2^i)\Big)$$

$$+ k\|\xi I(B < |\xi| < A)\|_{q_2}^{q_2}\exp\Big(K_2\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\Big)\Big)$$

$$\leqslant K_2 x^{-q_2}(n^{q_2/2}\|\xi I(B < |\xi| < A)\|_2^{q_2}\exp\Big(K_2\sum_{i=0}^{[\log n]}\rho(2^i)\Big)$$

$$+ n\cdot\|\xi I(B < |\xi| < A)\|_{q_2}^{q_2}\exp\Big(K_2\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\Big).$$

下面来估计 $I_4$. 记 $\mathscr{F}_i=\sigma\{\xi_j, j\leqslant 2(i+1)k\}, i=0,1,\cdots,p_1$, $\mathscr{F}_{-1}$为平凡 $\sigma$-域. 记

$$U_i = Y_i - E(Y_i|\mathscr{F}_{i-1}), G_i = \sum_{j=0}^{i}U_j,$$

$$H_i = \sum_{j=1}^{i}E(Y_j|\mathscr{F}_{j-1}), i = 0,1,\cdots,p_1.$$

容易看出

$$I_4\leqslant P\Big\{\max_{0\leqslant i\leqslant p_1}|G_i|\geqslant\frac{x}{24}\Big\}+P\Big\{\max_{0\leqslant i\leqslant p_1}|H_i|\geqslant\frac{x}{24}\Big\} \tag{8.4.15}$$

$$=: I_7+I_8.$$

因为$\{U_i,\mathscr{F}_i,i\geqslant 0\}$是鞅差序列,故由鞅差序列的极大值不等式、Marcinkiewicz-Zygmund 不等式和引理2.2.5,我们有

$$I_7\leqslant\Big(\frac{x}{24}\Big)^{-q_2}\Big(\frac{q_2}{q_2-1}\Big)^{q_2}E|G_{p_1}|^{q_2} \tag{8.4.16}$$

$$\leqslant K_3x^{-q_2}\cdot p_1^{q_2/2}\max_{0\leqslant i\leqslant p_1}E|U_i|^{q_2}$$

$$\leqslant K_3x^{-q_2}\cdot p_1^{q_2/2}\Big(k^{q_2/2}\|\xi I(B < |\xi| < A)\|_2^{q_2}$$

$$\exp(K_3\sum_{i=0}^{[\log n]}\rho(2^i)\Big)$$

$$+ k\|\xi I(B < |\xi| < A)\|_{q_2}^{q_2}\exp\Big(K_3\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\Big)\Big)$$

$$\leqslant K_3x^{-q_2}(n^{q_2/2}\|\xi I(B<|\xi|<A)\|_2^{q_2}\exp\Big(K_3\sum_{i=0}^{[\log n]}\rho(2^i)\Big)$$

$$+\Big(\frac{n}{k}\Big)^{q_2/2}k\cdot E|\xi|^{q_2}I(B < |\xi| < A)$$

$$\exp\left(K_3\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\right).$$

模仿引理2.2.2的证明并利用引理2.2.5，存在常数 $K_4=K_4(\rho(\cdot))$，使得

(8.4.17) $$E\left(\sum_{j=i+1}^{i+m}E(Y_j|\mathscr{F}_{j-1})\right)^2$$
$$\leqslant K_4mk\rho^2(k)E\xi^2I(B<|\xi|<A)$$
$$\cdot\log^2(2m)\exp\left(K_4\sum_{i=0}^{[\log n]}\rho(2^i)\right).$$

由引理4.1.2，

(8.4.18) $$E\max_{1\leqslant l\leqslant p_1}H_l^2\leqslant 3K_{4p_1}k\rho^2(k)\log^4(2p_1)E\xi^2I(B<|\xi|<A)$$
$$\cdot\exp\left(K_4\sum_{i=0}^{[\log n]}\rho(2^i)\right).$$

因此我们得到

(8.4.19) $$I_8\leqslant K_4x^{-2}n\rho^2(k)\log^4\left[\frac{n}{k}\right]E\xi^2I(B<|\xi|<A)$$
$$\cdot\exp\left(K_4\sum_{i=0}^{[\log n]}\rho 2^i)\right).$$

（当$[n/k]\leqslant 2$时，$p_1=0$，因此 $I_8=0$，也即(8.4.19)成立；当$[n/k]>2$时，由(8.4.18)可推出(8.4.19).）从(8.4.15)、(8.4.16)和(8.4.19)得到

(8.4.20) $$I_4\leqslant K_5x^{-q_2}\{n^{q_2/2}\|\xi_iI(B<|\xi_i|<A\|_2^{q_2}$$
$$\exp\left(K_5\sum_{i=0}^{[\log n]}\rho(2^i)\right)$$
$$+\left(\frac{n}{k}\right)^{q_2/2}k\|\xi_iI(B<|\xi_i|<A)\|_{q_2}^{q_2}$$
$$\exp\left(K_5\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\right)\}$$
$$+K_5x^{-2}n\rho^2(k)\log^4\left[\frac{n}{k}\right]\|\xi_iI(B<|\xi_i|<A)\|_{q_2}^{q_2}$$
$$\|\xi_iI(B<|\xi_i|<A)\|_2^2$$

$$\cdot \exp\Big(K_5\sum_{i=0}^{[\log n]}\rho(2^i)\Big).$$

利用同样证法，对 $I_5$ 我们也有(8.4.20). 引理8.4.1证毕.

**定理8.4.1的证明.**

由假设 $l(x),\beta(x)l(\beta(x))/x, x^{2-\varepsilon_0}/\beta(x)l(\beta(x))$ 都是拟单调不减的，因此存在常数 $c>0$，使得对任意的 $t\geqslant x>0$，

(8.4.21) $\qquad l(x)\leqslant Cl(t),$

(8.4.22) $\qquad \beta(x)l(\beta(x))/x\leqslant C\beta(t)l(\beta(t))/t,$

(8.4.23) $\qquad x^{2-\varepsilon_0}/\beta(x)l(\beta(x))\leqslant Ct^{2-\varepsilon_0}/\beta(t)l(\beta(t))$

由(8.4.22)、(8.4.23)和 $\beta(x)$ 的单调性，我们有

(8.4.24) $$\frac{\alpha(x)}{xl(x)}\geqslant\frac{1}{C}\frac{\alpha(t)}{tl(t)},$$

(8.4.25) $$\frac{\alpha^{2-\varepsilon_0}(x)}{xl(x)}\leqslant C\frac{\alpha^{2-\varepsilon_0}(t)}{tl(t)}$$

令 $c_n=(\log n)^{-8/\varepsilon_0}$，取引理8.4.1中的 $A=\alpha(n),B=\alpha(nC_n),k=[nC_n],g(x)=\beta(x)l(\beta(x)),x=\alpha(\varepsilon n)$. (8.4.7)和(8.4.8)对于充分大的 $n$ 是被满足的，事实上，对于 $n\geqslant 1/\varepsilon$，由(8.4.24)和(8.4.21)我们有

$$\frac{4Cn\alpha(n)}{nl(n)}E\beta(X_1)l(\beta(X_1))I(|X_1|\geqslant\alpha(n))$$

$$\leqslant\frac{4C^2\alpha(\varepsilon n)}{\varepsilon l(\varepsilon n)}E\beta(X_1)l(\beta(X_1))I(|X_1|\geqslant\alpha(n))$$

$$\leqslant\frac{4C^3\alpha(\varepsilon n)}{\varepsilon l(1)}E\beta(X_1)l(\beta(X_1))I(|X_1|\geqslant\alpha(n)).$$

注意到 $\lim\limits_{n\to\infty}E\beta(X_1)l(\beta(X_1))I(|X_1|\geqslant\alpha(n))=0$，对充分大的 $n$ 我们得

$$\frac{4Cn\alpha(n)}{nl(n)}E\beta(X_1)l(\beta(X_1))I(|X_1|\geqslant\alpha(n))\leqslant\alpha(\varepsilon n).$$

这就证明了(8.4.7)是被满足的. 类似地我们能够验证(8.4.8). 由引理8.4.1，令 $q_1=q_2=2$，我们有

(8.4.26) $$\sum_{n=1}^{\infty}\frac{l(n)}{n}P\{\max_{i\leqslant n}\beta(S_i)\geqslant\varepsilon n\}$$

$$= \sum_{n=1}^{\infty} \frac{l(n)}{n} P\{\max_{i\leqslant n} |S_i| \geqslant \alpha(\varepsilon n)\}$$

$$\leqslant \sum_{n=1}^{\infty} l(n) P\{|X_1| \geqslant \alpha(n)\}$$

$$+ c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} EX_1^2 I(|X_1| \leqslant \alpha(nC_n)\log^2 n$$

$$+ c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} EX_1^2 I(|X_1| \leqslant \alpha(n))$$

$$+ c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} \rho^2(nC_n)(\log\log n)^4 EX_1^2 I(|X_1| \leqslant \alpha(n))$$

$$=: J_1 + J_2 + J_3 + J_4.$$

由(8.4.21)

(8.4.27) $$J_1 \leqslant \sum_{n=1}^{\infty} l(n) \sum_{j=n}^{\infty} P\{j \leqslant \beta(X_1) < j+1\}$$

$$= \sum_{j=1}^{\infty} \sum_{n=1}^{j} l(n) P\{j \leqslant \beta(X_1) < j+1\}$$

$$\leqslant c\sum_{j=1}^{\infty} jl(j) P\{j \leqslant \beta(X_1) < j+1\}$$

$$\leqslant cE\beta(X_1) l(\beta(X_1)) < \infty.$$

由(8.4.23)、(8.4.25)和 $x/\alpha^{\theta}(x)$ 及 $\alpha^2(x)/x$ 的单调性，我们有

(8.4.28) $$J_2 \leqslant c\sum_{n=1}^{\infty} \frac{l(n)\log^2 n\alpha^2(nC_n)}{\alpha^2(\varepsilon n) nC_n l(nC_n)} E\beta(X_1) l(\beta(X_1))$$

$$\leqslant c\varepsilon^{-2/\theta} \sum_{n=1}^{\infty} \frac{\log^2 n}{n} \frac{\alpha^{\varepsilon_0}(nC_n)}{\alpha^{\varepsilon_0}(n)} E\beta(X_1) l(\beta(X_1))$$

$$\leqslant c\sum_{n=1}^{\infty} \frac{\log^2 n}{n} C_n^{\varepsilon_0/2}$$

$$\leqslant c\sum_{n=1}^{\infty} \frac{1}{n\log^2 n} < \infty.$$

因为 $\sum_{n=1}^{\infty} \rho(2^n) < \infty$，我们有 $\rho(n) \leqslant c\log^{-1} n$. 因此

$$(8.4.29)\qquad J_4 \leqslant c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)}\log^{-3/2} n E X_1^2 I(|X_1| \leqslant \alpha(n))$$

$$\leqslant c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} \frac{\alpha^2(n)}{nl(n)}\log^{-3/2} n E\beta(X_1)l(\beta(X_1))$$

$$\leqslant c\sum_{n=1}^{\infty} \frac{1}{n\log^{3/2} n} < \infty.$$

最后,我们来估计 $J_5$. 从(8.4.25)和 $x/\alpha^2(x)\downarrow$,

$$\sum_{j=n}^{\infty} \frac{l(j)}{\alpha^2(j)} = \sum_{j=n}^{\infty} \frac{l(j)j}{\alpha^{2-\varepsilon_0}(j)} \frac{j^{\varepsilon_0/2}}{\alpha^{\varepsilon_0}(j)} \frac{1}{j^{1+\varepsilon_0/2}}$$

$$\leqslant \frac{cl(n)n^{1+\varepsilon_0/2}}{\alpha^2(n)}\sum_{j=n}^{\infty} j^{-1-\varepsilon_0/2}$$

$$\leqslant \frac{c}{\varepsilon_0}\cdot\frac{nl(n)}{\alpha^2(n)}.$$

因此

$$(8.4.30)\qquad J_5 \leqslant c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(n)}\alpha^2(1)$$

$$+ \sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(n)}\sum_{j=1}^{n} EX_1^2 I(j < \beta(X_1) \leqslant j+1)$$

$$\leqslant c\sum_{j=1}^{\infty}\sum_{n=j}^{\infty} \frac{l(n)}{\alpha^2(n)} EX_1^2 I(j < \beta(X_1) \leqslant j+1)$$

$$\leqslant c\sum_{j=1}^{\infty} \frac{jl(j)}{\alpha^2(j)} EX_1^2 I(j < \beta(X_1) \leqslant j+1)$$

$$\leqslant cE\beta(X_1)l(\beta(X_1)) < \infty.$$

从(8.4.27)-(8.4.30)得证定理8.4.1.

**定理8.4.2的证明.**

由假设,$l(x)$,$x^q/\beta(x)l(\beta(x))$和$\beta(x)l(\beta(x))/x^2$是拟单调不减的,因此存在常数 $c>0$使得(8.4.21)被满足,且对任意的 $t\geqslant x>0$我们有

$$(8.4.31)\qquad \frac{\alpha^q(x)}{xl(x)} \leqslant c\frac{\alpha^q(t)}{tl(t)},\ \frac{\beta(x)l(\beta(x))}{x^2} \leqslant c\frac{\beta(t)l(\beta(t))}{t^2}.$$

令 $c_n = n^{-\varepsilon_1}$,其中 $\varepsilon_1 = \dfrac{\delta}{2(1+\delta)}$,又令 $\beta = \alpha(nC_n)$,$k=n$,$g(x)=\beta(x)$

$l(\beta(x))$. 取引理8.4.1中的 $x=\alpha(\varepsilon n)$. 那么(8.4.7)和(8.4.8)被满足. 事实上, 由 $l(x)\geqslant x^{\delta}$, 对充分大的 $n$ 我们有

$$\frac{48C\alpha(nC_n)}{C_n l(nC_n)}E\beta(X_1)l(\beta(X_1))$$

$$\leqslant\frac{48C\alpha(\varepsilon n)}{C_n(nC_n)^{\delta}}E\beta(X_1)l(\beta(X_1))$$

$$\leqslant\frac{48C\alpha(\varepsilon n)}{n^{\delta/2}}E\beta(X_1)l(\beta(X_1)).$$

因此对充分大的 $n$, (8.4.8)被满足. 同样方式可证(8.4.7)也被满足.

取引理8.4.1中的 $q_1=q_0+q+4, q_2=r$, 则有

(8.4.32)
$$\sum_{n=1}^{\infty}\frac{l(n)}{n}P\{\max_{i\leqslant n}\beta(S_i)\geqslant\varepsilon n\}$$

$$\leqslant c\sum_{n=1}^{\infty}l(n)P\{|X_1|\geqslant\alpha(n)\}$$

$$+c\sum_{n=1}^{\infty}\frac{l(n)}{n\alpha^{q_1}(n)}\Big(n^{q_1/2}+n\exp\Big(K\sum_{i=0}^{[\log n]}\rho^{2/q_1}(2^i)\Big)$$

$$\cdot\log^{q_1}nE|X_1|^{q_1}I(|X_1|\leqslant\alpha(nC_n))\Big)$$

$$+c\sum_{n=1}^{\infty}\frac{l(n)}{n\alpha^{r}(n)}\Big(n^{r/2}(E|X_1|^2I(|X_1|$$

$$\geqslant\alpha(nC_n)))^{r/2}+nE|X_1|^rI(|X_1|\leqslant\alpha(n))\Big)$$

$$=:L_1+L_2+L_3.$$

类似于定理8.4.1的证明中对 $J_1$ 的估计, 我们有

(8.4.33)
$$L_1<\infty.$$

注意到 $\alpha^2(x)/x\uparrow$ 及 $\exp\Big(K\sum_{i=1}^{[\log n]}\rho^{2/q_1}(2^i)\Big)\log^{q_1}n$ 是缓变的, 利用(8.4.31)和条件(8.4.3), 我们有

(8.4.34)
$$L_2\leqslant c\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^{q_1}(n)}n^{q_1/2}E|X_1|^qI(|X_1|\leqslant\alpha(nC_n))$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{l(n)n^{q_1/2}\alpha^{q_1}(nC_n)}{\alpha^{q_1}(n)l(nC_n)nC_n}$$

$$\leqslant c\sum_{n=1}^{\infty}n^{\varepsilon_1/2-1}\frac{\alpha^{q_1-q}(nC_n)}{\alpha^{q_1-q}(n)}$$

$$\leqslant c\sum_{n=1}^{\infty}n^{\varepsilon_1/2-1}C_n^2$$

$$\leqslant c\sum_{n=1}^{\infty}n^{-1-\varepsilon_1}<\infty.$$

下面来估计 $L_3$. 从(8.4.31)可得

(8.4.35)
$$\sum_{n=1}^{\infty}\frac{l(n)}{n}\left(\frac{nEX_1^2I(|X_1|\geqslant\alpha(nC_n))}{\alpha^2(n)}\right)^{r/2}$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{l(n)}{n}\frac{\alpha^r(nC_n)}{C_n^{r/2}l^{r/2}(nC_n)\alpha^r(n)}$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{l(n)n}{\alpha^r(n)}\frac{\alpha^r(nC_n)}{nC_nl(nC_n)}\frac{1}{nC_n^{r/2-1}l^{r/2-1}(nC_n)}$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{1}{nC_n^{r/2-1}(nC_n)^{(r/2-1)\delta}}$$

$$\leqslant c\sum_{n=1}^{\infty}n^{-1-\left(\frac{r-1}{4}\right)\delta}<\infty.$$

由(8.4.35)和(8.4.31)我们得

(8.4.36)
$$I_3\leqslant c\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^r(n)}E|X_1|^rI(|X_1|\leqslant\alpha(n))$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^r(n)}\cdot\alpha^r(1)+\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^r(n)}$$

$$\cdot\sum_{j=1}^{n}E|X_1|^rI(j\leqslant\beta(X_1)<j+1)$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{nl(n)}{\alpha^q(n)}\cdot\frac{n^{\frac{r-q}{2}}}{\alpha^{r-q}(n)}\frac{\alpha^r(1)}{n^{1+(r-q)/2}}$$

$$+\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\frac{nl(n)}{\alpha^q(n)}\frac{n^{(r-q)/2}}{\alpha^{r-q}(n)}\frac{1}{n^{1+(r-q)/2}}$$

$$E|X_1|^rI(j\leqslant\beta(X_1)<j+1)$$

$$\leqslant 1+c\sum_{j=1}^{\infty}\frac{jl(j)}{\alpha^r(j)}E|X_1|^rI(j\leqslant\beta(X_1)<j+1)$$

$$\leqslant cE\beta(X_1)l(\beta(X_1)) < \infty.$$

从(8.4.32)—(8.4.36)得证定理8.4.2.

为了证明推论8.4.1和8.4.2,我们需要下列引理.

**引理8.4.2** 设 $h(x)$ 是正的缓变函数,那么对任意的 $\varepsilon>0$, $x^\varepsilon h(x)$ 是拟单调不减函数.

**证** 由缓变函数的性质,我们有

$$\lim_{N\to\infty}\sup_{2^N\leqslant x\leqslant 2^{N+1}}\frac{h(x)}{h(2^{N+1})}=\lim_{N\to\infty}\inf_{2^N\leqslant x\leqslant 2^{N+1}}\frac{h(x)}{h(2^{N+1})}=1.$$

因此存在 $N$ 使得对 $m\geqslant N$ 成立

$$\sup_{2^m\leqslant x\leqslant 2^{m+1}}\frac{h(x)}{h(2^{m+1})}\leqslant 2^\varepsilon,\ 2^{-\varepsilon}\leqslant\inf_{2^m\leqslant x\leqslant 2^{m+1}}\frac{h(x)}{h(2^{m+1})}. \tag{8.4.37}$$

对任意的 $t\geqslant x\geqslant 2^N$,令 $M_1, M_2\geqslant N$ 使得

$$2^{M_1}\leqslant x<2^{M_1+1},\quad 2^{M_2}\leqslant t<2^{M_2+1}. \tag{8.4.38}$$

从(8.4.37)和(8.4.38)我们有

$$\begin{aligned}
x^\varepsilon h(x)&\leqslant x^\varepsilon\sup_{2^{M_1}\leqslant s<2^{M_1+1}}\frac{h(s)}{h(2^{M_1+1})}\cdot h(2^{M_1+1})\\
&\leqslant x^\varepsilon 2^{\varepsilon(M_2-M_1+1)}h(2^{M_2+1})\\
&\leqslant x^\varepsilon 2^{\varepsilon(M_2-M_1+2)}h(t)\\
&\leqslant t^\varepsilon 2^{\varepsilon(M_2-M_1+1)}\cdot 2^{\varepsilon(M_2-M_1+2)}h(t)\\
&=2^\varepsilon\cdot t^\varepsilon h(t).
\end{aligned}$$

这就证明了 $x^\varepsilon h(x)$ 是一个拟单调不减函数.

**推论8.4.1的证明.** 从 $p\alpha>1$ 和引理8.4.2可知 $n^{p\alpha-1}h(n)$ 是拟单调不减的. 如果 $1\leqslant p<2$,那么 $x^p h(x)$ 和 $x^{2-p}h(x)$ 都是拟单调不减的. 因此从定理8.4.1得证推论8.4.1如果 $p>2$,那么 $x^{p-2}h(x)$ 和 $x^{r-p}/h(x)$ 都是拟单调不减的,因此从定理8.4.2得证推论8.4.1. 如果 $p=2$,注意到 $\lim\limits_{n\to\infty}n^{-\varepsilon}EX_1^2I(|X_1|\leqslant n^\alpha)=0$ 对任意 $\varepsilon>0$ 都成立,推论8.4.1可以通过重复定理8.4.2的证明得到.

推论8.4.2的证明. 显然 $x^{p-1}/h(x)$ 是增函数. 由引理8.4.2, $x^{2-p}/h(x)$ 是拟单调不减的. 因此推论8.4.2从定理8.4.1得到.

## §8.5 $\alpha$混合序列的完全收敛性

对于$\alpha$混合序列的完全收敛性，Hipp(1979)曾给出过下列结果：

**定理8.5.0** 设$1/2<\alpha\leqslant 1, 2<r\leqslant\infty, 1/\alpha<p<r$，又设$\{X_n, n\geqslant 1\}$是强平稳$\alpha$混合的随机变量序列，$EX_1=0, E|X_1|^r<\infty$. 假设

(8.5.1) 对某个 $\theta>\left[2-\frac{r}{r-p}\right]\cdot\frac{p\alpha}{p\alpha-1}, \sum_{n=1}^{\infty}\alpha^{1/\theta}(n)<\infty.$

那么对任意的$\epsilon>0$

(8.5.2) $$\sum_{n=1}^{\infty}n^{p\alpha-2}P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant\epsilon n^{\alpha}\right\}<\infty.$$

然而，当$r=\infty$也即$X_1$有界时Berbee(1987)通过反例指出Hipp的结论是不成立的.

邵启满(1993c)证明了下列定理：

**定理8.5.1** 设$1/2<\alpha\leqslant 1, 1/\alpha\leqslant p<r\leqslant\infty$. 又设$\{X_n, n\geqslant 1\}$是$\alpha$混合的随机变量序列，$EX_n=0, \sup_n E|X_n|^r<\infty$. 假设对某个$\beta>rp/(r-p)$

(8.5.3) $$\alpha(n)=O(n^{-r(p\alpha-1)/(r-p)}\log^{-\beta}n),$$

那么(8.5.2)成立.

在$p=\alpha=1$时，定理8.5.1的一个直接推论是

**推论8.5.1** 设$1<r\leqslant\infty$，$\{X_n, n\geqslant 1\}$是$\alpha$混合的随机变量序列，$EX_n=0, \sup_n E|X_n|^r<\infty$. 假设对某个$\beta>r/(r-1)$

$$\alpha(n)=O(\log^{-\beta}n),$$

那么对任意的$\epsilon>0$

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant\epsilon n\right\}<\infty.$$

特别地我们有

$$S_n/n\to 0\quad\text{a.s.}$$

定理8.5.1的证明需要用到下列引理.

**引理8.5.1** 设$\{X_n,n\geqslant 1\}$是随机变量序列，对任一$n\geqslant 1$，$EX_n=0$。那么对任意的$a\geqslant 1$，$x\geqslant 1$，$c>0$和满足

$$1\leqslant k\leqslant x/(64ac\log x) \tag{8.5.4}$$

和

(8.5.5) 对某个$s\geqslant 2$，$\left(\sum_{i=1}^{n}\|X_iI(|X_i|\leqslant c)\|_s^2\right)\sum_{i=0}^{k}\alpha^{1-2/s}(i)\leqslant x^2/(32^3a\log x)$的整数$k$成立着

$$P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant x\right\}\leqslant 4x^{-1}\sum_{i=1}^{n}E|X_i|I(|X_i|>c)+4x^{-a}+32^3ncx^{-1}\alpha(k). \tag{8.5.6}$$

**证** 令

$$\overline{X}_i=X_iI(|X_i|\leqslant c)-EX_iI(|X_i|\leqslant c),$$

$$Y_{i,1}=\sum_{j=1+2ik}^{(2i+1)k\wedge n}\overline{X}_j,\ i=0,1,\cdots,q_1:=\left[\frac{n}{2k}-\frac{1}{2}\right],$$

$$Y_{i,2}=\sum_{j=1+(2i+1)k}^{2(i+1)k\wedge n}\overline{X}_j,\ i=0,1,\cdots,q_2:=\left[\frac{n}{2k}-1\right],$$

$$\overline{S}_i=\sum_{j=1}^{i}\overline{X}_j,\quad T_{i,1}=\sum_{j=0}^{i}Y_{j,1},\quad T_{i,2}=\sum_{j=0}^{i}Y_{j,2}.$$

容易看出

$$\max_{1\leqslant i\leqslant n}|S_i|\leqslant\max_{1\leqslant i\leqslant n}|\overline{S}_i|+\sum_{i=1}^{n}|X_i|I(|X_i|>c)+\sum_{i=1}^{n}E|X_i|I(|X_i|>c),$$

且进一步

$$P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant x\right\}\leqslant P\left\{\max_{1\leqslant i\leqslant n}|\overline{S}_i|\geqslant x/2\right\}+4x^{-1}\sum_{i=1}^{n}E|X_i|I(|X_i|>c). \tag{8.5.7}$$

由(8.5.4)，

$$\max_{1\leqslant i\leqslant n}|\overline{S}_i|\leqslant\max_{0\leqslant i\leqslant q_1}|T_{i,1}|+\max_{0\leqslant i\leqslant q_2}|T_{i,2}|+2kc$$

$$\leqslant \max_{0\leqslant i\leqslant q_1}|T_{i,1}| + \max_{0\leqslant i\leqslant q_2}|T_{i,2}| + x/32,$$

因此我们有

$$(8.5.8)\quad P\left\{\max_{1\leqslant i\leqslant n}|\overline{S}_i|\geqslant x/2\right\}\leqslant P\left\{\max_{0\leqslant i\leqslant q_1}|T_{i,1}|\geqslant x/8\right\} + P\left\{\max_{0\leqslant i\leqslant q_2}|T_{i,2}|\geqslant x/8\right\} =: I_1+I_2.$$

首先估计 $I_1$，$I_2$的估计是完全类似的. 记

$$G_{-1} = \sigma(\Omega), G_i = \sigma(X_j, 1\leqslant j\leqslant (2i+1)k),$$

$$u_i = Y_{i,1} - E(Y_{i,1}|G_{i-1}), U_i = \sum_{j=0}^{i} u_j, i = 0,1,\cdots$$

那么

$$(8.5.9)\quad I_1 \leqslant P\left\{\sum_{i=0}^{q_1}|E(Y_{i,1}|G_{i-1})| \geqslant x/16\right\} + P\left\{\max_{0\leqslant i\leqslant q_1}|U_i| \geqslant x/16\right\} =: I_3 + I_4,$$

而且$\{U_i, G_i, i\geqslant 0\}$是鞅. 此外对每一 $i\geqslant 0$，$|u_i|\leqslant 2kc$，注意到对任意实数 $t$ 和 $i\geqslant 1$

$$E(e^{tu_i}|G_{i-1}) = 1 + \sum_{l=2}^{\infty}E\left(\frac{(tu_i)^l}{l!}|G_{i-1}\right)$$

$$\leqslant 1 + t^2E(u_i^2|G_{i-1})\sum_{l=0}^{\infty}\frac{(|t|2kc)^l}{l!}$$

$$\leqslant \exp(t^2e^{2|t|kc}E(u_i^2|G_{i-1}))$$

$$\leqslant \exp(t^2e^{2|t|kc}E(Y_{i,1}^2|G_{i-1})),$$

从而可知对任一 $t$，$\left\{\exp\left(tU_i - t^2e^{2|t|kc}\sum_{j=0}^{i}E(Y_{j,1}^2|G_{j-1})\right), G_i, i \geqslant 0\right\}$是一非负上鞅. 因此由非负上鞅的极大不等式，对 $y > 0$，

$$(8.5.10)\quad P\left\{\max_{0\leqslant i\leqslant q_1}\exp\left(tU_i - t^2e^{2|t|kc}\sum_{j=0}^{i}E(Y_{j,1}^2|G_{j-1})\right)\geqslant y\right\}\leqslant \frac{1}{y}.$$

在上式中取 $t=(32a\log x)/x$. 由(8.5.10)和(8.5.4)，我们有

(8.5.11) $$P\Big\{\max_{0\leqslant i\leqslant q_1} U_i \geqslant x/16\Big\}$$

$$\leqslant P\Big\{\max_{0\leqslant i\leqslant q_1}\exp\Big(tU_i - t^2e^{2|t|kc}\sum_{j=0}^{i}E(Y_{j,1}^2|G_{j-1})\Big)$$

$$\geqslant \exp\Big(\frac{xt}{16} - t^2e^{2|t|kc}\sum_{j=0}^{i}E(Y_{j,1}^2|G_{j-1})\Big)\Big\}$$

$$\leqslant P\Big\{\sum_{j=0}^{q_1}E(Y_{j,1}^2|G_{j-1}) \geqslant \frac{x^2}{4(32)^2 a\log x}\Big\}$$

$$+ P\Big\{\max_{0\leqslant i\leqslant q_1}\exp\Big(tU_i - t^2e^{2|t|kc}\sum_{j=0}^{i}E(Y_{j,1}^2|G_{j-1})\Big)$$

$$\geqslant \exp\Big(\frac{xt}{16} - \frac{t^2e^{2|t|kc}x^2}{4(32)^2 a\log x}\Big)\Big\}$$

$$\leqslant P\Big\{\sum_{j=0}^{q_1}E(Y_{j,1}^2|G_{j-1}) \geqslant \frac{x^2}{4(32)^2 a\log x}\Big\}$$

$$+ \exp\Big(-\frac{xt}{16} + \frac{t^2e^{2|t|kc}x^2}{4(32)^2 a\log x}\Big)$$

$$\leqslant P\Big\{\sum_{j=0}^{q_1}E(Y_{j,1}^2|G_{j-1}) \geqslant \frac{x^2}{4(32)^2 a\log x}\Big\} + x^{-a}.$$

利用引理1.2.4，由(8.5.5)可得

$$\sum_{j=0}^{q_1}EY_{j,1}^2 \leqslant 4\Big(\sum_{i=0}^{k}\alpha^{1-2/s}(i)\Big)\sum_{j=0}^{q_1}\sum_{i=2jk+1}^{(2j+1)k}\|X_iI(|X_i|\leqslant c)\|_s^2$$

$$\leqslant 4\Big(\sum_{i=0}^{k}\alpha^{1-2/s}(i)\Big)\sum_{i=0}^{n}\|X_iI(|X_i|\leqslant c)\|_s^2$$

$$\leqslant \frac{x^2}{8(32)^2 a\log x}.$$

因此

(8.5.12) $$P\Big\{\sum_{j=0}^{q_1}E(Y_{j,1}^2|G_{j-1}) \geqslant \frac{x^2}{4(32)^2 a\log x}\Big\}$$

$$\leqslant \frac{8(32)^2 a\log x}{x^2}\sum_{j=0}^{q_1}E|E(Y_{j,1}^2|G_{j-1}) - EY_{j,1}^2|.$$

记 $\xi_j = E(Y_{j,1}^2|G_{j-1}) - EY_{j,1}^2$，再次应用引理1.2.4，有

(8.5.13) $$E|\xi_j| = E(Y_{j,1}^2 - EY_{j,1}^2)\mathrm{sgn}\xi_j \leqslant 4(kc)^2\alpha(k).$$

将(8.5.13)代入(8.5.12),由(8.5.4)即得

(8.5.14) $$P\left\{\sum_{i=0}^{q_1} E(Y_{i,1}^2 | G_{i-1}) \geqslant \frac{x^2}{4(32)^2 a \log x}\right\}$$

$$\leqslant \frac{(32)^2 anc^2 k\alpha(k)\log x}{x^2} \leqslant \frac{(32)^2 nc\alpha(k)}{x}.$$

将它与(8.5.11)相结合得到

$$P\left\{\max_{0\leqslant i\leqslant q_1} U_i \geqslant \frac{x}{16}\right\} \leqslant x^{-a} + (32)^2 ncx^{-1}\alpha(k).$$

类似地又有

$$P\left\{\max_{0\leqslant i\leqslant q_1} U_i \leqslant -\frac{x}{16}\right\} \leqslant x^{-a} + (32)^2 ncx^{-1}\alpha(k).$$

因此

(8.5.15) $$I_4 \leqslant 2x^{-a} + 2(32)^2 ncx^{-1}\alpha(k).$$

类似于(8.5.13),可有

$$E|E(Y_{i,1} | G_{i-1})| = EY_{i,1}\mathrm{sgn}(E(Y_{i,1}|G_{i-1})) \leqslant 4kc\alpha(k),$$

因此

(8.5.16) $$I_3 \leqslant 64ncx^{-1}\alpha(k).$$

从(8.5.9),(8.5.15)和(8.5.16)得到

(8.5.17) $$I_1 \leqslant 2x^{-a} + 3(32)^2 ncx^{-1}\alpha(k).$$

类似地,我们也有

(8.5.18) $$I_2 \leqslant 2x^{-a} + 3(32)^2 ncx^{-1}\alpha(k).$$

由(8.5.7),(8.5.8),(8.5.17)和(8.5.18),得证(8.5.6).

**引理8.5.2** 设$\{X_n, n\geqslant 1\}$是$\alpha$混合序列,且对任一$i\geqslant 1$和某个$1<\nu\leqslant\infty$,$C_0\geqslant 1$,$\tau\geqslant 0$和实的$\lambda$,

$$EX_i = 0, \|X_i\|_\nu \leqslant D \text{ 且 } \alpha(i) \leqslant C_0 i^{-\tau}\log^{-\lambda} i.$$

那么存在一有限的正常数$K=K(\nu,\tau,\lambda,C_0)$,使得对任意的$x\geqslant KDn^{1/2}\log^{1+|\lambda|/2} n$

(8.5.19) $$P\{\max_{1\leqslant i\leqslant n}|S_i| \geqslant x\} \leqslant Kn(D/x)^{\nu(\tau+1)/(\nu+\tau)}$$

$$\log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)}(x/D).$$

**证** 不失一般性,假设$D=1$.只需证明:存在常数$K$,使得对

任意的 $x\geqslant Kn^{1/2}\log^{1+|\lambda|/2}n$

$$P\{\max_{1\leqslant j\leqslant n}|S_j|\geqslant x\}\leqslant Knx^{-\nu(\tau+1)/(\nu+\tau)}\log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)}x. \tag{8.5.20}$$

取引理8.5.1中的

$$C=2x^{\tau/(\nu+\tau)}\log^{(\lambda-\tau)/(\nu+\tau)}x, a=r+2, k=[x/(64ac\log x)].$$

假设

$$Knx^{-\nu(\tau+1)/(\nu+\tau)}\log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)}x\leqslant 1. \tag{8.5.21}$$

不然的话,(8.5.20)是平凡的.如果引理8.5.1的条件被满足,那么(8.5.20)直接从(8.5.6)得出.所以我们只需验证(8.5.5).今后我们将用 $K_1$ 表示仅与 $\nu,\tau,\lambda$ 和 $C_0$ 有关的有限正常数,它在不同的地方可取不同的值.如果$1<\nu\leqslant 2$,那么由(8.5.21)

$$\begin{aligned}
&\left\{\sum_{i=1}^{n}\|X_iI(|X_i|\leqslant c)\|_2^2\right\}\sum_{i=1}^{k}\alpha^{1-2/\nu}(i)\\
&\leqslant 2nkc^{2-\nu}\leqslant xnc^{1-\nu}/(32a\log x)\\
&=\frac{x^2}{(32)^3a\log x}(32)^2 2^{1-\nu}nx^{-\nu(\tau+1)/(\nu+\tau)}\log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)}x\\
&\leqslant\frac{x^2}{(32)^3a\log x}.
\end{aligned} \tag{8.5.22}$$

当 $\nu>2$ 时,由于(8.5.21)和 $x\geqslant Kn^{1/2}\log^{1+|\lambda|/2}n$,我们有

$$\begin{aligned}
&\left\{\sum_{i=1}^{n}\|X_iI(|X_i|\leqslant c)\|_\nu^2\right\}\sum_{i=1}^{k}\alpha^{1-2/\nu}(i)\\
&\leqslant n\left\{1+C_1\sum_{i=3}^{k}i^{-\tau(1-2/\nu)}\log^{-\lambda(1-2/\nu)}i\right\}\\
&\leqslant K_1n(\log^{1-\lambda(1-2/\nu)}k+k^{-\tau(1-2/\nu)+1}\log^{-\lambda(1-2/\nu)}k)\\
&\leqslant K_1n(\log^{1-\lambda}x+(x/(c\log x))^{-\tau(1-2/\nu)+1}\log^{-\lambda(1-2/\nu)}x)\\
&\leqslant\frac{x^2}{(32)^3a\log x}\{K_1nx^{-2}\log^{2+\lambda(1-2/\nu)}x\\
&\quad+K_1nx^{-\nu(\tau+1)/(\nu+\tau)}\log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)}x\}\\
&\leqslant\frac{x^2}{(32)^3a\log x},
\end{aligned} \tag{8.5.23}$$

至此,从(8.5.22)和(8.5.23)即知(8.5.5)被满足.引理8.5.2证毕.

定理8.5.1的证明.由引理8.5.2,对任意的 $\varepsilon>0$ 存在一正常

数 $K$，使得

$$
\begin{aligned}
&P\left\{\max_{1\leqslant i\leqslant n}|S_i|\geqslant \epsilon n^{\alpha}\right\}\\
&\leqslant K n^{1-\alpha r(1+r(p-1)/(r-p))/(r+r(p-1)/(r-p))}\\
&\quad\cdot\log^{(1-r)(\beta-r(p-1)/(r-p))/(r+r(p-1)/(r-p))} n\\
&=K n^{1-p\alpha}\log^{1-(r-p)(\beta-rp/(r-p))/r} n,
\end{aligned}
$$

由此，并注意到(8.5.5)得证(8.5.6)．由它直接产生定理的结论．

推论8.5.1中 $S_n/n\to 0$ a.s.，只须证明$\max\limits_{i\leqslant 2^k}|S_i|/2^k\to 0$即可．后一断语显然成立．

下面的例子说明，如果用 $\beta\geqslant r/(r-p)$代替 $\beta>rp/(r-p)$，定理8.5.1不成立

**例8.5.1** 令$1<p<r$，$1/p\leqslant\alpha\leqslant 1$．记

$$a=\frac{\alpha(r-p)}{r(1-\alpha)+p\alpha-1},\quad b=\frac{\alpha(p-1)}{r-p},\quad d=\frac{-1}{r-1-\alpha(r-p)},$$

$$g(x)=x^{a}\log^{d}x,\quad x\geqslant 0,$$

$$G(0)=0,\quad G(n)=\sum_{i=1}^{n}[g(i)],\quad n=1,2,\cdots,$$

$$f(x)=(g(x))^{r(p-1)/(r-p)}(\log g(x))^{r/(r-p)})\log\log g(x),\quad x\geqslant 0.$$

又设$\{Y_n,n\geqslant 1\}$是独立随机变量序列，满足

$$P\{Y_n=\pm f^{1/r}(n)\}=\frac{1}{2f(n)},\quad P\{Y_n=0\}=1-\frac{1}{f(n)}.$$

对 $G(j-1)<n\leqslant G(j)$，定义 $X_n=Y_j$．序列$\{X_n,n\geqslant 1\}$有下列性质：

(8.5.24)　$EX_n=0,\quad E|X_n|^r=1,$

(8.5.25)　$\alpha(n)=O(n^{-r(p-1)/(r-p)}\log^{-r/(r-p)}n(\log\log n)^{-1}),$

(8.5.26)对任意的 $\epsilon>0$，$\sum\limits_{n=1}^{\infty}n^{p\alpha-2}P\{|S_n|\geqslant\epsilon n^{\alpha}\}=\infty.$

**例8.5.2** 令 $r>1$．记

$$a=(r-1)/r,\quad g(n)=[n^{a-1}\exp(n^{a})],\quad G(n)=\sum_{i=1}^{n}g(i).$$

设$\{Y_n,n\geqslant 1\}$是独立随机变量序列，满足

$$P\{Y_n=\pm n^{1/r}\log^{1/r}n\}=\frac{1}{2n\log n},\quad P\{Y_n=0\}=1-\frac{1}{n\log n}.$$

对 $G(j-1)<n\leqslant G(j)$，定义 $X_n=Y_j$. 序列 $\{X_n,n\geqslant 1\}$ 有下列性质：

$$EX_n=0,\quad E|X_n|^r=1,$$

$$\alpha(n)=O(\log^{-r/(r-1)}(\log\log n)^{-1}),$$

$$\text{对任意的 }\varepsilon>0,\ \sum_{n=1}^{\infty}\frac{1}{n}P\{|S_n|\geqslant\varepsilon n\}=\infty.$$

**注8.5.1** 我们现在可来讨论定理8.5.0的结论在 $r<\infty$ 时是否成立. 假设 $1<p\leqslant r/2$. 于是(8.5.1)对 $\theta>8$($p$ 充分大)是满足的. 但例8.5.1告诉我们混合速度至少为 $n^{-(p-1)/2}$. 这就是说定理8.5.0很可能是不成立的. 但是，例8.5.1中的 $\{X_n,n\geqslant 1\}$ 并不是强平稳的. 邵启满(1993c)猜测：存在一个强平稳 $\alpha$ 混合序列，满足(8.5.24)，(8.5.26)且 $\alpha(n)=O(n^{-r(p-1)/(r-p)}\log^{-1}n)$. 他又猜测定理8.5.1中的假设 $\beta>rp/(r-p)$ 可以用 $\beta>r/(r-p)$ 代替.

## §8.6 混合序列部分和的完全收敛性的进一步讨论

令 $\{X_n,n\geqslant 1\}$ 是 $i.i.d.$ 随机变量序列. 记 $S_n=\sum_{i=1}^{n}X_i$. 假设 $H(t)$ 和 $\psi(t)$ 是定义在 $(0,\infty)$ 上的正函数，当 $t\to\infty$ 时 $H(t)\uparrow\infty$. 记 $\hat{\psi}(t)=\int_0^t\psi(u)du$, $t>0$,

$$\nu(\varepsilon)\sum_{n=1}^{\infty}I((|S_n|\geqslant\varepsilon H(n)),$$

$$\eta(\varepsilon)=\sup_{n\geqslant 1}\{(|S_n|-\varepsilon H(n))^{+}/\varepsilon\},$$

$$\chi(\varepsilon)=\sup\{n\geqslant 1;\ |S_n|\geqslant\varepsilon H(n)\},$$

其中 $\varepsilon$ 是一任意正数. Prohorov 曾提出过三个问题：

(1) 对任意 $\varepsilon>0$

$$\sum_{n=1}^{\infty}\psi(n)P\{|S_n|\geqslant\varepsilon H(n)\}<\infty \tag{8.6.1}$$

是否等价于对任意 $\varepsilon>0$

$$E\hat{\psi}(\nu(\varepsilon))<\infty?$$

(2) 对任意 $\varepsilon>0$

$$(8.6.2)\qquad \sum_{n=1}^{\infty}\psi(n)P\left\{\max_{1\leq k\leq n}|S_k|\geq \varepsilon H(n)\right\}<\infty$$

是否等价于对任意 $\varepsilon>0$

$$E\dot{\psi}(\mathrm{inv}H(\eta(\varepsilon)))<\infty?$$

(3) 对任意 $\varepsilon>0$

$$(8.6.3)\qquad \sum_{n=1}^{\infty}\psi(n)P\left\{\sup_{k\geq n}|S_k|/H(k)\geq \varepsilon\right\}<\infty$$

是否等价于对任意 $\varepsilon>0$

$$E\dot{\psi}(\chi(\varepsilon))<\infty?$$

令

$$M_1=\{\psi:\psi(x)>0,x\in[1,\infty],\text{存在 }\delta_1,\delta_2>0,\text{使得当 }x\to\infty\text{ 时 }x^{1-\delta_1}\psi(x)\uparrow,e^{-\delta_2 x}\psi(x)\downarrow\}.$$

Sirazgimov 和 Gafrov(1987)讨论了这些问题且在下列条件下:

$$(8.6.4)\qquad \limsup_{x\to\infty}H(Cx)/H(x)<\infty,\forall\ C>1;$$

$$(8.6.5)\qquad \liminf_{x\to\infty}P\{S_n\geq-\varepsilon H(n)\}>0,\forall\ \varepsilon>0$$

和

$$\psi\in M_1\text{或当 }x\to\infty\text{ 时 }\psi(x)\uparrow,$$

证明了下面三个结论是等价的:

$$(8.6.6)\ P(\psi,H,\varepsilon):=\sum_{n=1}^{\infty}\psi(n)P\{S_n\geq\varepsilon H(n)\}<\infty,\forall\varepsilon>0,$$

$$(8.6.7)\ M(\psi,H,\varepsilon):=\sum_{n=1}^{\infty}\psi(n)P\left\{\max_{1\leq k\leq n}S_k\geq\varepsilon H(n)\right\}<\infty,\forall\varepsilon>0,$$

$$(8.6.8)\ S(\psi,H,\varepsilon):=\sum_{n=1}^{\infty}\psi(n)P\left\{\sup_{k\geq n}S_k/H(k)\geq\varepsilon\right\}<\infty,\forall\varepsilon>0.$$

苏淳(1989)研究了这些问题,给出了如下结果.假设满足下列条件:

(A) 存在常数 $C>0,t_0>0$,使对任意的 $t\geq t_0,\psi(t)\leq c\dot{\psi}(t)$;

(B) $t\psi(t)\uparrow\infty(t\to\infty)$,存在 $c'>0,t_0>0$,使对任意的 $t\geq t_0$

$$\int_0^t u\psi(u)du\geq c't^2\psi(t);$$

(C) $0<\beta_1\leqslant H(t)/t\leqslant\beta_2<\infty$或者 $H(t)/t\uparrow\infty(t\to\infty)$或者 $H(t)/t\downarrow 0$且 $H^2(t)/t\uparrow\infty(t\to\infty)$；

(D) 存在正整数 $N$ 和 $\delta_2>0$,使对任意的整数 $k>0$

$$\sum_{n=1}^{N^{k+1}}\psi(n)\geqslant(1+\delta_2)\sum_{n=1}^{N^k}\psi(n).$$

那么(8.6.1)⇔(8.6.2)⇔(8.6.3)⇔$E\hat{\psi}(x(\varepsilon))<\infty$.

王岳宝(1993)对强平稳的 $\rho$ 混合序列讨论了 Prohorov 的三个问题.记

(8.6.9) $\varphi^*(1)=\sup\limits_k\ \sup\limits_{A\in\mathscr{F}_1^k,B\in\mathscr{F}_{k+1}^\infty,P(A)P(B)>0}\ \max$

$\{|P(B|A)-P(B)|,|P(A|B)-P(A)|\}$,

(8.6.10) $\psi^*(x)=\dfrac{1}{x}\left(x\psi(x)-\left[\dfrac{x}{2}\right]\psi\left(\left[\dfrac{x}{2}\right]\right)\right),x\geqslant1,$

$M_1^*=\{\psi:\psi(x)>0,x\geqslant1,$

$n\psi(n)\to\infty,n\psi(n)>[n/2]\psi([n/2])\}$.

他证明了下列定理.

**定理8.6.1** 设$\{X_n,n\geqslant1\}$是强平稳 $\rho$ 混合序列,满足

(8.6.11) 对任意 $\varepsilon>0,\liminf\limits_{n\to\infty}P\{S_n\geqslant-\varepsilon H(n)\}>\varphi^*(1)$.

又设 $\psi\in M_1^*,H(n)$满足(8.6.4),

(8.6.12) $$\sum_{n=1}^{\infty}\psi(n)\rho(n)<\infty$$

且对 $\psi\in M_1^*-M_1$

(8.6.13) $P\left\{\sup\limits_{k\geqslant n}\dfrac{S_k}{H(k)}\geqslant\varepsilon\right\}\geqslant CP\left\{\inf\limits_{k\geqslant n}\dfrac{S_k}{H(k)}\leqslant-c\varepsilon\right\},$

$$n=1,2,\cdots,$$

其中 $c$ 是一常数.那么对任意的 $\varepsilon>0$,(8.6.6)⇔(8.6.7)⇔

(8.6.8)[1] $$S(\psi^*,H,\varepsilon)<\infty.$$

定理8.6.1的证明通过几个引理给出.

**引理8.6.1** 如果对任意的 $x>0$

(8.6.11)′ $\liminf\limits_{n\to\infty}\ \inf\limits_{k\leqslant n-1}P\{S_n-S_k\geqslant-x\}>\varphi^*(1).$

那么存在常数 $C>0$使对任意的 $y$ 和 $n\in N$

(8.6.14) $$P\left\{\max_{k\leqslant n}S_k \geqslant y\right\}\leqslant CP\{S_n \geqslant y - x\}.$$

**证** 由 $\varphi^*(1)$的定义和(8.6.11)′我们有

$$P\{S_n \geqslant y - x\} \geqslant P\left\{S_n \geqslant y - x,\max_{k\leqslant n}S_k \geqslant y\right\}$$

$$= \sum_{k=1}^{n}P\left\{S_n \geqslant y - x,S_k \geqslant y,\max_{i\leqslant k-1}S_i < y\right\}$$

$$\geqslant \sum_{k=1}^{n}P\left\{S_n - S_k \geqslant - x,S_k \geqslant y,\max_{i\leqslant k-1}S_i < y\right\}$$

$$\geqslant \sum_{k=1}^{n}\left[P\{S_n - S_k \geqslant - x\}P\left\{S_k \geqslant y,\max_{i\leqslant k-1}S_i < y\right\}\right.$$

$$\left. - \varphi^*(1)P\left\{S_k \geqslant y,\max_{i\leqslant k-1}S_i < y\right\}\right]$$

$$\geqslant CP\{\max_{k\leqslant n}S_k \geqslant y\},n = 1,2,\cdots.$$

**引理8.6.2** 设$\{X_n,n\geqslant1\}$是强平稳序列,满足(8.6.11),又设 $n\psi(n)\geqslant C,n=1,2,\cdots$. 那么(8.6.6)$\Leftrightarrow$(8.6.7).

**证** 只需证明(8.6.6)$\Rightarrow$(8.6.7). 取(8.6.14)中的 $y=\varepsilon H(n)$,$x=\varepsilon H(n)/2$.如能验证(8.6.11)′就足够了.首先我们有

(8.6.15) $$P\{S_n \geqslant \varepsilon H(n)\} \to 0,n \to \infty.$$

否则,应有$\{n_i\}$,$r$ 和 $\varepsilon_0>0$使得

$$P\{S_{n_i} \geqslant \varepsilon_0H(n_i)\} \geqslant r > 0.$$

不失一般性可设 $n_{i+1}\geqslant3n_i,i=1,2,\cdots$. 于是,由(8.6.11),对任意的$2n_i\leqslant n\leqslant3n_i,i=1,2,\cdots$对某$r_1>0$我们有

$$P\left\{S_n \geqslant \frac{\varepsilon_0}{2}H(n)\right\}\geqslant P\left\{S_{n_i} \geqslant \varepsilon_0H(n),S_n - S_{n_i} \geqslant - \frac{\varepsilon_0}{2}H(n)\right\}$$

$$\geqslant P\{S_{n_i} \geqslant \varepsilon_0H(n)\}\left[P\left\{S_n - S_{n_i} \geqslant - \frac{\varepsilon_0}{2}H(n)\right\}- \varphi^*(1)\right]$$

$$\geqslant P\{S_{n_i} \geqslant \varepsilon_0H(n)\}\left[P\left\{S_{n-n_i} \geqslant - \frac{\varepsilon_0}{2}H(n - n_i)\right\}- \varphi^*(1)\right]$$

$$\geqslant r_1 > 0.$$

从 $n\psi(n)\geqslant c>0,n=1,2,\cdots$,可得

$$P(\psi,H,\varepsilon_0/2)\geqslant\sum_{i=1}^{\infty}\sum_{m=2ni}^{3ni}\psi(m)P\{S_m\geqslant\varepsilon_0H(m)/2\}$$

$$\geqslant r_1\sum_{i=1}^{\infty}\sum_{m=2ni}^{3ni}m\psi(m)/m$$

$$\geqslant r_1C\sum_{i=1}^{\infty}\sum_{m=2ni}^{3ni}\frac{1}{m}=\infty,$$

与(8.6.6)矛盾. 因此从(8.6.15)和(8.6.11)得到: 对任意 $\varepsilon>0$

$$\liminf_{n\to\infty}\inf_{k\leqslant n-1}P\{S_n-S_k\geqslant-2\varepsilon H(n)\}$$

$$\geqslant\liminf_{n\to\infty}\inf_{k\leqslant n-1}P\{S_n\geqslant-\varepsilon H(n),S_k<\varepsilon H(n)\}$$

$$\geqslant\liminf_{n\to\infty}\inf_{k\leqslant n-1}(P\{S_n\geqslant-\varepsilon H(n)\}-P\{S_k\geqslant\varepsilon H(n)\})$$

$$>\varphi^*(1).$$

引理8.6.2证毕.

**引理8.6.3** 设 $\{X_n\}$ 是随机变量序列, $H(x)$ 满足(8.6.4)且 $n\psi(n)>[n/2]\psi([n/2])$, $n=1,2,\cdots$, 那么(8.6.7)⇒(8.6.8)′.

**证** 注意到 $n\psi(n)\uparrow$ 和 $H(n)/H(2n)\geqslant\delta$, 我们有

$$\sum_{n=2}^{\infty}\frac{n\psi(n)-[n/2]\psi([n/2])}{n}P\left\{\sup_{i\geqslant n}\frac{S_i}{H(i)}\geqslant\varepsilon\right\}$$

$$\leqslant\sum_{j=1}^{\infty}\sum_{n=2^j}^{2^{j+1}-1}\frac{n\psi(n)-[n/2]\psi([n/2])}{n}P\left\{\sup_{i\geqslant 2^{j-1}}\frac{S_i}{H(i)}\geqslant\varepsilon\right\}$$

$$\leqslant\sum_{j=1}^{\infty}\frac{1}{2^j}\sum_{n=2^j}^{2^{j+1}-1}(n\psi(n)-[n/2]\psi([n/2]))$$

$$\sum_{k=j}^{\infty}P\left\{\sup_{2^k\leqslant i<2^{k+1}}\frac{S_i}{H(i)}\geqslant\varepsilon\right\}$$

$$\leqslant\sum_{j=1}^{\infty}\left(\frac{1}{2^j}\sum_{n=2^j}^{2^{j+1}-1}n\psi(n)-\frac{1}{2^{j-1}}\sum_{n=2^{j-1}}^{2^j-1}n\psi(n)\right)$$

$$\cdot\sum_{k=j}^{\infty}P\left\{\sup_{i\leqslant 2^k}S_i\geqslant\varepsilon H(2^{k-1})\right\}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{n=2^k}^{2^{k+1}-1} n\psi(n) P\left\{\sup_{i\leqslant 2^k} S_i \geqslant \varepsilon H(2^{k-1})\right\}$$

$$\leqslant 2\sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \psi(n) P\left\{\sup_{i\leqslant 2^k} S_i \geqslant \varepsilon\delta^2 H(2^{k+1})\right\}$$

$$\leqslant 2\sum_{n=2}^{\infty} \psi(n) P\left\{\sup_{i\leqslant n} S_i \geqslant \varepsilon\delta^2 H(n)\right\}.$$

由 $\varepsilon$ 的任意性和(8.6.7),即得引理8.6.3.

**引理8.6.4** 在定理8.6.1的条件下,(8.6.8)′⇒(8.6.7).

**证** 对任给的 $\varepsilon>0$,记 $\varepsilon_1=\max\{\varepsilon, c\varepsilon\}$. 由(8.6.13)我们有

(8.6.16) $$(1+C^{-1})P\left\{\sup_{i\geqslant 2^j} S_i/H(i)\geqslant\varepsilon\right\}$$

$$\geqslant P\left\{\sup_{i\geqslant 2^j} S_i/H(i)\geqslant\varepsilon\right\}+P\left\{\inf_{i\geqslant 2^j} S_i/H(i)\leqslant -c\varepsilon\right\}$$

$$\geqslant P\left\{\sup_{i\geqslant 2^j} |S_i|/H(i)\geqslant\varepsilon_1\right\}.$$

由(8.6.4),存在 $0<\delta<1$,使对任意的 $x\geqslant 1$ 有

(8.6.17) $$\delta H(2x)\leqslant H(x).$$

由此我们有

$$\{\sup_{i\geqslant 2^j} |S_i|/H(i)\geqslant\varepsilon_1\}=\bigcup_{i=j}^{\infty}\bigcup_{k=2^i}^{2^{i+1}-1}\{|S_k|\geqslant\varepsilon_1 H(k)\}$$

$$\supset\bigcup_{i=j}^{\infty}\bigcup_{k=2^i}^{2^{i+1}-1}\left\{|S_k|\geqslant\frac{\varepsilon_1}{\delta}H(2^i)\right\}\cup\bigcup_{i=j}^{\infty}\left\{|S_{2^i}|\geqslant\frac{\varepsilon_1}{\delta}H(2^i)\right\}$$

$$\supset\bigcup_{i=j}^{\infty}\bigcup_{k=2^i}^{2^{i+1}-1}\left\{|S_k-S_{2^i}|\geqslant\frac{2\varepsilon_1}{\delta}H(2^i)\right\}$$

$$=\left\{\sup_{i\geqslant j}\max_{2^i\leqslant k\leqslant 2^{i+1}}\frac{|S_k-S_{2^i}|}{H(2^i)}\geqslant\frac{2\varepsilon_1}{\delta}\right\}.$$

结合(8.6.16)就得

(8.6.18) $$P\left\{\sup_{i\geqslant 2^j} S_i/H(i)\geqslant\varepsilon_1\delta\right\}$$

$$\geqslant(1+C^{-1})^{-1}P\left\{\sup_{i\geqslant j}\max_{2^i\leqslant k<2^{i+1}}\frac{|S_k-S_{2^i}|}{H(2^i)}\geqslant 2\varepsilon_1\right\}.$$

记

$$B_i = \left\{ \max_{2^i \leqslant k < 2^{i+1}} \frac{S_k - S_{2^i}}{H(2^i)} \geqslant 2\varepsilon_1 \right\}, A_i = \bigcap_{l=i+1}^{\infty} B_l^c,$$

$$B_i^* = \left\{ \max_{2^i \leqslant k < 2^{i+1} - 2^{i-1}} \frac{S_k - S_{2^i}}{H(2^i)} \geqslant 2\varepsilon_1 \right\}.$$

容易验证：任何混合序列都服从0-1律. 因为$\{B_i, i, o.\} \in \bigcap_{n=1}^{\infty} \sigma(X_k, k \geqslant n)$，故有$P(B_i, i, o.) = 1$或0. 如果$P(B_i, i, o.) = 1$，那么

$$P\left\{ \sup_{i \geqslant j} \max_{2^i \leqslant k \leqslant 2^{i+1}} \frac{S_k - S_{2^i}}{H(2^i)} \geqslant 2\varepsilon_1 \right\} = P\{\bigcup_{i=j}^{\infty} B_i\} = 1, j = 1, 2, \cdots.$$

由这一不等式，(8.6.18)和$M_1^*$的性质，我们有

(8.6.19) $$S(\psi^*, H, \varepsilon_0)$$

$$\geqslant \sum_{j=2}^{\infty} 2^{-j} \Big( \sum_{n=2^{j-1}}^{2^j - 1} n\psi(n) - 2 \sum_{n=2^{j-2}}^{2^{j-1}-1} n\psi(n) \Big)$$

$$\cdot P\left\{ \sup_{i \geqslant 2^j} \frac{S_i}{H(i)} \geqslant \varepsilon_0 \right\}$$

$$\geqslant c \sum_{j=2}^{\infty} \Big( 2^{-j} \sum_{n=2^{j-1}}^{2^j - 1} n\psi(n) - 2^{-(j-1)} \sum_{n=2^{j-2}}^{2^{j-1}-1} n\psi(n) \Big)$$

$$= c \lim_{N\to\infty} \Big( 2^{-N} \sum_{n=2^{N-1}}^{2^N - 1} n\psi(n) - \frac{1}{2}\psi(1) \Big) = \infty.$$

事实上，对任给$M > 0$存在$N_0$，当$N \geqslant N_0, n \geqslant 2^{N_0 - 1}$有$n\psi(n) \geqslant 2M$，所以$2^{-N} \sum_{n=2^{N-1}}^{2^N - 1} n\psi(n) \geqslant M$. (8.6.19)与(8.6.8)′矛盾. 因此

$$P\{B_i, i. o.\} = 0,$$

也就是说，当$i\to\infty$时$P(\bigcup_{l=i+1}^{\infty} B_l) \to 0$. 这就证明了当$i\to\infty$时$P(A_i) \to 1$. 将这一结果与(8.6.17)相结合，并注意到平稳性及$\rho$混合的定义，对充分大的$j$我们有

(8.6.20) $$P\left\{ \sup_{i \geqslant j} \max_{2^i \leqslant k \leqslant 2^{i+1}} \frac{S_k - S_{2^i}}{H(2^i)} \geqslant 2\varepsilon_1 \right\} = P\{\bigcup_{i=j}^{\infty} B_i\}$$

$$\geqslant P\{\bigcup_{i=j}^{\infty} B_i A_i\} = \sum_{i=j}^{\infty} P(B_i A_i) \geqslant \sum_{i=j}^{\infty} P(B_i^* A_i)$$

$$\geqslant \sum_{i=j}^{\infty}\{P(B_i^*)P(A_i)-\rho(2^{i-1})\}$$

$$\geqslant c\sum_{i=j}^{\infty}\{P(B_i^*)-c\rho(2^{i-1})\}$$

$$\geqslant c\sum_{i=j}^{\infty}\left\{P\left\{\max_{k\leqslant 2^{i-1}}S_k\geqslant\frac{2\varepsilon_1}{\delta}H(2^{i-1})\right\}-c\rho(2^{i-1})\right\}.$$

从 $n\psi(n)>[n/2]\psi([n/2])$ 可得

$$\sum_{n=2^{j-1}}^{2^j-1}n\psi(n)\geqslant\sum_{n=2^{j-2}}^{2^{j-1}-1}n\psi(n).$$

将这一结论与(8.6.18)，(8.6.20)及(8.6.12)相结合，对任意的 $\varepsilon>0$ 我们有

$$\infty>S(\psi^*,H,\varepsilon\delta)$$

$$\geqslant c\sum_{j=2}^{\infty}2^{-j}\left(\sum_{n=2^{j-1}}^{2^j-1}n\psi(n)-2\sum_{n=2^{j-2}}^{2^{j-1}-1}n\psi(n)\right)$$

$$\cdot P\left\{\sup_{i\geqslant j}\max_{2^i\leqslant k<2^{i+1}}\frac{S_k}{H(k)}\geqslant\varepsilon_1\delta\right\}$$

$$\geqslant c\sum_{j=2}^{\infty}2^{-j}\left(\sum_{n=2^{j-1}}^{2^j-1}n\psi(n)-2\sum_{n=2^{j-2}}^{2^{j-1}-1}n\psi(n)\right)$$

$$P\left\{\sup_{i\geqslant j}\max_{2^i\leqslant k<2^{i+1}}\frac{S_k-S_{2^i}}{H(2^i)}\geqslant 2\varepsilon_1\right\}$$

$$\geqslant c\sum_{i=2}^{\infty}\sum_{j=2}^{i}\left(2^{-j}\sum_{n=2^{j-1}}^{2^j-1}n\psi(n)-2^{-(j-1)}\sum_{n=2^{j-2}}^{2^{j-1}-1}n\psi(n)\right)P(B_i^*A_i)$$

$$\geqslant c\sum_{i=2}^{\infty}2^{-i}\sum_{n=2^{i-2}}^{2^{i-1}-1}n\psi(n)P(B_i^*A_i)$$

$$\geqslant c\sum_{i=2}^{\infty}\sum_{n=2^{i-2}}^{2^{i-1}-1}\psi(n)\left(P\left\{\max_{k\leqslant 2^{i-1}}S_k\geqslant\frac{2\varepsilon_1}{\delta}H(2^{i-1})\right\}-c\rho(2^{i-1})\right)$$

$$\geqslant c\sum_{n=1}^{\infty}\psi(n)P\left\{\max_{k\leqslant n}S_k\geqslant\frac{2\varepsilon_1}{\delta}H(n)\right\}-c\sum_{n=1}^{\infty}\psi(n)\rho(n)$$

$$=cM(\psi,H,2\varepsilon_1/\delta)-c\sum_{n=1}^{\infty}\psi(n)\rho(n).$$

引理8.6.4证毕.

定理8.6.1从引理8.6.1—8.6.4得到.

**注8.6.1** 条件(8.6.13)仅在引理8.6.4证明中被用到.当考虑双侧尾概率级数时,条件(8.6.13)就可被免除.

对独立随机变量序列 Prohorov 问题的进一步讨论由苏淳(1986)和邵启满(1988)等给出.

# 第九章　强逼近

关于混合随机变量序列的重对数律和强不变原理在 60 年代已被若干作者所讨论. Philipp 和 Stout 的专著(1975)系统地应用 Strasseen 鞅嵌入方法于相依随机变量部分和的强逼近,对 $\alpha$ 混合序列函数,强平稳 $\varphi$ 混合序列,缺项三角级数,一类 Gauss 序列及马氏链可加泛函等建立了强不变原理. 对强平稳 $\varphi$ 混合序列及强平稳 $\rho$ 混合序列,Berkes 和 Philipp(1979)、Dabrowski(1982)及 Bradley(1985)分别作了讨论,获得了进一步的结果. 但这些结果已被陆传荣和邵启满作了本质上的改进. 在邵启满和陆传荣(1986)中,他们对 $\varphi$ 混合序列获得了一个更好的逼近阶. 这些将在 §9.1 中介绍. 对强平稳 $\varphi$ 混合随机变量序列 $\{X_n, n\geqslant 1\}$, $EX_1=0$, $EX_1^2<\infty$ 且 $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$ 时,Heyde 和 Scott(1973)首先给出了部分和 $S_n$ 用 Wiener 过程 $W(n)$ 强逼近的阶为 $O((n\log\log n)^{1/2})$. 邵启满(1989,1993b)改进这一结果,对 $\varphi$ 混合和 $\rho$ 混合序列,当 2 阶矩有限且混合速度为 $O((\log n)^{-1-\varepsilon})$ 时,给出了同样的强逼近阶. 由此可推得重对数律. 当 $2+\delta(\delta>0)$ 阶矩存在时,他给出了进一步的结果. 由此可推出钟重对数律. 这些将被讨论于 §9.2 中. 邵启满和陆传荣(1987)及邵启满(1989a)也研究了 $\alpha$ 混合序列的强逼近. 这些也改进了前人的结果,将被讨论于 §9.3 中.

## §9.1　$\varphi$ 混合序列的强逼近

设 $\{X_n, n\geqslant 1\}$ 是 $\varphi$ 混合随机变量序列, $EX_n=0$. 令 $S_n=\sum_{k=1}^{n}X_k$, $S(t)=S_{[t]}(t>0)$. 在本节中,我们首先讨论在应用

Strassen 的鞅嵌入方法时，对 $\varphi$ 混合序列部分和的强逼近的速度有多快. 由邵启满和陆传荣(1986)所得到的强逼近速度已接近对于 $i.i.d.$ 情形理想的阶 $O((n\log\log n)^{1/4}(\log n)^{1/2})$. 证明着下述定理.

**定理 9.1.1** (邵启满，陆传荣 1986). 设 $\{X_n,n\geq 1\}$ 是 $\varphi$ 混合序列，$EX_n=0$. 假设

(i) 对某 $c>0$，$\sigma_n^2=ES_n^2\geq cn$，

(ii) $\sup\limits_{n\geq 1}E|X_n|^4<\infty$，

(iii) $\varphi(n)=O(1/n)$.

那么可在一较大的概率空间上，在其上有一标准 Wiener 过程 $\{W(t),t\geq 0\}$，不改变 $\{S(t),t\geq 0\}$ 的分布重新定义过程 $\{S(t),t\geq 0\}$，使得对任一 $\varepsilon>0$ 当 $t\to\infty$ 有

(9.1.1) $$S(t)-W(\sigma_t^2)=O(\sigma_t^{1/2}(\log\sigma_t)^{9/4+\varepsilon})\quad \text{a.s.}$$

其中 $\sigma_t^2=\sigma_{[t]}^2$.

从定理 9.1.1 我们即可写出下述推论.

**推论 9.1.1** 设 $\{X_n,n\geq 1\}$ 是平稳 $\varphi$ 混合随机变量序列，$EX_1=0$. 若

(9.1.2) $$\sigma^2=EX_1^2+2\sum_{k=2}^{\infty}EX_1X_k$$

绝对收敛，不妨设 $\sigma=1$，那么在条件(ii)和(iii)下，有

(9.1.3) $$S(t)-W(t)=O(t^{1/4}(\log t)^{9/4+\varepsilon})\quad \text{a.s.}$$

为证明定理 9.1.1，我们首先给出一个基本命题. 命题的证明指明了如何对混合序列应用 Strassen 鞅嵌入方法以获得强逼近结果.

设 $\mathscr{F}_0=\{\phi,\Omega\}$，$\{\mathscr{F}_n,n\geq 0\}$ 是一不减的 $\sigma$ 域序列. 假设 $X_n$ 是 $\mathscr{F}_n$ 可测的，$n=1,2,\cdots$. 对每一 $n\geq 1$ 定义

(9.1.4) $$\begin{aligned}Y_n&=\sum_{k=0}^{2^n}\{E(X_{n+k}|\mathscr{F}_n)-E(X_{n+k}|\mathscr{F}_{n-1})\}\\&=X_n+u_n-u_{n-1}-v_n\end{aligned}$$

其中

(9.1.5) $u_n = \sum_{k=1}^{2^n} E(X_{n+k} | \mathscr{F}_n)$, $v_n = \sum_{k=2^{n-1}}^{2^n} E(X_{n+k} | \mathscr{F}_{n-1})$,

易见$\{Y_n, \mathscr{F}_n, n \geqslant 1\}$是一鞅差序列.

**命题 9.1.1** 设$\{X_n, n \geqslant 1\}$是随机变量序列, $EX_n = 0$, $\sup_n E|X_n|^{2+\delta} < \infty (0 < \delta \leqslant 2)$. 设$\mathscr{F}_n = \sigma\{X_k, 1 \leqslant k \leqslant n\}$是自然$\sigma$域序列. 若下述条件被满足:

(a) 对某$c > 0$, $\sigma_n^2 = ES_n^2 \geqslant cn$,

(b) $\| u_n \|_{2+\delta} = O(1)$, $\sum_{n=1}^{\infty} \| u_n \|_{2+\delta} < \infty$,

(c) 记$T_m(n) = \sum_{m < k \leqslant m+n} (Y_k^2 - EY_k^2)$, 对某$\lambda \geqslant 0$

(9.1.6) $$E|T_m(n)|^{(2+\delta)/2} = O(n(\log n)^\lambda),$$

那么存在一个概率空间, 在其上有一标准 Wiener 过程$\{W(t), t \geqslant 0\}$及与$\{S(t), t \geqslant 0\}$同分布的过程, 仍记作$\{S(t)\}$, 使对任给$\varepsilon > 0$, 当$t \to \infty$时有

(9.1.7) $$S(t) - W(\sigma_t^2) = O(\sigma_t^{2/(2+\delta)} (\log \sigma_t)^{1+\varepsilon+(1+\lambda)/(2+\delta)}) \quad \text{a.s.}$$

特别地, 对$\delta = 2$我们有

(9.1.8) $$S(t) - W(\sigma_t^2) = O(\sigma_t^{1/2} (\log \sigma_t)^{(5+\lambda)/4+\varepsilon}) \quad \text{a.s.}$$

**证** 1) 首先我们来证对每一$\varepsilon > 0$,

(9.1.9) $$S(t) - \sum_{k \leqslant t} Y_k = O(t^{1/(2+\delta)} (\log t)^{(1+\varepsilon)/(2+\delta)}) \quad \text{a.s.}$$

事实上, 注意到

$$S(t) - \sum_{k \leqslant t} Y_k = \sum_{k \leqslant t} v_k - u_{[t]},$$

利用条件(b)及 Borel-Cantelli 引理, 就得(9.1.9)成立.

现在对鞅差序列$\{Y_n, \mathscr{F}_n\}$应用 Skorohod-Strassen 鞅嵌入定理, 存在一概率空间, 在其上有一标准 Wiener 过程$\{W(t), t \geqslant 0\}$及非负停时序列$\{T_n\}$, 使得

$$\left\{W\left(\sum_{j \leqslant n} T_j\right), n \geqslant 1\right\} 和 \left\{\sum_{j \leqslant n} Y_j, n \geqslant 1\right\}$$

有相同的联合分布. 所以不失一般性, 可重新定义

$$Y_n = W\Big(\sum_{j\leqslant n} T_j\Big) - W\Big(\sum_{j<n} T_j\Big),$$

于这一新的概率空间上. 现令

$$\mathscr{F}_n = \sigma\Big\{W\Big(\sum_{j\leqslant k} T_j\Big); 1\leqslant k\leqslant n\Big\},$$

$$\mathscr{G}_n = \sigma\Big\{W(t); 0\leqslant t\leqslant \sum_{j\leqslant n} T_j\Big\}.$$

容易看出 $\mathscr{F}_n \subseteq \mathscr{G}_n$, $T_n$ 是 $\mathscr{G}_n$ 可测的, 且对每一 $n\geqslant 1$

$$(9.1.10)\quad E(T_n|\mathscr{G}_{n-1}) = E(Y_n^2|\mathscr{G}_{n-1}) = E(Y_n^2|\mathscr{F}_{n-1}) \quad \text{a.s.}$$

其次, 对每一 $1<p\leqslant 2$ 有

$$E|T_n|^p \leqslant E|Y_n|^{2p}.$$

2) 写

$$(9.1.11)\quad S(t) - W(\sigma_t^2) = S(t) - \sum_{k\leqslant t} Y_k + W\Big(\sum_{k\leqslant t} T_k\Big) - W(\sigma_t^2).$$

为估计(9.1.11)右边第二个差, 我们需要估计 $\sum_{k\leqslant t} T_k - \sigma_t^2$. 写

$$(9.1.12)\quad \sum_{k\leqslant t} T_k - \sigma_t^2 = \sum_{k\leqslant t}\{T_k - E(T_k|\mathscr{G}_{k-1})\} + \sum_{k\leqslant t}\{E(Y_k^2|\mathscr{F}_{k-1}) - Y_k^2\} + \Big\{\sum_{k\leqslant t} Y_k^2 - \sigma_t^2\Big\}.$$

我们来证

$$(9.1.13)\quad \sum_{k\leqslant t} T_k - \sigma_t^2 = O(\sigma_t^{4/(2+\delta)}(\log\sigma_t)^{1+\varepsilon+2(1+\lambda)/(2+\delta)}) \quad \text{a.s.}$$

令 $R_j = Y_j^2 - E(Y_j^2|\mathscr{F}_{j-1})$. 那么 $\{R_j, \mathscr{F}_j\}$ 是鞅差序列, 由 b) 得

$$E|R_j|^{(2+\delta)/2} \leqslant 16E|Y_j|^{2+\delta} = O(1).$$

因此由鞅的基本定理(参见 Chow 1965), 我们有

$$(9.1.14)\quad \sum_{k\leqslant t} R_k = O(t^{2/(2+\delta)}(\log t)^{(2+\varepsilon)/(2+\delta)}) \quad \text{a.s.}$$

类似地有

$$(9.1.15)\quad \sum_{k\leqslant t}(T_k - E(T_k|\mathscr{G}_{k-1})) = O(t^{2/(2+\delta)}(\log t)^{(2+\varepsilon)/(2+\delta)}) \quad \text{a.s.}$$

对于(9.1.12)右边的第三项, 由条件(c), Moricz 定理及 Borel-Cantelli 引理易证

$$(9.1.16)\quad \sum_{k\leqslant t}(Y_k^2 - EY_k^2) = O(t^{2/(2+\delta)}(\log t)^{1+\varepsilon+2(1+\lambda)/(2+\delta)}) \quad \text{a.s.}$$

注意到$\{Y_k,\mathscr{F}_k\}$是鞅差序列，由条件(a)，(b)及 Schwarz 不等式有

$$(9.1.17)\quad \sum_{k\leqslant t} EY_k^2-\sigma_t^2=E\Big(\sum_{k\leqslant t}Y_k\Big)^2-\sigma_t^2$$

$$=2E\Big(\sum_{k\leqslant t}X_k\Big)\Big(u_t-\sum_{k\leqslant t}v_k\Big)+E\Big(u_t-\sum_{k\leqslant t}v_k\Big)^2$$

$$=O(\sigma_t).$$

因此由(9.1.16)，(9.1.17)和条件(a)，我们有

$$(9.1.18)\quad \sum_{k\leqslant t}Y_k^2-\sigma_t^2=O(\sigma_t^{4/(2+\delta)}(\log\sigma_t)^{1+\varepsilon+2(1+\lambda)/(2+\delta)})\quad \text{a.s.}$$

从(9.1.14)，(9.1.15)和(9.1.18)就得(9.1.13).

现在由(9.1.13)并按照 Hanson 和 Russo(1983)的定理 3.2.B 的证明，可得

$$W\Big(\sum_{k\leqslant t}T_k\Big)-W(\sigma_t^2)=O(\sigma_t^{2/(2+\delta)}(\log\sigma_t)^{1+\varepsilon+(1+\lambda)/(2+\delta)})\quad \text{a.s.}$$

结合(9.1.11)和(9.1.9)得证命题 9.1.1 成立.

**定理 9.1.1 的证明**　我们来验证命题 9.1.1 的条件(b)和(c)被满足. 由引理 1.2.8 和引理 2.2.8，我们有

$$\|u_n\|_{2+\delta}^{2+\delta}=E|u_n|^{2+\delta}=E(u_n(\mathrm{sgn}u_n)|u_n|^{1+\delta})$$

$$=E(X_{n+1}|u_n|^{1+\delta}\mathrm{sgn}u_n)$$

$$+\sum_{i=0}^{n-1}E\Big(\sum_{k=2^i-1}^{2^{i+1}}X_{n-k}|u_n|^{1+\delta}\mathrm{sgn}u_n\Big)$$

$$=O\Big(\|u_n\|_{2+\delta}^{1+\delta}+\sum_{i=0}^{n-1}\varphi^{(1+\delta)/(2+\delta)}(2^i)2^{i/2}\|u_n\|_{2+\delta}^{1+\delta}\Big)$$

$$=O(\|u_n\|_{2+\delta}^{1+\delta}).$$

这就得证

$$(9.1.19)\qquad \|u_n\|_{2+\delta}=O(1).$$

类似地有

$$\|v_n\|_{2+\delta}=O(\varphi^{(1+\delta)/(2+\delta)}(2^{n-1})2^{(n-1)/2}).$$

因此

$$(9.1.20)\qquad \sum_{n=1}^{\infty}\|v_n\|_{2+\delta}<\infty.$$

所以条件(b)被满足.

对条件(c),令

$$(9.1.21)\quad T_m(n)=\sum_{m<k\leqslant m+n}(Y_k^2-EY_k^2),\tau_n=\sup_m\|T_m(n)\|_2.$$

易见对每一正整数 $m$,$n$ 有

$$(9.1.22)\qquad \|T_m(n)\|_2\leqslant\|T_m([n/2])+T_{m+[n/2])+l}([n/2])\|_2+2\tau_l+2\tau_1,$$

其中 $l=[2n(\log(2n))^{-2-\epsilon}]$,$\epsilon>0$. 注意到

$$(9.1.23)\quad E(T_m([n/2])+T_{m+[n/2]+l}([n/2]))^2\leqslant 2\tau_{[n/2]}^2+2E\{T_m([n/2])T_{m+[n/2]+l}([n/2])\},$$

通过一个初等的计算,从(9.1.4)和引理 2.2.8 得

$$\begin{aligned}
&ET_m([n/2])T_{m+[n/2]+l}([n/2])\\
&=ET_m([n/2])\Big(\sum_k Y_k\Big)^2\\
&=ET_m([n/2])\Big\{\Big(\sum_k X_k\Big)^2+2\sum_k X_k\Big(u_{N_1}-u_{N_2}-\sum_k v_k\Big)\\
&\quad+\Big(u_{N_1}-u_{N_2}-\sum_k v_k\Big)^2\Big\}\\
&\leqslant 2\varphi^{1/2}(l)\|T_m([n/2])\|_2\Big\|\sum_k X_k\Big\|_4^2\\
&\quad+\|T_m([n/2])\|_2\Big(\|u_{N_1}\|_4+\|u_{N_2}\|_4+\sum_k\|v_k\|_4\Big)\Big\|\sum_k X_k\Big\|_4\\
&\quad+\|T_m([n/2])\|_2\Big(\|u_{N_1}\|_4^2+\|u_{N_2}\|_4^2+\sum_k\|v_k\|_4^2\Big).
\end{aligned}$$

其中 $\sum_k$ 是对 $m+[n/2]+l<k\leqslant m+2[n/2]+l$ 求和,又

$$N_1=m+2[n/2]+l,\quad N_2=m+[n/2]+l.$$

利用引理 2.2.8,(9.1.12),(9.1.13)和条件(i),(ii),存在常数 $c$ 使对每一 $m,n\geqslant 1$ 成立

$$(9.1.24)\qquad ET_m([n/2])T_{m+[n/2]+l}([n/2])\leqslant c\tau([n/2])n^{1/2}(\log n)^{1+\epsilon}.$$

由(9.1.22),(9.1.23)和(9.1.24)式得

$$(9.1.25)\quad \|T_m(n)\|_2\leqslant 2^{1/2}\tau([n/2])+cn^{1/2}(\log n)^{1+\epsilon}.$$

$$+\tau([2n(\log n)^{-2-\varepsilon}]).$$

最后，从(9.1.25)并应用归纳法有

$$\tau_n \leqslant C_0 n^{1/2}(\log n)^{2+\varepsilon},$$

其中 $C_0=\max(\exp(2^{2/\varepsilon}),2c)$. 这就证明了条件(c)对 $\lambda=4+\varepsilon$ 成立. 定理 9.1.1 证毕.

**注 9.1.1** 设$\{X_n,n\geqslant1\}$是 $\varphi$ 混合序列，$EX_n=0$，且定理 9.1.1 的条件(i)被满足. 假设对某 $0<\delta\leqslant2$

$$\sup_n E|X_n|^{2+\delta}<\infty$$

且 $\varphi(n)$以多项式速度趋于 0. 邵启满和陆传荣(1986)也给出了下述结果：

1) 若 $0<\delta<2$ 且 $\varphi(n)=O(n^{-\alpha})$某 $\alpha>1$，那么对任给的 $\varepsilon>0$

$$S(t)-W(\sigma_t^2)=O(\sigma_t^{2/(2-\delta)}(\log\sigma_t)^{1-\varepsilon+(1+\lambda)/(2-\delta)})\quad \text{a.s.}$$

其中 $\lambda=2\log 4/\log\theta^{-1}$，$\theta=1-2(\alpha-1)/(\alpha(2+\delta))>0$.

2) 若 $0<\delta\leqslant2$ 且 $\varphi(n)=O(n^{-\alpha})$，$(2+\delta)/(2(1+\delta))<\alpha\leqslant1$，那么对任给 $\varepsilon>0$

$$S(t)-W(\sigma_t^2)=O(\sigma_t^{1-\alpha\delta/(2+\delta)+\varepsilon})\quad \text{a.s.}$$

## § 9.2 $\rho$ 混合序列的强逼近

在本章的引言中，我们已提到一个有兴趣的问题，即当仅设 $\sup_n EX_n^2<\infty$和 $\varphi(n)=O((\log n)^{-a})$，$a>1$(或 $\rho(n)=O((\log n)^{-a})$，$a>1$)时，混合序列的部分和过程 $S(t)$用 Wiener 过程强逼近的速度有多快? 在这一节中，我们讨论了 $\rho$ 混合序列，给出它的强逼近的阶为 $O((n\log\log n)^{1/2})$. 由此得出对这一 $\rho$ 混合序列重对数律成立. 为了获得钟重数律对 $\rho$ 混合序列成立，我们需要假设$\sup_n E|X_n|^{2+\delta}<\infty$.

**定理 9.2.1** (邵启满 1989a，1993b). 设$\{X_n,n\geqslant1\}$是强平稳 $\rho$ 混合序列，$EX_1=0$，$EX_1^2<\infty$. 假设

(i) $\sigma_n^2=ES_n^2\to\infty$ $(n\to\infty)$，

(ii) 对某 $\varepsilon'>0, \rho(n)=O((\log n)^{-1-\varepsilon'})$,

那么可在一较大概率空间上,在其上有一 Wiener 过程,不改变 $\{S(t), t\geqslant 0\}$ 的分布重新定义过程 $\{S(t), t\geqslant 0\}$,使得当 $t\to\infty$ 时有

$$(9.2.1)\qquad S(t)-W(\sigma_t^2)=O(\sigma_t(\log\log t)^{1/2})\quad \text{a. s.}$$

**定理 9.2.2** 设 $\{X_n, n\geqslant 1\}$ 是强平稳 $\varphi$ 混合序列, $EX_1=0$, $EX_1^2<\infty$. 假设

(i) $\sigma_n^2\to\infty\ (n\to\infty)$,

(ii) $\sum\limits_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$.

那么(9.2.1)也成立.

借助 Wiener 过程的重对数律,从上面的定理即可写出下一推论.

**推论 9.2.1** 在定理 9.2.1 或 9.2.2 的条件下,我们有

$$(9.2.2)\qquad \limsup_{n\to\infty}|S_n|/\sqrt{2\sigma_n^2\log\log\sigma_n^2}=1\quad \text{a. s.}$$

下述强逼近定理可推得钟重对数律

**定理 9.2.3** (邵启满 1989a,1993b). 设 $\{X_n, n\geqslant 1\}$ 是强平稳 $\rho$ 混合序列, $EX_1=0$, 对某 $\delta>0$, $E|X_1|^{2+\delta}<\infty$. 假设

(i) $\sigma_n^2\to\infty\ (n\to\infty)$,

(ii) 对某 $r>1/2, \rho(n)=O((\log n)^{-r})$.

那么对任一 $0<\theta<r/2-1/4$ 有

$$(9.2.3)\qquad S(t)-W(\sigma_t^2)=O(\sigma_t(\log t)^{-\theta})\quad \text{a. s.}$$

**推论 9.2.2** 在定理 9.2.3 的条件下,我们有(9.2.2)和

$$(9.2.4)\qquad \liminf_{N\to\infty}\left(\frac{8\log\log N}{\pi^2\sigma_N^2}\right)^{1/2}\max_{1\leqslant n\leqslant N}|S_n|=1\quad \text{a. s.}$$

**注 9.2.1** 对平稳 $\varphi$ 混合序列 $\{X_n, n\geqslant 1\}$, $EX_1=0$, $E|X_1|^{2+\delta}<\infty$, $\sigma_n^2\to\infty$ 且 $\varphi(n)=O((\log n)^{-r})$(某 $r>(2+\delta)/(2(1+\delta))$),定理 9.2.3 和推论 9.2.2 的结果也成立.

**注 9.2.2** 在这些定理和推论中平稳性的假设可被同分布条件代替. 顺便指出,本节的结果改进了 Bradley(1985), Dabrowski

(1982)及 Berkes 和 Philipp(1979)有关结果.

在这些定理的证明中,我们将应用 Bernstein 分段法和 Strassen 鞅嵌入法. 令

(9.2.5) $\qquad \bar{X}_k = X_k I(X_k^2 \geqslant k) - EX_k I(X_k^2 \geqslant k),$

(9.2.6) $\qquad \hat{X}_k = X_k I(X_k^2 < k) - EX_k I(X_k^2 < k).$

设

$$\bar{S}(n) = \sum_{k=1}^{n} \bar{X}_k, \qquad \hat{S}(n) = \sum_{k=1}^{n} \hat{X}_k,$$

$$\bar{S}_k(n) = \sum_{i=k+1}^{k+n} \bar{X}_i, \qquad \hat{S}_k(n) = \sum_{i=k+1}^{k+n} \hat{X}_i.$$

首先我们将证明

(9.2.7) $\qquad \bar{S}(n) = O(n^{1/2}) \quad \text{a.s.}$

然后我们定义整数的区组 $H_1, I_1, H_2, I_2, \cdots$,其中 $H_k$ 含 $h_k$ 个,$I_k$ 含 $i_k$ 个相继的整数,且相邻的区组间无空隙,这里

(9.2.8) $\qquad h_k = \text{Card}H_k = [ak^{a-1}\exp(k^a)],$

$\qquad\qquad i_k = \text{Card}I_k = [ak^{a-1}\exp(k^a/2)]$

且 $0<a<1$ 待下面确定. 令

$$N_k := \sum_{j\leqslant k}\text{Card}(H_j \cup I_j) \sim \exp(k^a),$$

$$u_k := \sum_{j\in H_k} \hat{X}_j, \quad v_k := \sum_{j\in I_k} \hat{X}_j,$$

$$\xi_k := u_k - E(u_k | \mathscr{F}_{k-1}) \text{ 其中 } \mathscr{F}_k = \sigma(X_i, i \leqslant N_{k-1} + h_k)$$

$$m_n := \{k : n \in H_k \cup I_k\} - 1 \sim (\log n)^{1/a}.$$

显然有

(9.2.9) $\qquad S_n = \hat{S}(n) + \bar{S}(n).$

(9.2.10)
$$\hat{S}(n) = \sum_{i=1}^{m_n} \xi_i + \sum_{i=1}^{m_n} E(u_i | \mathscr{F}_{i-1}) + \sum_{i=1}^{m_n} v_i + \sum_{i=N_{m_n}+1}^{n} \hat{X}_i.$$

由这两个表示式,为证定理 9.2.1,我们首先指出此时仅需验

证(9.2.7)和下述关系式成立:

$$(9.2.11)\qquad \sum_{i=1}^{n} v_i = O\left(\exp\left(n^{\alpha}/2\right)\right) \quad \text{a.s.}$$

$$(9.2.12)\qquad \max_{N_n < j \leqslant N_{n+1}} \left| \sum_{i=N_n+1}^{j} \hat{X}_i \right| = O(\exp(n^{\alpha}/2)) \quad \text{a.s.}$$

$$(9.2.13)\qquad \sum_{i=1}^{n} E(u_i | \mathscr{F}_{i-1}) = O(\exp\left(n^{\alpha}/2\right)) \quad \text{a.s.}$$

$$(9.2.14)\qquad \sum_{i=1}^{\infty} e^{-(1+\delta_1)^{i\alpha}} E|\xi_i|^{2(1+\delta_1)} < \infty \quad 某\ \delta_1 \in (0,1).$$

$$(9.2.15)\qquad \sum_{i=1}^{n} (E(\xi_i^2 | \mathscr{F}_{i-1}) - E\xi_i^2) = O(\exp(n^{\alpha})) \quad \text{a.s.}$$

$$(9.2.16)\qquad \sum_{i=1}^{m_n} E\xi_i^2 - ES_n^2 = o(n) \quad \text{a.s.}$$

**定理 9.2.1 的证明.**

假设(9.2.7)和(9.2.11)-(9.2.16)成立. 记 $\tilde{\sigma}_n^2 = \sum_{i=1}^{m_n} E\xi_i^2$. 那么

$$(9.2.17)\qquad S(n) - W(\sigma_n^2) = \sum_{i=1}^{m_n} \xi_i - W(\tilde{\sigma}_n^2) + W(\tilde{\sigma}_n^2) - W(\sigma_n^2) + O(n^{1/2}) \quad \text{a.s.}$$

应用 Strassen 鞅嵌入方法于鞅差 $\{\xi_k, \mathscr{F}_k, k \geqslant 1\}$, 存在停时序列 $\{T_n, n \geqslant 1\}$ 使得

$$\left\{W\left(\sum_{i=1}^{n} T_i\right), n \geqslant 1\right\} \overset{d}{=\!=} \left\{\sum_{i=1}^{n} \xi_i, n \geqslant 1\right\}.$$

重新定义

$$\xi_1 = W(T_1), \xi_n = W\left(\sum_{i=1}^{n} T_i\right) - W\left(\sum_{i=1}^{n-1} T_i\right), n \geqslant 2.$$

类似于(9.1.10), 我们有

$$E(T_n | \mathscr{G}_{n-1}) = E(\xi_n^2 | \mathscr{G}_{n-1}) = E(\xi_n^2 | \mathscr{F}_{n-1})$$

写

$$(9.2.18)\qquad \sum_{i=1}^{m_n} \xi_i - W(\tilde{\sigma}_n^2) = W\left(\sum_{i \leqslant m_n} T_i\right) - W\left(\sum_{i \leqslant m_n} ET_i\right)$$

和

$$(9.2.19)\quad \sum_{i=1}^{n}(T_i - ET_i) = \sum_{i=1}^{n}(T_i - E(T_i|\mathscr{G}_{i-1})) + \sum_{i=1}^{n}(E(\xi_i^2|\mathscr{F}_{i-1}) - \xi_i^2) + \sum_{i=1}^{n}(\xi_i^2 - E\xi_i^2).$$

由 Chow(1968)鞅的基本定理和(9.2.14),级数

$$\sum_{i=1}^{\infty} e^{-n^a}\xi_n^2 < \infty \quad \text{a.s.}$$

可推得级数

$$\sum_{i=1}^{\infty} e^{-n^a}(\xi_n^2 - E\xi_n^2)$$

也概率为 1 地收敛. 从 Kronecker 引理即得

$$\sum_{i\leqslant n}(\xi_i^2 - E\xi_i^2) = O(\exp(n^a)) \quad \text{a.s.}$$

同样有 $\sum_{i\leqslant n}(T_i - E(T_i|\mathscr{G}_{i-1})) = o(\exp(n^a))$. 因此结合(9.2.15)有

$$\sum_{i=1}^{n}(T_i - ET_i) = O(\exp(n^a)) \quad \text{a.s.}$$

应用 Wiener 过程滞后增量的 Hanson-Russo 定理(参见 Hanson 和 Russo1983 定理 3.2$B$)和(9.2.18),我们得

$$(9.2.20)\quad \sum_{i=1}^{m_n}\xi_i - W(\tilde{\sigma}_n^2) = O((n\log\log n)^{1/2}) \quad \text{a.s.}$$

类似地,从(9.2.16)我们有 $W(\tilde{\sigma}_n^2) - W(\sigma_n^2) = O((n\log\log n)^{1/2})$ a.s. 结合(9.2.20)和(9.2.17),就得定理 9.2.1 成立.

现在为完成定理 9.2.1 的证明,还需验证(9.2.7),(9.2.11)-(9.2.16)成立. 它们将由下述一组引理来完成. 首先,对任意随机变量给出三个引理. 由引理 9.2.1 可推出(9.2.7).

**引理 9.2.1** 设$\{X_n\}$是强平稳的随机变量序列,$EX_1^2<\infty$. 那么有

$$(9.2.21)\quad \sum_{i=1}^{n}\{|X_i|I(|X_i|\geqslant i^{1/2}) + E|X_i|I(|X_i|\geqslant i^{1/2})\}$$

$$= o(n^{1/2}) \quad \text{a.s.}$$

**证** 我们有

$$\begin{aligned}
&\sum_{i=1}^{\infty} i^{-1/2} E|X_i| I(X_i^2 \geqslant i) \\
&\leqslant \sum_{j=1}^{\infty} \sum_{i=1}^{j} i^{-1/2} E|X_1| I(j-1 < X_1^2 \leqslant j) \\
&\leqslant 4 \sum_{j=1}^{\infty} (j-1)^{1/2} E|X_1| I(j-1 < X_1^2 \leqslant j) \\
&\leqslant 4 \sum_{j=1}^{\infty} E X_1^2 I(j-1 < X_1^2 \leqslant j) \\
&= 4 E X_1^2 < \infty.
\end{aligned}$$

所以 $\sum\limits_{i=1}^{\infty} i^{-1/2} |X_i| I(X_i^2 \geqslant i) < \infty$ a.s.，那么从 Kronecker 引理即得(9.2.21)成立.

**引理 9.2.2** 若 $E|X| < \infty$，那么对任一 $0 < b \leqslant 1, \varepsilon > 0$

$$\sum_{k=1}^{\infty} k^{b-1} e^{-\varepsilon k^b} E|X|^{1+\varepsilon} I(|X| < e^{k^b}) < \infty.$$

**证** 易见上式左边等于

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^{b-1} e^{-\varepsilon k^b} E|X|^{1+\varepsilon} I(|X| < 1) \\
&\quad + \sum_{k=1}^{\infty} k^{b-1} e^{-\varepsilon k^b} \sum_{l=1}^{k} E|X|^{1+\varepsilon} I(e^{(l-1)^b} \leqslant |X| < e^{l^b}) \\
&\leqslant c + c \sum_{l=1}^{\infty} E|X|^{1+\varepsilon} I(e^{(l-1)^b} \leqslant |X| < e^{l^b}) e^{-\varepsilon l^b} \\
&\leqslant cE|X| < \infty.
\end{aligned}$$

**引理 9.2.3** 设$\{c_n, n \geqslant 1\}$是单调不增的正数列，那么对任一实数序列$\{\eta_n, n \geqslant 1\}$有

(9.2.22) $$\max_{1 \leqslant i \leqslant n} \Big|\sum_{j=1}^{i} \eta_j\Big| c_i \leqslant 2 \max_{1 \leqslant i \leqslant n} \Big|\sum_{j=1}^{i} c_j \eta_j\Big|.$$

**证** 记 $D_i = \sum\limits_{j=1}^{i} c_j \eta_j$. 我们有

$$\eta_j = (D_j - D_{j-1})/c_j = \sum_{i=1}^{j}\left(\frac{1}{c_i} - \frac{1}{c_{i-1}}\right)(D_j - D_{j-1}),$$

其中 $1/c_0=0$. 所以

$$\sum_{j=1}^{k}\eta_j = \sum_{j=1}^{k}\sum_{i=1}^{j}\left(\frac{1}{c_i} - \frac{1}{c_{i-1}}\right)(D_j - D_{j-1})$$

$$= \sum_{i=1}^{k}\left(\frac{1}{c_i} - \frac{1}{c_{i-1}}\right)(D_k - D_{i-1}).$$

由此即得

$$(9.2.23)\qquad \max_{1\leqslant k\leqslant n} c_k\left|\sum_{j=1}^{k}\eta_j\right| \leqslant \max_{1\leqslant k\leqslant n}\ \max_{1\leqslant i\leqslant k}|D_k - D_{i-1}|$$

$$\leqslant 2\max_{1\leqslant k\leqslant n}\left|\sum_{j=1}^{k}c_j\eta_j\right|.$$

引理证毕.

**引理 9.2.4** 设 $\{X_n, n\geqslant 1\}$ 是 $\rho$ 混合序列, $EX_n=0$, $EX_n^2<\infty$. 设 $u_k, v_k$ 如上. 记 $u_k(n)=\sum_{i=k+1}^{k+n}u_i$, $v_k(n)=\sum_{i=k+1}^{k+n}v_i$. 假设

$$\sum_{n=1}^{\infty}\rho(2^n) < \infty.$$

那么存在常数 $C=C(\rho(\cdot))$ 使对任一 $k\geqslant 0, n\geqslant 1$ 有

$$(9.2.24)\qquad Eu_k^2(n) \leqslant C\left(\sum_{i=k+1}^{k+n}Eu_i^2\right),$$

$$(9.2.25)\qquad Ev_k^2(n) \leqslant C\left(\sum_{i=k+1}^{k+n}Ev_i^2\right).$$

**证** 只对(9.2.24)给出证明. 当 $n=1$ 时显然. 当 $n\geqslant 2$ 时, 记 $n_1=n-[n/2]$, $n_2=[n/2]$. 那么由 $\rho(\cdot)$ 的定义有

$$(9.2.26)\qquad Eu_k^2(n) = Eu_k^2(n_1) + Eu_{k+n_1}^2(n_2) + 2Eu_k(n_1)u_{k+n_1}(n_2)$$

$$\leqslant (Eu_k^2(n_1) + Eu_{k+n_1}^2(n_2))(1+\rho(i_{n_1})).$$

设 $c_1=1$, $c_n=c_{n_1}(1+\rho(i_{n_1}))$ $(n\geqslant 2)$. 易见 $c_n$ 是单调不减的. 从(9.2.26)得对任一 $k\geqslant 0, n\geqslant 1$

$$Eu_k^2(n) \leqslant c_n\left(\sum_{i=k+1}^{k+n}Eu_i^2\right).$$

现在, 我们来证 $c_n$ 是有界序列. 注意到

$$c_{2^m} = c_{2^{m-1}}(1 + \rho(i_{2^{m-1}}))$$
$$\leqslant c_{2^{m-1}}\exp(\rho(i_{2^{m-1}})) \leqslant \exp\left\{\sum_{j=0}^{m-1}\rho(i_{2^j})\right\}.$$

从 $i_k$ 的定义，对充分大 $j$ 有

$$\rho(i_{2^j}) \leqslant \rho(2^{2^{ja}}),$$

且进一步由 $\rho(\cdot)$的单调性有

$$\sum_{j=0}^{m-1}\rho(i_{2^j}) \leqslant \sum_{j=0}^{m-1}\rho(2^{2^{ja}}) \leqslant \sum_{j=0}^{m-1}\rho(2^j) < \infty.$$

这就证明了 $c_{2^m}\leqslant c<\infty$. 从 $c_n$ 的单调性即得$\{c_n\}$是有界的. 引理证毕.

下述引理证明了(9.2.11)成立.

**引理 9.2.5** 设$\{X_n,n\geqslant 1\}$是 $\rho$ 混合序列，$EX_n=0$，$EX_n^2<\infty$. 那么

$$\sum_{j=1}^{n}v_j = O(\exp(n^a/3)) \quad \text{a. s.} \tag{9.2.27}$$

**证** 从 $q=2$ 时的引理 2.2.5 和条件(ii)有

$$E\left(\sum_{j=k+1}^{k+n}v_j\right)^2 \leqslant cn \max_{1\leqslant j\leqslant n}Ev_j^2$$
$$\leqslant cn^a\exp(n^a/2).$$

由 Borel-Cantelli 引理得(9.2.27). 引理证毕.

下述引理给出了(9.2.12).

**引理 9.2.6** 假设定理 9.2.1 的条件被满足且

$$0 < a < \varepsilon'/(8 + \varepsilon'). \tag{9.2.28}$$

那么

$$\max_{N_k<j\leqslant N_{k+1}}\left|\sum_{i=N_k}^{j}\hat{X}_i\right| = O(\exp(k^a/2)) \quad \text{a. s.} \tag{9.2.29}$$

**证** 由 Borel-Cantelli 引理，仅需证明对任给 $\varepsilon>0$

$$\sum_{k=1}^{\infty}P\left\{\max_{N_k<j\leqslant N_{k+1}}\left|\sum_{i=N_k}^{j}\hat{X}_i\right| \geqslant \varepsilon\exp(k^a/2)\right\}< \infty. \tag{9.2.30}$$

令 $d_k=N_{k+1}-N_k\sim ak^{a-1}$，设

$$B = k^{-a(16+\varepsilon')/\varepsilon'}\exp(k^a/2), m = [k^{-a(16+\varepsilon')/\varepsilon'}\exp(k^a)].$$

对任给 $\varepsilon>0$ 对充分大 $k$ 有

$$\frac{48m}{B}EX_1^2I(|X_1|\geqslant B)\leqslant\varepsilon\exp(k^a/2) \tag{9.2.31}$$

从引理 4.3.2 即得

$$P\left\{\max_{N_k<j\leqslant N_{k+1}}\left|\sum_{i=N_k}^{j}\hat{X}_i\right|\geqslant\varepsilon\exp(k^a/2)\right\}$$

$$\leqslant c\left\{\exp\left(-\frac{8+\varepsilon'}{8}k^a\right)(d_k^{(8-\varepsilon')/8}\right.$$

$$+d_k\log^{2+\varepsilon'/4}(2d_k)E|X_1|^{2+\varepsilon'/4}I(|X_1|\leqslant B)$$

$$+d_kE|X_1|^{2+\varepsilon'/4}I(|X_1|\leqslant N_{k+1}))$$

$$\left.+\exp(-k^a)d_k(1+\rho^2(m)\log^4[d_k/m])\right\}$$

$$\leqslant c\left\{k^{-(1-a)(8+\varepsilon')/8}+k^{-1-a}+k^{a-1}\right.$$

$$\left.\exp(-\varepsilon'k^a/8)E|X_1|^{2+\varepsilon'/4}I(|X_1|^2\leqslant 2e^{k^a})\right\}$$

$$+c\{k^{a-1-2a(1+\varepsilon')}\log^4k\}.$$

由引理 9.2.2 和(9.2.28)得(9.2.30)成立.

下述引理给出了(9.2.13).

**引理 9.2.7** 假设定理 9.2.1 的条件(ii)被满足. 那么我们有

$$\sum_{k=1}^{n}E(u_k|\mathscr{F}_{k-1})=O(\exp(n^a/2))\quad\text{a.s.} \tag{9.2.32}$$

**证** 首先我们证明对任给 $k\geqslant0,n\geqslant1$ 和实数列 $\{c_n\}$ 有

$$EG_k^2(n)\leqslant c\left\{\sum_{j=k+1}^{k+n}\rho^2(i(j/2))c_j^2Eu_j^2\right\}\log^2(2n), \tag{9.2.33}$$

其中 $G_k(n)=\sum\limits_{j=k+1}^{k+n}c_jE(u_j|\mathscr{F}_{j-1})$ 且 $i(x)$ 是 $i_k$ 的线性内插函数. 从(9.2.24),存在常数 $c'$ 使对任给 $k\geqslant0,n\geqslant1$ 有

$$E\left|\sum_{i=k+1}^{k+n}c_iu_i\right|^2\leqslant c'\left(\sum_{i=k+1}^{k+n}c_i^2Eu_i^2\right). \tag{9.2.34}$$

令 $c=100c'\log^{-2}(3/2)$. 对 $n$ 应用归纳法. 当 $n=1$ 时

$$EG_k^2(1)=c_{k+1}^2E(E(u_{k+1}|\mathscr{F}_k))^2$$

$$=c_{k+1}^2E(u_{k+1}E(u_{k+1}|\mathscr{F}_k))$$

$$\leqslant c_{k+1}^2\rho(i_k)\|u_{k+1}\|_2\|E(u_{k+1}|\mathscr{F}_k)\|_2$$

即

$$EG_k^2(1)\leqslant\rho^2(i_k)Eu_{k+1}^2.$$

因此 $n=1$ 时(9.2.33)成立. 假设对任一小于 $n$ 的整数(9.2.33)成立. 我们来证(9.2.33)对 $n$ 也成立. 记 $n_1=n-[n/2]$, $n_2=[n/2]$. 由定义

$$\begin{aligned}EG_k^2(n)&=EG_k^2(n_1)+EG_{k+n_1}^2(n_2)+2EG_k(n_1)G_{k+n_1}(n_2)\\&=EG_k^2(n_1)+EG_{k+n_1}^2(n_2)+2EG_k(n_1)\Big(\sum_{j=k+n_1+1}^{k+n}c_ju_j\Big)\\&\leqslant EG_k^2(n_1)+EG_{k+n_1}^2(n_2)\\&\quad+2\rho(i_{k+n_1})\|G_k(n_1)\|_2\Big\|\sum_{j=k+n_1+1}^{k+n}c_ju_j\Big\|_2.\end{aligned}$$

由(9.2.34)和归纳假设,我们有

$$\begin{aligned}EG_k^2(n)&\leqslant c\Big\{\sum_{j=k+1}^{k+n}\rho^2(i(j/2))c_j^2Eu_j^2\Big\}\log^2(2n_1)\\&\quad+2\sqrt{cc'}\rho(i_{k+n_1})\Big\{\sum_{j=k+n_1+1}^{k+n}c_j^2Eu_j^2\Big\}^{1/2}\\&\quad\cdot\Big\{\sum_{j=k+1}^{k+n_1}\rho^2(i(j/2))c_j^2Eu_j^2\Big\}^{1/2}\log(2n_1)\\&\leqslant(c(\log2n_1)^2+\sqrt{cc'}\log(2n_1))\Big\{\sum_{j=k+1}^{k+n}\rho^2(i(j/2))c_j^2Eu_j^2\Big\}\\&\leqslant c\Big\{\sum_{j=k+1}^{k+n}\rho^2(i(j/2))c_j^2Eu_j^2\Big\}\log^2(2n).\end{aligned}$$

得证(9.2.33)成立.

由(9.2.33), $\sum_{i=1}^{\infty}\rho(2^i)<\infty$ 和引理 2.2.2 即得

$$\begin{aligned}&P\Big\{\Big|\sum_{i=1}^{n}E(u_i|\mathscr{F}_{i-1})\Big|\geqslant\varepsilon\exp(n^\alpha/2)\Big\}\\&\leqslant ce^{-n^\alpha}\Big\{\sum_{i=1}^{n}i^{-2\alpha}Eu_i^2\Big\}\log^2(2n)\end{aligned}$$

$$\leqslant ce^{-n^a}\left\{\sum_{i=1}^{n} i^{-a-1}e^{i^a}\right\}\log^2(2n)$$

$$\leqslant cn^{-2a}\log^2(2n).$$

设 $n_k=[k^{1/a}]$. 由 Borel-Cantelli 引理有

$$\sum_{i=1}^{n_k} E(u_i|\mathscr{F}_{i-1}) = O(\exp(n_k^a/2)) \quad \text{a.s.} \tag{9.2.35}$$

其次，从引理 9.2.3，引理 2.2.2，引理 4.1.2 和(9.2.33). 我们有

$$P\left\{\max_{n_k<j\leqslant n_{k+1}}\left|\sum_{i=n_k}^{j} e^{-i^a/2}E(u_i|\mathscr{F}_{i-1})\right|\geqslant \varepsilon\right\}$$

$$\leqslant P\left\{\max_{n_k<j\leqslant n_{k+1}}\left|\sum_{i=n_k}^{j} e^{-i^a/2}E(u_i|\mathscr{F}_{i-1})\right|\geqslant \varepsilon/2\right\}$$

$$\leqslant c\left\{\sum_{j=n_k+1}^{n_{k+1}} j^{-2a}e^{-j^a}Eu_j^2\right\}\log^4(n_{k+1}-n_k)$$

$$\leqslant ck^{-2}(\log k)^4.$$

由它可推出

$$\max_{n_k<j\leqslant n_{k+1}}\left|\sum_{i=n_k}^{j} e^{-i^a/2}E(u_i|\mathscr{F}_{i-1})\right| \to 0 \quad \text{a.s.} \tag{9.2.36}$$

这样从(9.2.35)和(9.2.36)得证(9.2.32)成立.

下述引理给出了(9.2.14).

**引理 9.2.8** 假设 $a$ 满足(9.2.28)，且定理 9.2.1 的条件(ii)被满足. 那么对 $\delta_1=\varepsilon'/4$，我们有

$$\sum_{k=1}^{\infty} e^{(1-\delta_1)k^a}E|\xi_k|^{2+2\delta_1} < \infty. \tag{9.2.37}$$

**证** 从引理 2.2.5，引理 9.2.2 和条件(ii)有

$$E|\xi_k|^{2+2\delta_1} \leqslant cE|u_k|^{2+2\delta_1}$$

$$\leqslant c\{(k^{a-1}\exp(k^a))^{1+\delta_1} + k^{a-1}\exp(k^a)$$

$$E|X_1|^{2+2\delta_1}I(X_1^2\leqslant 2\exp(k^a))\}.$$

那么由 $0<a<\varepsilon'/(8+\varepsilon')$ 和引理 9.2.2，得证(9.2.37)成立.

下述引理给出了(9.2.15).

**引理 9.2.9** 假设 $a$ 满足(9.2.28)和定理 9.2.1 的条件(ii).

那么

$$(9.2.38)\qquad \sum_{j=1}^{n}(E(\xi_j^2|\mathscr{F}_{j-1})-E\xi_j^2)=O(\exp(n^a))\quad \text{a.s.}$$

**证** 我们有

$$(9.2.39)\qquad \left|\sum_{j=1}^{n}(E(\xi_j^2|\mathscr{F}_{j-1})-E\xi_j^2)\right|$$

$$\leqslant\left|\sum_{j=1}^{n}(E(u_j^2|\mathscr{F}_{j-1})-Eu_j^2)\right|$$

$$+\sum_{j=1}^{n}(E^2(u_j|\mathscr{F}_{j-1})+E(E(u_j|\mathscr{F}_{j-1}))^2).$$

首先来证

$$(9.2.40)\qquad \sum_{j=1}^{n}(E^2(u_j|\mathscr{F}_{j-1})+E(E(u_j|\mathscr{F}_{j-1}))^2)$$

$$=o(\exp(n^a))\quad \text{a.s.}$$

事实上，由 $\rho(\cdot)$的定义我们有

$$E(E(u_j|\mathscr{F}_{j-1}))^2=E(u_jE(u_j|\mathscr{F}_{j-1}))$$

$$\leqslant\rho(i_{j-1})\,\|u_j\|_2\,\|E(u_j|\mathscr{F}_{j-1})\|_2.$$

所以

$$E(E(u_j|\mathscr{F}_{j-1}))^2\leqslant\rho^2(i_{j-1})Eu_j^2\leqslant cj^{-2a(1+\varepsilon)+a-1}e^{j^a},$$

由此我们有

$$\sum_{j=1}^{\infty}E(E(u_j|\mathscr{F}_{j-1}))^2/e^{j^a}\leqslant c\sum_{j=1}^{\infty}j^{-1-a(1+2\varepsilon)}<\infty.$$

由 Kronecker 引理即得(9.2.40)成立.

其次，我们来证

$$(9.2.41)\qquad \sum_{j=1}^{n}(E(u_j^2|\mathscr{F}_{j-1})-Eu_j^2)=O(\exp(n^a))\quad \text{a.s.}$$

记 $\bar{u}_i=u_i^2I(|u_i|\leqslant e^{i^a/2})$. 容易看出

$$(9.2.42)\qquad \left|\sum_{i=1}^{n}(E(u_i^2|\mathscr{F}_{i-1})-Eu_i^2)\right|$$

$$\leqslant\left|\sum_{i=1}^{n}(E(\bar{u}_i|\mathscr{F}_{i-1})-E\bar{u}_i)\right|$$

$$+\sum_{i=1}^{n}(E(u_i^2I(|u_i|\geqslant e^{i^\alpha/2})|\mathscr{F}_{i-1}))$$
$$+Eu_i^2I(|u_i|\geqslant e^{i^\alpha/2})).$$

从引理 2.2.5 和引理 9.2.2 即得

$$\sum_{j=1}^{\infty}E(E(u_j^2I(|u_j|\geqslant e^{j^\alpha/2})|\mathscr{F}_{j-1}))/e^{j^\alpha}$$
$$=\sum_{j=1}^{\infty}e^{-j^\alpha}Eu_j^2I(|u_j|\geqslant e^{j^\alpha/2})$$
$$\leqslant c\sum_{j=1}^{\infty}e^{-(1+\delta_1/2)j^\alpha}E|u_j|^{2+\delta_1}$$
$$\leqslant c\sum_{j=1}^{\infty}e^{-(1+\delta_1/2)j^\alpha}\{(j^{\alpha-1}e^{j^\alpha})^{1+\delta_1/2}$$
$$+e^{j^\alpha}j^{\alpha-1}E|X_1|^{2+\delta_1}I(X_1^2\leqslant 2e^{j^\alpha})\}<\infty.$$

由 Kronecker 引理即得

$$(9.2.43)\qquad \sum_{j=1}^{n}(E(u_j^2I(|u_j|\geqslant e^{j^\alpha/2})|\mathscr{F}_{j-1})$$
$$+Eu_j^2I(|u_j|\geqslant e^{j^\alpha/2}))=o(\exp(n^\alpha))\quad \text{a. s.}$$

另一方面，如引理 9.2.7 中(9.2.33)式同样证法，存在常数 $C$ 使对任一实数列 $\{c_n,n\geqslant 1\}$ 有

$$E\Big(\sum_{j=k+1}^{k+n}E(\bar{u}_j-E\bar{u}_j|\mathscr{F}_{j-1})\Big)^2$$
$$\leqslant C\Big(\sum_{j=k+1}^{k+n}\rho(i(j/2))\quad E\bar{u}_j^2\Big)(\log(2n))^2.$$

那么由引理 2.2.2 和条件(ii)，我们有

$$P\Big\{\Big|\sum_{j=1}^{n}E(u_j-E\bar{u}_j|\mathscr{F}_{j-1})\Big|\geqslant \varepsilon e^{n^\alpha}\Big\}$$
$$\leqslant ce^{-2n^\alpha}\Big(\sum_{j=1}^{n}j^{-2\alpha(1+\varepsilon')}Eu_j^2\Big)(\log n)^2$$
$$\leqslant ce^{-2\alpha^\alpha}\Big(\sum_{j=1}^{n}j^{-2\alpha(1+\varepsilon')}e^{j^\alpha}\Big)(\log n)^2$$

$$\leqslant cn^{-2a(1+\varepsilon')}(\log n)^2.$$

由引理 9.2.7 证明的后半部分的类似讨论即得

$$\sum_{j=1}^{n} E(\bar{u}_j - E\bar{u}_j \mid \mathscr{F}_{j-1}) = O(\exp(n^{\varepsilon})) \quad \text{a.s.}$$

结合(9.2.42),(9.2.43)得(9.2.41).引理 9.2.9 证毕.

下述引理给出了(9.2.16).

**引理 9.2.10** 若定理 9.2.1 的条件被满足,我们有

$$\sum_{i=1}^{m_n} E\xi_i^2 - ES_n^2 = o(n). \tag{9.2.44}$$

**证** 记 $\hat{u}_j = \sum_{l \in H_j} X_l, \hat{v}_j = \sum_{l \in I_j} X_l$. 我们有

$$ES_n^2 = E\Big(\sum_{j=1}^{m_n} \hat{u}_j + \sum_{j=1}^{m_n} \hat{v}_j + \sum_{j=N(m_n)+1}^{n} X_j\Big)^2 \tag{9.2.45}$$

$$= E\Big(\sum_{j=1}^{m_n} \hat{u}_j\Big)^2 + E\Big(\sum_{j=1}^{m_n} \hat{v}_j + \sum_{j=N(m_n)+1}^{n} X_j\Big)^2$$

$$+ 2E\Big(\sum_{j=1}^{m_n} \hat{u}_j\Big)\Big(\sum_{j=1}^{m_n} \hat{v}_j + \sum_{j=N(m_n)+1}^{n} X_j\Big).$$

由引理 9.2.4,$N_k$ 及 $m_n$ 的定义即得

$$E\Big(\sum_{j=1}^{m_n} \hat{v}_j + \sum_{j=N(m_n)+1}^{n} X_j\Big)^2 = O\Big(\sum_{j=1}^{m_n} E\hat{v}_j^2 + (n - N_{m_n})\Big) \tag{9.2.46}$$

$$= O\Big(n(\log n)^{\frac{a-1}{a}}\Big).$$

与

$$E\Big(\sum_{j=1}^{m_n} \hat{u}_j\Big)^2 = E\Big(\sum_{j=1}^{m_n} u_j + \sum_{j=1}^{m_n} (\hat{u}_j - u_j)\Big)^2.$$

从引理 9.2.4 和引理 2.2.2 即得

$$E\Big(\sum_{j=1}^{m_n} (\hat{u}_j - u_j)\Big)^2 \leqslant c\Big(\sum_{j=1}^{m_n} E(\hat{u}_j - u_j\Big)^2$$

$$\leqslant c\sum_{j=1}^{m_n} j^{a-1} e^{j^a} EX_1^2 I(|X_1|^2 \geqslant N_{j-1})$$

$$= o(n).$$

另一方面

$$\sum_{j=1}^{m_n} E\xi_j^2 - E\Big(\sum_{j=1}^{m_n} u_j\Big)^2$$

$$= E\Big(\sum_{j=1}^{m_n} \xi_j\Big)^2 - E\Big(\sum_{j=1}^{m_n} u_j\Big)^2$$

$$= E\Big(\sum_{j=1}^{m_n} E(u_j|\mathscr{F}_{j-1})\Big)^2 + 2E\Big(\sum_{j=1}^{m_n} u_j\Big)\sum_{j=1}^{m_n} E(u_j|\mathscr{F}_{j-1}).$$

从(9.2.33)即得

$$E\Big(\sum_{j=1}^{m_n} E(u_j|\mathscr{F}_{j-1})\Big)^2 \leqslant c\Big(\sum_{j=1}^{m_n} j^{-2(1+\varepsilon)a} Eu_j^2\Big)(\log m_n)^2$$

$$\leqslant cn(\log n)^{-2(1+\varepsilon)}(\log\log n)^2.$$

结合上面诸关系式得证(9.2.44)成立.

现在完成了定理 9.2.1 的证明.

定理 9.2.2 的证明与定理 9.2.1 的证明相同,只需以引理 2.2.10代替引理 2.2.5 和引理 4.3.2. 定理 9.2.3 的证明与定理 9.2.1 的证明类似. 我们只需对$\{X_n\}$直接应用 Bernstein 分段法,其细节从略.

## §9.3 α 混合序列的强逼近

邵启满和陆传荣(1986)改进了 Philipp 和 Stout(1975)关于 α 混合序列的强逼近结果,获得了在应用 Strassen 鞅嵌入方法时较理想的强逼近阶.

**定理 9.3.1** 设$\{X_n, n\geqslant 1\}$是 α 混合序列,$EX_n=0$,函数 $g(x)$ 使得 $g(x)/x^{2+\delta}$是单调增加趋于∞的. 记

$$\|X\|_g = \inf\{t > 0, Eg(|X|/t) \leqslant 1\}.$$

若$\sup\limits_n \|X_n\|_g<\infty$且下述条件被满足

(i) $\sigma_n^2=ES_n^2\geqslant cn$,

(ii) 对某 $0<\delta\leqslant 2$，$\sum_{n=1}^{\infty}\alpha(n)^{\frac{1}{2+\delta}}\mathrm{invg}\left(\frac{1}{\alpha(n)}\right)<\infty$.

那么

(9.3.1) $S(t)-W(\sigma_t^2)=O(\sigma_t^{2/(2+\delta)})(\log\sigma_t^2)^{1+(1+\lambda)/(2+\delta)})$ a.s.

其中 $\lambda=(\log 2)/\log((2+\delta)/\delta)\leqslant 1$.

**证** 写

(9.3.2)
$$Y_n=\sum_{k=0}^{\infty}\{E(X_{k+n}|\mathscr{F}_n)-E(X_{k+n}|\mathscr{F}_{n-1})\}$$
$$=X_n+u_n-u_{n-1},$$

其中 $u_n=\sum_{k=0}^{\infty}E(X_{k+n}|\mathscr{F}_{n+1})$. 让我们来验证命题 9.1.1 的诸条件. 条件(a)是被假设的. 为验证条件(b). 记

$$u_{kn}=E(X_{k+n}|\mathscr{F}_{n-1}).$$

在引理 1.2.3 中令 $f(x)=x^{(2+\delta)/(1+\delta)}$，我们有

$$E|u_{kn}|^{2+\delta}=E(X_{k+n}u_{kn}|u_{kn}|^{\delta})$$
$$\leqslant c\,\mathrm{invg}\left(\frac{1}{\alpha(k)}\right)\alpha(k)^{1/(2+\delta)}\|X_{k+n}\|_g\|u_{kn}\|_{2+\delta}^{1+\delta}.$$

所以

$$\|u_{kn}\|_{2+\delta}\leqslant c\,\mathrm{invg}(1/\alpha(k))\alpha(k)^{1/(2+\delta)}.$$

从(ii)即得命题 9.1.1 的条件(b)被满足.

记

$$T_m(n)=\sum_{m<k\leqslant m+n}(Y_k^2-EY_k^2),\tau_n=\sup_m E|T_m(n)|^{(2+\delta)/2}.$$

我们用归纳法来证

(9.3.3)
$$\tau_n\leqslant cn(\log n)^{\lambda}.$$

其中 $\lambda=\left(\log\frac{2+M_\delta}{2}\right)/\log\frac{2+\delta}{\delta}\leqslant 1, 1\leqslant M_\delta\leqslant 2, 0<\delta\leqslant 2$. 令 $\theta=\delta/(2+\delta)$, $n_1=n-[n^\theta]$, $n_2=[n^\theta]$. 从一初等不等式：对 $1<p\leqslant 2$, $1\leqslant C_p\leqslant 2$ 有

$$|1+x|^p\leqslant 1+px+C_p|x|^p.$$

由此我们有

$$(9.3.4)\quad E|T_m(n)|^{\frac{2+\delta}{2}} \leqslant E|T_m(n_1)|^{\frac{2+\delta}{2}} + M_\delta E|T_{m+n_1}(n_2)|^{\frac{2+\delta}{2}}$$

$$+ \left(\frac{2+\delta}{2}\right) E|T_m(n_1)|^{\frac{\delta}{2}}|T_{m+n_1}(n_2)|$$

$$\leqslant \tau_{n_1} + M_\delta \tau_{n_2} + \left(\frac{2+\delta}{2}\right) E\Big( |T_m(n_1)|^{\frac{\delta}{2}}$$

$$\cdot \Big\{\Big(\sum_{j=N_2}^{m+n} X_j + u_{N_1} - u_{N_2}\Big)^2$$

$$- E\Big(\sum_{j=N_2}^{m+n} X_j + u_{N_1} - u_{N_2}\Big)^2\Big\}\Big)$$

$$=: \tau_{n_1} + M_\delta \tau_{n_2} + \frac{2+\delta}{2} I,$$

其中 $N_1 = m+n+1, N_2 = m+n_1+1$. 注意到

$$(9.3.5)\quad I = E|T_m(n_1)|^{\frac{\delta}{2}} \sum_{j=N_2}^{m+n} (X_j^2 - EX_j^2)$$

$$+ E|T_m(n_1)|^{\frac{\delta}{2}}((u_{N_1} - u_{N_2})^2 - E(u_{N_1} - u_{N_2})^2)$$

$$+ 2E|T_m(n_1)|^{\frac{\delta}{2}}\Big\{\Big(\sum_{j=N_2}^{m+n} X_j\Big)(u_{N_1} - u_{N_2})$$

$$- E\Big(\sum_{j=N_2}^{m+n} X_j\Big)(u_{N_1} - u_{N_2})\Big\}$$

$$+ 2E|T_m(n_1)|^{\frac{\delta}{2}} \sum_{1\leqslant i,j,i+j\leqslant n_2} (X_{j+N_2-1} X_{i+j+N_2-1}$$

$$- EX_{j+N_2-1} X_{i+j+N_2-1})$$

$$=: I_1 + I_2 + 2I_3 + 2I_4.$$

从引理 1.2.3(取 $g_1(x^2) = g(x), f(x) = x^{(2+\delta)/\delta}$)和条件(ii),有

$$I_1 \leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}} \sum_{k=0}^{\infty} \mathrm{inv} g_1\left(\frac{1}{\alpha(k)}\right) \mathrm{inv} f\left(\frac{1}{\alpha(k)}\right) \alpha(k)$$

$$\leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}} \sum_{k=0}^{\infty} \left(\mathrm{inv} g\left(\frac{1}{\alpha(k)}\right) \alpha(k)^{\frac{1}{2+\delta}}\right)^2$$

$$\leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}.$$

由 Hölder 不等式和条件(b)有

$$I_2 \leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}(\|u_{N_1}\|_{2+\delta}^2 + \|u_{N_2}\|_{2+\delta}^2)$$

$$\leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}.$$

又由 Hölder 不等式和引理 1.2.3,我们有

$$I_3 = E|T_m(n_1)|^{\frac{\delta}{2}}\Big\{\sum_{j=N_2+1}^{N_1-1} X_j \sum_{k=1}^{2N_1} X_{N_1+k}$$

$$- \sum_{j=N_2+1}^{N_1-1} u_{N_2} X_j - E\sum_{j=N_2+1}^{N_1-1} X_j \sum_{k=1}^{2N_1} X_{N_1+k} + E\sum_{j=N_2+1}^{N_1-1} u_{N_2} X_j\Big\}$$

$$\leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}} \sum_j \sum_k \mathrm{inv}f$$

$$\left(\frac{1}{\alpha(N_1+k-j)}\right)\alpha(N_1+k-j)^{\frac{2}{2+\delta}}$$

$$\leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}} \sum_j \sum_k (\mathrm{inv}g$$

$$\left(\frac{1}{\alpha(N_1+k-j)}\right)\alpha(N_1+k-j)^{\frac{1}{2+\delta}})^2$$

$$\leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}},$$

其中 $f(x^{(2+\delta)/2}) = g(x)$, $\mathrm{inv}f(x) = (\mathrm{inv}g(x))^{(2+\delta)/2}$. 对于 $I_4$,有

$$I_4 = E|T_m(n_1)|^{\frac{\delta}{2}}\Big(\sum_{j=1}^{n_2}\sum_{1\leqslant i<j} + \sum_{i=1}^{n_2}\sum_{1\leqslant j<i}\Big)$$

$$(X_{j+N_2-1}X_{i+j+N_2-1} - EX_{j+N_2-1}X_{i+j+N_2-1})$$

$$\leqslant c\sum_{j=1}^{n_2}\sum_{1\leqslant i<j}\alpha(j)^{\frac{2}{2+\delta}}\mathrm{inv}f\left(\frac{1}{\alpha(j)}\right)(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}$$

$$+ c\sum_{i=1}^{n_2}\sum_{1\leqslant j<i}\alpha(i)^{\frac{2}{2+\delta}}\mathrm{inv}f\left(\frac{1}{\alpha(i)}\right)(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}},$$

其中 $f(x^2) = g(x)$, $\mathrm{inv}f(x) = (\mathrm{inv}g(x))^2$. 容易看出

$$I_4 \leqslant c\sum_{i=1}^{n_2}\sum_{1\leqslant j<i}\alpha(i)^{\frac{1}{2+\delta}}\mathrm{inv}g\left(\frac{1}{\alpha(i)}\right)(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}$$

$$\leqslant c(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}.$$

把这些与(9.3.4),(9.3.5)相结合我们有

$$\tau_n \leqslant \tau_{n_1} + M_\delta \tau_{n_2} + c\tau_{n_1}^{\delta/(2+\delta)}.$$

因此命题 9.1.1 的条件(c)被满足,定理 9.3.1 证毕.

**推论 9.3.1** 设$\{X_n, n\geqslant 1\}$是 $\alpha$ 混合序列,$g(x)=x^r$,$r>2+\delta$,$0<\delta\leqslant 2$.若

(i) $\sigma_n^2=ES_n^2\geqslant cn$ 某 $c>0$,

(ii) $\sum_{n=1}^{\infty} \alpha(n)^{\frac{1}{2+\delta}-\frac{1}{r}}<\infty$.

那么

$$S_n - W(\sigma_n^2) = O(\sigma_n^{\frac{2}{2+\delta}}(\log\sigma_n)^{1+(1+\lambda)/(2+\delta)}) \quad \text{a.s.}$$

特别地,若 $\delta=2$ 且$\lim\limits_{n\to\infty}\sigma_n^2/n=\sigma^2$(不失一般性,假设 $\sigma^2=1$),我们有

$$S_n - W(n) = O(n^{1/4}(\log n)^{3/2}) \quad \text{a.s.}$$

进一步,若$\sup\limits_n|X_n|<\infty$, $\sum \alpha(n)^{1/4}<\infty$ 且 $\sigma_n^2=n\sigma^2$,那么

$$S_n - W(n) = O(n^{1/4}(\log n)^{5/4+\varepsilon}). \quad \text{a.s}$$

邵启满(1989a)给出了下述定理和推论,它改进了 Bradley (1983)及 Dehling(1983)的结论且减弱有关条件.

**定理 9.3.2** 设$\{X_n, n\geqslant 1\}$是 $\alpha$ 混合序列,$EX_n=0$,对某 $\delta>0$,$\sup\limits_n E|X_n|^{2+\delta}<\infty$.假设

$$\sup_{n\geqslant 1}\sup_{k\geqslant 0} E|S_k(n)|^{2+\delta}/n^{1+\delta/2} < \infty, \tag{9.3.6}$$

$$\alpha(n) = O((\log n)^{-r}), r > 1 + 2/\delta. \tag{9.3.7}$$

那么对任一 $0<\theta<\frac{1}{4}\left(\frac{r\delta}{2+\delta}-1\right)$,我们有

$$S(t) - W(\sigma_t^2) = O(t^{1/2}(\log t)^{-\theta}) \quad \text{a.s.}$$

**推论 9.3.2** 设$\{X_n, n\geqslant 1\}$是 $\alpha$ 混合序列,$EX_n=0$,$\sup\limits_n E|X_n|^{2+\delta}<\infty$(某 $\delta>0$).假设 $\alpha(n)=O(n^{-r})$,$r>1+2/\delta$ 且

$$\lim_{n\to\infty}\sigma_n^2/n = \sigma^2 > 0.$$

那么$\{X_n\}$服从重对数律和钟重对数律.

定理 9.3.2 和推论 9.3.2 的证明从略.

# 第十章　部分和的增量

在第九章中，我们研究了混合序列的部分和用一个 Wiener 过程去强逼近时的有关结果. 但是，借助于这些定理并不能得到类似于 $i.i.d.$ 序列情形时关于部分和增量的理想结果(参见 Csörgö和 Révész 1981). 在这一章中，我们通过直接估计增量大小的途径，对 $\varphi$ 混合序列(可以是不平稳甚至是不同分布的)建立了这类结果. 在对混合系数趋于零的速度的限制下，获得了与独立随机变量情形时的理想结果相接近的若干定理. 所用的方法也可用于处理其它类型的混合序列

## §10.1　几个引理

为了研究有关增量的 a.s. 大小，我们需要建立若干指数型的概率不等式. 为此，首先给出下面的结果(参见 Stout1974，引理 5.4.1及其推论).

**引理10.1.1**　设$\{Z_n,\mathscr{F}_n,n\geqslant 1\}$是上鞅，$EZ_n=0$. 记 $Z_0=0,U_i=Z_i-Z_{i-1}(i\geqslant 1)$. 假设存在$0\leqslant C<\infty$，对一切 $i\geqslant 1,U_i\leqslant C$. 固定 $\lambda>0$，使得 $\lambda C\leqslant 1$. 记

$$M_n=\exp(\lambda Z_n)\exp\left\{-(\lambda^2/2)(1+\lambda C/2)\sum_{i=1}^{n}E(U_i^2|\mathscr{F}_{i-1})\right\},n\geqslant 1,M_0=1\quad \text{a.s.}$$

那么$\{M_n,\mathscr{F}_n,n\geqslant 0\}$是一非负上鞅，且对任意的 $\alpha>0$，

$$P\{\sup_{n\geqslant 0}M_n>\alpha\}\leqslant\alpha^{-1}.$$

设$\{Yn\}$是 $\varphi$ 混合随机变量序列，混合系数 $\varphi_n=\varphi(n)\downarrow 0$. 不失一般性，假设对每一 $n,EY_n=0$. 记 $T_n=\sum_{i=1}^{n}Y_i,\tau_n^2=ET_n^2$. 本节中，我

们总假设下列条件是被满足的：

(a) $|Y_n|\leqslant b_n<\infty$；

(b) 存在$0<\sigma^2<\sigma'^2<\infty$，使得对每一 $m\geqslant 0$和充分大的 $n$，$\sigma^2 n\leqslant E(Y_{m+1}+\cdots+Y_{m+n})^2\leqslant\sigma'^2 n$.

设 $p,q,k$ 是正整数，满足 $p=p_n\leqslant n$，$q=q_n=0(p_n)$，$q_n\uparrow\infty$，$k=k_n=[n/(p_n+q_n)]$，记 $b=\max\limits_{1\leqslant i\leqslant n}b_i$.

**引理10.1.2** 假设

$$(10.1.1)\qquad \varphi_q pb^2=0(1),\ \sum_{j=1}^{k}\varphi^{1/2}(jp)=0(1).$$

又设 $x=x_n$ 和充分小的 $\varepsilon>0$，它们满足

$$(10.1.2)\qquad \frac{4}{\varepsilon}bn\varphi_q\leqslant x\leqslant\frac{\varepsilon\tau_n^2}{pb}$$

和

$$(10.1.3)\qquad x^2/n\to\infty.$$

那么对充分大的 $n$

$$(10.1.4)\quad P\{\max_{1\leqslant i\leqslant n}|T_i|\geqslant x\}\leqslant 3\exp\left\{-\frac{(1-6\varepsilon)x^2}{2\tau_n^2}\right\}.$$

如果用

$$(10.1.5)\qquad x>\varepsilon\tau_n^2/pb$$

代替(10.1.2)，则有

$$(10.1.6)\quad P\{\max_{1\leqslant i\leqslant n}|T_i|\geqslant x\}\leqslant 3\exp\left\{-\frac{\varepsilon(1-5\varepsilon)x}{2pb}\right\}.$$

**证** 下面总假设 $n$ 是充分大的整数. 首先，我们来证明(10.1.4). 定义

$$\xi_i=\sum_{j=i(p+q)+1}^{(i+1)p+iq}Y_j,\ \eta_i=\sum_{j=(i-1)p+iq+1}^{(i+1)(p+q)}Y_j,\ i=0,1,\cdots,k-1,$$

$$\eta_k=\sum_{j=k(p+q)+1}^{n}Y_j.$$

设 $\sigma$ 域 $\mathscr{F}_{-1}=\{\phi,\Omega\}$，$\mathscr{F}_i=\sigma\{Y_j,j\leqslant(i+1)p+iq\}$，$i=0,1\cdots,k-1$. 定义鞅差 $\gamma_i=\xi_i-E(\xi_i|\mathscr{F}_{i-1})$，$i=0,1,\cdots,k-1$. 写

$$(10.1.7)\quad P\{|T_n|\geqslant x\}\leqslant P\left\{\left|\sum_{i=0}^{k-1}\xi_i\right|\geqslant\left(1-\frac{\varepsilon}{2}\right)x\right\}$$

$$+P\left\{\left|\sum_{i=0}^{k}\eta_i\right|\geqslant\frac{1}{2}\varepsilon x\right\}=:J_1+J_2.$$

考虑 $J_1$. 写

$$(10.1.8)\quad J_1\leqslant P\left\{\left|\sum_{i=0}^{k-1}\gamma_i\right|\geqslant(1-\varepsilon)x\right\}$$

$$+P\left\{\left|\sum_{i=0}^{k-1}E(\xi_i|\mathscr{F}_{i-1})\right|\geqslant\frac{1}{2}\varepsilon x\right\}=:J_{11}+J_{12}.$$

由引理2.2.8(注意到 $|\xi_i|\leqslant pb$),对任意的 $B_{i-1}\in\mathscr{F}_{i-1}$,

$$|E\xi_i I_{B_{i-1}}|\leqslant 2\varphi_q pbp(B_{i-1}),i=0,1,\cdots,k-1.$$

由此推得

$$(10.1.9)\ |E(\xi_i|\mathscr{F}_{i-1})|\leqslant 2\varphi_q pb\quad \text{a.s.}\quad i=0,1,\cdots,k-1.$$

于是利用条件(10.1.2)就有

$$(10.1.10)\qquad J_{12}=0.$$

下面我们来估计 $J_{11}$. 因为 $\{r_i,\mathscr{F}_i,i=0,1,\cdots,k-1\}$ 具有鞅差性,利用引理10.1.1并注意到对充分大的 $n$, $|\gamma_i|\leqslant(1+\varepsilon)pb$ a.s. ,对于 $0<\lambda\leqslant((1+\varepsilon)pb)^{-1}$

$$\zeta_j:=\exp\left\{\lambda\sum_{i=0}^{j}\gamma_i\right\}\exp\left\{-\frac{\lambda^2}{2}\left(1+\frac{1}{2}(1+\varepsilon)pb\lambda\right)\sum_{i=0}^{j}E(r_i^2|\mathscr{F}_{i-1})\right\}$$

具有非负上鞅性. 写

$$(10.1.11)\quad P\left\{\sum_{i=0}^{k-1}\gamma_i\geqslant(1-\varepsilon)x\right\}=P\left\{\zeta_{k-1}\geqslant\exp(\lambda(1-\varepsilon)x)\right.$$

$$\left.\cdot\exp\left\{-\frac{\lambda^2}{2}\left(1+\frac{1}{2}(1+\varepsilon)pb\lambda\right)\sum_{i=0}^{k-1}E(\gamma_i^2|\mathscr{F}_{i-1})\right\}\right\}.$$

我们来估计 $\sum_{i=0}^{k-1}E(r_i^2|\mathscr{F}_{i-1})$. 写

$$\sum_{i=0}^{k-1}E(\gamma_i^2|\mathscr{F}_{i-1})=\sum_{i=0}^{k-1}E(\xi_i^2|\mathscr{F}_{i-1})-\sum_{i=0}^{k-1}(E(\xi_i|\mathscr{F}_{i-1}))^2.$$

通过类似于对(10.1.9)的证明,我们有 $|E(\xi_i^2|\mathscr{F}_{i-1})-E\xi_i^2|\leqslant 2\varphi_q(pb)^2$ a.s. 因此,由条件(b)和(10.1.1),得到

$$\sum_{i=0}^{k-1} E(\xi_i^2 | \mathscr{F}_{i-1}) = (1 + 0(1)) \sum_{i=0}^{k-1} E\xi_i^2 \quad \text{a. s.}$$

此外，不等式(10. 1. 9)推出

$$\sum_{i=0}^{k-1} (E(\xi_i | \mathscr{F}_{i-1}))^2 \leqslant 4\varphi_q^2 pb^2 n = 0(n) \quad \text{a. s.}$$

于是我们有

(10. 1. 12) $$\sum_{i=0}^{k-1} E(Y_i^2 | \mathscr{F}_{i-1}) = (1 + 0(1)) \sum_{i=0}^{k-1} E\xi_i^2 \quad \text{a. s.}$$

再来估计 $\sum_{i=0}^{k-1} E\xi_i^2$. 写

(10. 1. 13) $$\sum_{i=0}^{k-1} E\xi_i^2 = E\Big(\sum_{i=0}^{k-1} \xi_i\Big)^2 - 2 \sum_{0 \leqslant i < j \leqslant k-1} E\xi_i\xi_j.$$

由引理1. 2. 10, $|E\xi_i\xi_j| \leqslant 2\varphi((j-i-1)p)pb\sigma' p^{1/2}$. 因此对固定的 $i$，利用(10. 1. 1)中的假设，

$$\sum_{j \geqslant i+2} |E\xi_i\xi_j| \leqslant 2\sigma' \varphi_p^{1/2} p^{3/2} b \sum_{j=1}^{\infty} \varphi^{1/2}(jp) = 0(p),$$

对它关于 $i$ 求和即得阶$0(n)$. 此外再利用(10. 1. 1)又有

$$|E\xi_i\xi_{i+1}| \leqslant 2\varphi_q pb\sigma' p^{1/2} = 0(p),$$

求和后又得阶$0(n)$. 因此

(10. 1. 14) $$\sum_{i=0}^{k-1} E\xi_i^2 = E\Big(\sum_{i=0}^{k-1} \xi_i\Big)^2 + 0(n).$$

进一步再写

(10. 1. 15) $$E\Big(\sum_{i=0}^{k-1} \xi_i\Big)^2 = ET_n^2 - 2ET_n\Big(\sum_{i=0}^{k} \eta_i\Big) + E\Big(\sum_{i=0}^{k} \eta_i\Big)^2.$$

由条件(b)和(10. 1. 1)中的第二个等式，我们能够证明

$$E\Big(\sum_{i=0}^{k} \eta_i\Big)^2 = 0(kq + p).$$

事实上

$$E\Big(\sum_{i=0}^{k} \eta_i\Big)^2 \leqslant 2E\Big(\sum_{i=0}^{k-1} \eta_i\Big)^2 + 2E\eta_k^2,$$

其中 $E\eta_k^2 = 0(p)$而

$$E\Big(\sum_{i=0}^{k-1} \eta_i\Big)^2 = \sum_{i=0}^{k-1} E\eta_i^2 + 2 \sum_{0 \leqslant i < j \leqslant k-1} E\eta_i\eta_j$$

$$\leqslant \sigma'^2 kq + 2\sigma'^2 q \sum_{j=1}^{k-1} (k-j)\varphi^{1/2}(jp) = O(kq).$$

因此从(10.1.15)推出 $E\left(\sum_{i=0}^{k-1}\xi_i\right)^2=(1+0(1))\tau_n^2$. 将这一结果与(10.1.14)和(10.1.12)相结合即得

(10.1.16) $\quad \displaystyle\sum_{i=0}^{k-1} E(Y_i^2|\mathscr{F}_{i-1}) = (1+0(1))\tau_n^2 \quad \text{a.s.}$

将它代入(10.1.11)并利用引理10.1.1,对充分大的 $n$ 我们得

(10.1.17) $\displaystyle P\left\{\sum_{i=0}^{k-1} Y_i \geqslant (1-\varepsilon)x\right\} \leqslant \exp\{-\lambda(1-\varepsilon)x\}$

$$\exp\left\{\frac{\lambda^2}{2}\left(1+\frac{1}{2}(1+\varepsilon)pb\lambda\right)(1+\varepsilon)\tau_n^2\right\}.$$

取 $\lambda=x/((1+\varepsilon)\tau_n^2)$. 由(10.1.2)我们有 $\lambda\leqslant\varepsilon((1+\varepsilon)pb)^{-1}$. 将它代入(10.1.17)得到

(10.1.18) $\displaystyle P\left\{\sum_{i=0}^{k-1} Y_i \geqslant (1-\varepsilon)x\right\} \leqslant \exp\left\{-\frac{(1-4\varepsilon)x^2}{2\tau_n^2}\right\}.$

用 $-Y_j$ 代替 $Y_j$ 即有

$$J_{11} = P\left\{\left|\sum_{i=0}^{k-1} Y_i\right| \geqslant (1-\varepsilon)x\right\} \leqslant 2\exp\left\{-\frac{(1-4\varepsilon)x^2}{2\tau_n^2}\right\}.$$

将它与(10.1.10)结合得到

$$J_1 \leqslant 2\exp\left\{-\frac{(1-4\varepsilon)x^2}{2\tau_n^2}\right\}.$$

对于 $J_2$,注意到 $q_n=0(p_n)$,易见它的一个上界是$3^{-1}\exp\{-(1-4\varepsilon)x^2/(2\tau_n^2)\}$. 至此我们已证明了

(10.1.19) $\displaystyle P\{|T_n| \geqslant x\} \geqslant \frac{7}{3}\exp\left\{-\frac{(1-4\varepsilon)x^2}{2\tau_n^2}\right\}.$

为了从它推出(10.1.4),我们需要利用引理5.1.2. 首先因为 $\varphi_n \downarrow 0$,所以存在一个整数 $n_0$使得 $\varphi_{n_0}\leqslant 10^{-1}$. 由条件(b)和(10.1.3)可得对充分大的 $n$,

$$\max_{1\leqslant i\leqslant n} P\left\{|T_n - T_i| \geqslant \frac{\varepsilon x}{2(1+\varepsilon)}\right\} \leqslant \max_{1\leqslant i\leqslant n}$$

$$\frac{4(1+\varepsilon)^2 E(T_n-T_i)^2}{\varepsilon^2 x^2} \leqslant \frac{4(1+\varepsilon)^2\sigma'^2 n}{\varepsilon^2 x^2} \leqslant \frac{1}{10}.$$

因此引理5.1.2中的 $\eta$ 可以选为 $5^{-1}$. 此外，从(10.1.2)和(10.1.3)有

$$(10.1.20)\qquad \max_{1\leqslant i\leqslant n}|Y_i| \leqslant b \leqslant \frac{\varepsilon\sigma'^2 n}{px} \leqslant \frac{\varepsilon\sigma'^2 k}{x^2}x = 0(x).$$

因此对充分大的 $n$

$$P\{\max_{1\leqslant i\leqslant n}|Y_i| \geqslant \varepsilon x/(2(1+\varepsilon)(n_0-1))\} = 0.$$

应用这些结果和引理5.1.2我们得

$$P\{\max_{1\leqslant i\leqslant n}|T_i| \geqslant x\} \leqslant \frac{5}{4}P\left\{|T_n| \geqslant \frac{x}{1+\varepsilon}\right\} + \frac{5}{4}P$$

$$\left\{\max_{1\leqslant i\leqslant n}|Y_i| \geqslant \frac{\varepsilon x}{2(1+\varepsilon)(n_0-1)}\right\} \leqslant 3\exp\left\{-\frac{(1-6\varepsilon)}{2\tau_n^2}x^2\right\}.$$

这也就是(10.1.4).

现在来证明(10.1.6). 显然我们能假设有某个 $\delta>0$, $x/pb>\delta$. 如果条件(10.1.5)是被满足的，通过取(10.1.17)中的 $\lambda=\varepsilon((1+\varepsilon)pb)^{-1}$,我们有

$$(10.1.21)\quad P\left\{\sum_{i=0}^{k-1}\gamma_i \geqslant (1-\varepsilon)x\right\} \leqslant \exp\left\{-\frac{\varepsilon(1-4\varepsilon)x}{2pb}\right\}.$$

模仿从(10.1.18)到(10.1.4)的证明过程并用 $\max_{1\leqslant i\leqslant n}|Y_i|\leqslant b<x/(p\delta)=0(x)$ 代替(10.1.20)，由(10.1.21)即得(10.1.16). 引理证毕.

**引理10.1.3** 假设引理10.1.2中的(10.1.1)用下列条件代替

$$(10.1.1)'\qquad \varphi_p q b^2 = o(1),\ \sum_{j=1}^{k}\varphi^{1/2}(jp) = O(1).$$

那么对任意的 $\nu>o$，存在 $\alpha(\nu)$，使得对所有充分大的 $n$ 和满足 $\alpha\geqslant\alpha(\nu)$ 的 $\alpha$，当

$$(10.1.22)\qquad \varphi_q = o(\alpha/b)$$

且

$$(10.1.23)\qquad pb\alpha\tau_n = o(n)$$

时成立着

$$P\{T_n \geqslant \alpha\tau_n\} \geqslant \frac{1}{2}\exp\{-(1+\nu)\alpha^2/2\}.$$

**证** 对$0<\delta<1$写

$$(10.1.24)\quad P\{T_n\geqslant\alpha\tau_n\}\geqslant P\Big\{\sum_{i=0}^{k-1}\xi_i\geqslant(1+\delta)\alpha\tau_n\Big\}$$

$$-P\Big\{\sum_{i=0}^{k}\eta_i\leqslant-\delta\alpha\tau_n\Big\}=:I_1-I_2$$

和

$$(10.1.25)\quad I_1\geqslant P\Big\{\sum_{i=0}^{k-1}\gamma_i\geqslant(1+2\delta)\alpha\tau_n\Big\}$$

$$-P\Big\{\sum_{i=0}^{k-1}E(\xi_i|\mathscr{F}_{i-1})\leqslant-\delta\alpha\tau_n\Big\}=:I_{11}-I_{12}.$$

利用条件(10.1.22)并回顾(10.1.9)，对充分大的$n$，我们也有

$$(10.1.26)\quad I_{12}=0.$$

下面来估计$I_{11}$. 令$\varepsilon$是一个待定的小正数. 记$v=3\sqrt{\varepsilon}/(1-3\sqrt{\varepsilon})$，$u=(1+2\delta)\alpha/(1-v)$. 写

$$(10.1.27)\quad E\Big\{\exp\Big[u\Big(\sum_{i=0}^{k-1}\gamma_i\Big)/\tau_n\Big]|\mathscr{F}_{k-2}\Big\}$$

$$=\exp\Big\{u\Big(\sum_{i=0}^{k-2}\gamma_i\Big)/\tau_n\Big\}E\{\exp(u\gamma_{k-1}/\tau_n)|\mathscr{F}_{k-2}\}.$$

条件(10.1.23)推出

$$(10.1.28)\quad upb/\tau_n\to0.$$

因此

$$E\{\exp(u\gamma_{k-1}/\tau_n)|\mathscr{F}_{k-2}\}\geqslant1+\frac{u^2}{2\tau_n^2}\Big(1-\frac{upb}{\tau_n}\Big)E(\gamma_{k-1}^2|\mathscr{F}_{k-2})$$

$$\geqslant\exp\Big\{\frac{u^2}{2\tau_n^2}\Big(1-\frac{2upb}{\tau_n}\Big)E(\gamma_{k-1}^2|\mathscr{F}_{k-2})\Big\}.$$

将它代入(10.1.27)，则有

$$E\Big\{\Big[\exp\Big(u\Big(\sum_{i=0}^{k-1}\gamma_i\Big)/\tau_n\Big)\exp\Big(-\frac{u^2}{2\tau_n^2}\Big(1-\frac{2upb}{\tau_n}\Big)$$

$$E(\gamma_{k-1}^2|\mathscr{F}_{k-2})\Big)\Big]|\mathscr{F}_{k-2}\Big\}\geqslant\exp\Big\{u\Big(\sum_{i=0}^{k-2}\gamma_i\Big)/\tau_n\Big\}.$$

于是，用归纳法得出

$$E\left\{\exp\left[u\left(\sum_{i=0}^{k-1}\gamma_i\right)/\tau_n\right]\right.$$
$$\left.\exp\left[-\frac{u^2}{2\tau_n^2}\left(1-\frac{2upb}{\tau_n}\right)\sum_{i=0}^{k-1}E(\gamma_i^2\mid\mathscr{F}_{i-1})\right]\right\}\geqslant 1.$$

因为我们也有(10.1.16)，所以上面的不等式可改写为：对给定的$\varepsilon>0$，只要$n$充分大，成立着

$$(10.1.29)\quad E\left\{\exp\left[u\left(\sum_{i=0}^{k-1}\gamma_i\right)/\tau_n\right]\right\}\geqslant\exp\left\{\frac{u^2}{2}\left(1-\frac{2upb}{\tau_n}-\frac{\varepsilon}{4}\right)\right\}$$
$$\geqslant\exp\left\{\frac{u^2}{2}\left(1-\frac{\varepsilon}{2}\right)\right\}.$$

接下去的证明类似于 Stout(1974)中的定理5.2.2的证明的相应部分，只是某些细节略有不同(注意$v$和$u$的取法)。详细过程不在此叙述了.

除了上面的引理10.1.2和引理10.1.3之外，下述关于 Borel-Cantelli 引理的推广(见 Erdös1959)也将是需要的.

**引理10.1.4** 如果$C_1,C_2,\cdots$是一列事件，满足

$$\sum_{n=1}^{\infty}P(C_n)=\infty\text{ 且}\liminf_{n\to\infty}\sum_{k=1}^{n}\sum_{l=1}^{n}P(C_kC_l)/\left(\sum_{k=1}^{n}P(C_k)\right)^2=1,$$

那么

$$P\{C_n,i.o.\}=1.$$

## §10.2 矩母函数存在时增量有多大

设$\{X_n,n\geqslant1\}$是$\varphi$混合的随机变量序列，$EX_n=0,n\geqslant1$. 记$S_n=\sum_{k=1}^{n}X_k,\sigma_k^2=EX_k^2$.

**定理10.2.1** (林正炎1991). 假设上面定义的$\{X_n\}$满足下列条件：

(i) $\liminf_{n\to\infty}\inf_{m\geqslant0}E(X_{m+1}+\cdots+X_{m+n})^2/n>0$;

(ii) 存在$t_0$、$M>0$，使得对每一$k$和任意的$|t|\leqslant t_0$，$Ee^{tX_k}\leqslant M$;

(iii) 存在 $l>1$，使得 $\varphi_n=O(n^{-l})$.

又设 $\{a_n\}$ 是不减的正整数序列，满足

(a) 存在 $a>0$，使得 $a\log^d n\leqslant a_n\leqslant n$，其中 $d>(3l+1)/(l-1)$.

那么，若记 $\sigma_{nN}^2=E(X_{n+1}+\cdots+X_{n+a_N})^2$，$\beta_{nN}=\sigma_{nN}\{2[\log(N/\sigma_{nN}^2)+\log\log N]\}^{1/2}$，$S(n,k)=S_{n+k}-S_n$，成立着

(10.2.1) $$\limsup_{N\to\infty}\beta_{NN}^{-1}|S(N,a_N)|=1\quad \text{a.s.}$$

(10.2.2) $$\limsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\beta_{nN}^{-1}|S(n,a_N)|=1\quad \text{a.s.}$$

(10.2.3) $$\limsup_{N\to\infty}\max_{1\leqslant k\leqslant a_N}\beta_{NN}^{-1}|S(N,k)|=1\quad \text{a.s.}$$

(10.2.4) $$\limsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\beta_{nN}^{-1}|S(n,k)|=1\quad \text{a.s.}$$

进一步，如果条件(a)用下列条件代替

(b) 存在 $a>0$，使得 $an^{1/(l+1)}\leqslant a_n\leqslant n$ 且

$$\lim_{n\to\infty}\log(n/a_n)/\log\log n=\infty,$$

那么

(10.2.5) $$\lim_{N\to\infty}\max_{1\leqslant n\leqslant N}\beta_{nN}^{-1}|S(n,a_N)|=1\quad \text{a.s.}$$

(10.2.6) $$\lim_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\beta_{nN}^{-1}|S(n,k)|=1\quad \text{a.s.}$$

**证** 从条件(i)-(iii)，存在常数 $w_1$，和 $w_2$，$0<w_1\leqslant w_2<\infty$，使得对每一整数 $m\geqslant0$和大的 $n$

(10.2.7) $$w_1n\leqslant E(X_{m+1}+\cdots+X_{m+n})^2\leqslant w_2n.$$

其中右边的不等式是由于从引理1.2.8，对$1<l'<l$ 和 $i<j$

$$|EX_iX_j|\leqslant 2\varphi_{j-i}^{1/l'}E^{1/l'}|X_i|^{l'}E^{1-1/l'}|X_j|^{l'/(l'-1)}$$
$$=O((j-i)^{-l/l'}).$$

首先，我们证明(10.2.1)－(10.2.4). 显然，只需验证

(10.2.8) $$\limsup_{N\to\infty}\beta_{NN}^{-1}|S(N,a_N)|\geqslant1\quad \text{a.s.}$$

(10.2.9) $$\limsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\beta_{nN}^{-1}|S(n,k)|\leqslant1\quad \text{a.s.}$$

就够了. 对 $B>1/t_0$，定义

$$Y_n=X_nI(|X_n|<B\log n),Y'_n=Y_n-EY_n,T_n=\sum_{k=1}^nY'_k,$$

$$\lambda_{nN}^2 = E(Y_{n+1} + \cdots + Y_{n+a_N})^2, T(n,k) = T_{n+k} - T_n,$$

$$\alpha_n N = \lambda_{nN}\{2[\log(N/\lambda_{nN}^2) + \log\log N]\}^{1/2}.$$

从条件(ii)可知 $P\{X_n \neq Y_n, i.o.\}=0$且对 $t'$，当它满足 $t'<t_0$和 $t'B>1$时

$$|EY_n| \leqslant cn^{-t'B}.$$

因此当 $N\to\infty$时

(10.2.10) $$\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N} \beta_{nN}^{-1}|S(n,k) - T(n,k)| \to 0 \quad \text{a.s.}$$

此外，由(10.2.7)和 $Y_n$ 的定义，不难验证对 $n$ 一致地有

(10.2.11) $$\sigma_{nN}^2/\lambda_{nN}^2 \to 1, N\to\infty.$$

我们来证明(10.2.8)．它等价于

(10.2.12) $$\limsup_{N\to\infty} \alpha_{NN}^{-1}|T(N,a_N)| \geqslant 1 \quad \text{a.s.}$$

记 $h=2/(3l+1)$．定义 $p_n=[n^h]$，$q_n=[\alpha_n p_n]$，其中$\{\alpha_n\}$是一个以足够慢的速度趋于零的正数序列．由条件(iii)，(a)和 $p_n$ 及 $q_n$ 的定义，可以看出，只要 $\alpha_n$ 趋于0足够慢就有

$$\varphi_{q_{a_N}} P_{a_N}(\log N)^2 \leqslant c\alpha_{a_N}^{-l} p_{a_N}^{-l+1}(\log N)^2$$

$$\leqslant c\alpha_{a_N}^{-l} a_N^{-2(l-1)/(3l+1)}(\log N)^2 \to 0, N\to\infty.$$

借助于条件(iii)，我们又有

$$\sum_{j=1}^k \varphi^{1/2}(jp) \leqslant c\sum_{j=1}^k (jp)^{-l/2}$$

$$\leqslant ckn^{-l/2} \leqslant cn^{(3l-1)/(3l+1)-l/2} = o(1).$$

因此条件(10.1.1)，进而条件(10.1.1)′，是被满足的．类似地我们有

$$\varphi_{q_{a_n}} = o((\log(N/\lambda_{NN}^2) + \log\log N)^{1/2}/\log N)$$

和 $p_{a_n}(\log N)\alpha_{nN}/a_n = o(1)$．因此我们可以应用引理10.1.3(选取 $\nu=\varepsilon>0$)，只要 $N$ 充分大就有

(10.2.13) $$P(C_N) \geqslant \frac{1}{2}\exp\{-(1+\varepsilon)(1-\varepsilon)^2\log(N/\lambda_{NN}^2) + \log\log N\} \geqslant c\left(\frac{a_N}{N\log N}\right)^{1-\varepsilon},$$

其中 $C_N=\{\alpha_{NN}^{-1}T(N,a_N)\geqslant 1-\varepsilon\}$. 记 $n_1=1$,定义 $n_{k+1}=n_k+2a_{Nk}$.
由 $\varphi$ 混合性,我们有

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{m}P(C_{n_j}C_{n_k})/\left(\sum_{k=1}^{m}P(C_{n_k})\right)^2-1\right|$$

$$\leqslant\frac{\sum_{j=1}^{m}P(c_{n_j})(1-P(C_{n_j}))+4\sum_{j=1}^{m}\sum_{k=j+1}^{m}P(C_{n_j})\varphi(n_k-n_j-a_{n_j})}{\left\{\sum_{k=1}^{m}P(C_{n_k})\right\}^2}$$

$$\leqslant\left\{1+4\sum_{k=2}^{m}\varphi(n_k-n_1-a_{n_1})\right\}/\sum_{k=1}^{m}P(C_{n_k})$$

利用条件(iii)并回顾 $n_k$ 的定义,容易验证 $\sum_{k=2}^{\infty}\varphi(n_k-n_1-a_{n_1})<\infty$.
此外,因为

$$\sum_{n=n_k+1}^{n_{k+1}}(n\log n)^{-1}\leqslant(n_k\log n_k)^{-1}$$

$$(n_{k+1}-n_k)=(n_k\log n_k)^{-1}2a_{n_k}\leqslant cP(C_{n_k}).$$

故有 $\sum_{k=1}^{\infty}P(C_{n_k})=\infty$. 因此从引理10.1.4即有

$$P\{C_{n_k},i,o.\}=1.$$

由此得证(10.2.12),也就完成了(10.2.8)的证明.

进一步我们来证明(10.2.9). 从(10.2.10)和(10.2.11)可知,这时只需证明

(10.2.14) $\limsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\alpha_{nN}|T(n,k)|\leqslant 1\quad$ a.s.

就够了. 令 $r=r(\varepsilon)>1$是一个待定的正整数. 记 $R=[a_N/r]$, $n_r=R[n/R]$. 由(10.2.7)和(10.2.11)对充分大的 $n$ 我们有

(10.2.15) $$|\lambda_{nN}^2/E(Y_{n_r+1}+\cdots+Y_{(n+a_N)_r})^2-1|\leqslant\frac{2w_2(2r^{1/2}+1)}{w_1(r-1)}.$$

写

(10.2.16) $$|T_{n+k}-T_n|\leqslant|T_{n+k}-T_{(n+k)_r}|+|T_{(n+k)_r}-T_{n_r}|+|T_{n_r}-T_n|.$$

考虑(10.2.16)右边的第二项，我们有

$$(10.2.17)\qquad P\Big\{\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\alpha_{nN}^{-1}|T_{(n+k)_r}-T_{n_r}|\geqslant 1+\varepsilon/3\Big\}$$

$$\leqslant cr\frac{N}{a_N}\max_{1\leqslant n\leqslant N}P\Big\{\max_{1\leqslant k\leqslant a_N}|T_{(n+k)_r}-T_{n_r}|\geqslant(1+\varepsilon/3)\alpha_{nN}\Big\}.$$

显然，$|(n+a_N)_r-n_r|\leqslant a_N(1+1/r)$. 因为$(\log N)a_N\varphi_{q_{a_N}}=0(\alpha_{nN})$且$\alpha_{nN}=0(E(T_{(n+a_N)_r}-T_{n_r})^2/p_{a_N}\log N)$，我们可以应用引理10.1.2的(10.1.4). 注意到(10.2.15)并选取 $r$ 足够大，对充分大的 $N$，(10.2.17)的右边不超过

$$cr\frac{N}{a_N}\exp\{-(1-\varepsilon/3)(1+\varepsilon/3)^2\alpha_{nN}^2/E(T_{(n+a_N)_r}-T_{n_r})^2\}$$

$$\leqslant cr\frac{N}{a_N}\Big(\frac{a_N}{N\log N}\Big)^{1+\varepsilon/4}=cr\Big(\frac{a_N}{N}\Big)^{\varepsilon/4}(\log N)^{-(1+\varepsilon/4)}.$$

记 $N_1=1$，通过 $a_{N_{j+1}}=\min\{a_n,a_n\geqslant[\theta^j]\}(\theta>1)$，定义 $N_{j+1}$. 显然 $N_{j+1}\geqslant[\theta^j]$(因 $a_N\leqslant N$). 因此

$$\sum_{j=1}^{\infty}P\Big\{\max_{1\leqslant n\leqslant N_j}\max_{1\leqslant k\leqslant a_{N_j}}\alpha_{nN}^{-1},|T_{(n+k)_r}-T_{n_r}|\geqslant 1+\frac{\varepsilon}{3}\Big\}<\infty,$$

由它即得

$$(10.2.18)\quad \limsup_{j\to\infty}\max_{1\leqslant n\leqslant N_j}\max_{1\leqslant k\leqslant a_{N_j}}\alpha_{nN_j}^{-1}|T_{(n+k)_r}-T_{n_r}|\leqslant 1+\frac{\varepsilon}{3}\ \text{a.s.}$$

从条件(i)-(iii)和(10.2.11)以及 $N_j$ 的定义，不难看出存在一常数 $G>0$使得

$$(10.2.19)\qquad \left|\lim_{j\to\infty}\frac{\lambda_{nN_{j+1}}^2}{\lambda_{nN_j}^2}-1\right|\leqslant G(\theta-1)^{1/2}.$$

选取 $\theta$ 充分接近于1并注意到(10.2.18)里的两个 max 中的变程随着 $j$ 的增加而变大，从(10.2.18)我们得

$$(10.2.20)\quad \limsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\alpha_{nN}^{-1}|T_{(n+k)_r}-T_{n_r}|\leqslant 1+\frac{\varepsilon}{2}\ \text{a.s.}$$

转向考虑(10.2.16)右边的第一项. 写

$$(10.2.21)\quad P\Big\{\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\alpha_{nN}^{-1}|T_{n+k}-T_{(n+k)_r}|\geqslant\frac{\varepsilon}{6}\Big\}$$

$$\leqslant \frac{rN}{a_N} \max_n \max_k P\left\{ \max_{(n+k)_r < j \leqslant (n+k)_r + R} |T_j - T_{(n+k)_r}| \geqslant \frac{\varepsilon}{6} \alpha_{nN} \right\}.$$

类似于在(10.2.18)的证明中提到的，我们也能够利用引理10.1.2中的(10.1.4). 回顾(10.2.7)和(10.2.11)，选取 $r=r(\varepsilon)$ 足够大，那么对一切充分大的 $N$，(10.2.21)的右边不超过

$$3\frac{rN}{a_N}\exp\left\{-\frac{(1-6\varepsilon)\varepsilon^2\lambda_{nN}^2}{36E(T_{(n+k)_r+R}-T_{(n+k)_r})^2}\log\frac{N\log N}{\lambda_{nN}^2}\right\}$$

$$\leqslant \frac{3rN}{a_N}\exp\left\{-cr\varepsilon^2\log\frac{N\log N}{a_N}\right\} \leqslant \frac{3rN}{a_N}\left\{\frac{aN}{N\log N}\right\}^{cr\varepsilon^2} \leqslant c\log^{-2}N.$$

因此

$$\limsup_{j\to\infty} \max_{1\leqslant n\leqslant N_j} \max_{1\leqslant k\leqslant a_{N_j}} \alpha_{nN_j}^{-1} |T_{n+k} - T_{(n+k)_r}| \leqslant \frac{\varepsilon}{6} \quad \text{a. s.}$$

模仿从(10.2.18)推到(10.2.20)的步骤，我们得

(10.2.22) $$\limsup_{N\to\infty} \max_{1\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \alpha_{nN}^{-1} |T_{n+k} - T_{(n+k)_r}| \leqslant \frac{\varepsilon}{4} \quad \text{a. s.}$$

显然，我们又有

$$\limsup_{N\to\infty} \max_{1\leqslant n\leqslant N} \alpha_{nN}^{-1} |T_n - T_{n_r}| \leqslant \frac{\varepsilon}{4} \quad \text{a. s.}$$

因此(10.2.14)，进而(10.2.9)，得证.

最后，我们来证明(10.2.5)和(10.2.6)，为此目的，从已经证明的(10.2.2)和(10.2.4)，只需验证下式就够了：

(10.2.23) $$\liminf_{N\to\infty} \max_{1\leqslant n\leqslant N} \beta_{NN}^{-1} |S(n, a_N)| \geqslant 1 \quad \text{a. s.}$$

由(10.2.10)和(10.2.11)，(10.2.23)等价于

(10.2.24) $$\liminf_{N\to\infty} \max_{1\leqslant n\leqslant N} \alpha_{nN}^{-1} |T(n, a_N)| \geqslant 1 \quad \text{a. s.}$$

对充分大的 $N$ 我们有

$$P\left\{ \max_{1\leqslant n\leqslant N} \alpha_{nN}^{-1} |T(n, a_N)| \leqslant 1 - \varepsilon \right\}$$

$$\leqslant P\left\{ \max_{1\leqslant j\leqslant [N/a_N]^{1-\varepsilon/2}} \alpha_{2ja_N,N}^{-1} |T(2ja_N, a_N)| \leqslant 1 - \varepsilon \right\}$$

$$\leqslant \prod_{j=1}^{[N/a_N]^{1-\varepsilon/2}} P\{\alpha_{2ja_N,N}^{-1} |T(2ja_N, a_N)|$$

$$\leqslant 1-\varepsilon\}+\left(\frac{N}{a_N}\right)^{1-\varepsilon/2}\varphi(aN).$$

由(10.2.13)和条件($b$)，上式右端不超过

$$\left\{1-c\left(\frac{a_N}{N\log N}\right)^{1-\varepsilon}\right\}^{[N/aN]^{1-\varepsilon/2}}+c\left(\frac{N}{a_N}\right)^{-\varepsilon/2}$$

$$\leqslant 2\exp\left\{-c\left(\frac{N}{a_N}\right)^{-\varepsilon/2}(\log N)^{-(1-\varepsilon)}\right\}+c\left(\frac{N}{a_N}\right)^{-\varepsilon/2}\leqslant c(\log N)^{-2}.$$

最后的不等号是由于条件(b). 因此如果 $N_j$ 如上面定义就得

(10.2.25) $\qquad \liminf_{j\to\infty}\max_{1\leqslant n\leqslant N_j}\alpha_{nN_j}^{-1}|T(n_ja_{N_j})|\geqslant 1 \quad$ a.s.

考虑 $N_j<N\leqslant N_{j+1}$，我们有

$$\liminf_{N\to\infty}\max_{1\leqslant n\leqslant N}\alpha_{nN}^{-1}|T(n_ja_N)|$$

$$\geqslant\liminf_{j\to\infty}\max_{1\leqslant n\leqslant N_j}(\alpha_{nN_j}^{-1}|T(n,aN_j)|)(\alpha_nN_j\alpha_{nN}^{-1})$$

$$-\limsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N-a_{N_j}}\alpha_{nN}^{-1}|T(n_jk)|.$$

由(10.2.25)和(10.2.19)，存在 $G'>0$使得上式右边的第一项 a.s. $\geqslant 1-G'(\theta-1)^{1/2}$；而由(10.2.14)，存在 $G''>0$使得第二项 a.s. $\leqslant G''(1-1/\theta)^{1/2}$，令 $\theta\downarrow 1$即得(10.2.24). 定理证毕.

**注10.2.1** 当条件(iii)中的 $l$ 足够大时，换言之，$\varphi_n$ 以很高的速度趋于零时，$a_n$ 可以是$0(\log^{3+\varepsilon}n)$，其中 $\varepsilon$ 是任给的正数. 对于一个独立序列，相应的要求是 $a_n/\log n\to\infty(n\to\infty)$(见林正炎1988).

**注10.2.2** 在上述定理中，并不要求 $a_n/n$ 是不增的. 但是即使对独立序列，这一条件也曾是被假设的(见 Csörgö和Révész 1981). 事实上，这一条件是不实际的，因为在此条件下或者对所有的 $n$，$a_n=n$，或者对某个固定的 $m$，$a_n=m\wedge n$.

## §10.3 矩母函数不存在时增量有多大

设$\{X_n,n\geqslant 1\}$是上节开头提到的 $\varphi$混合序列.

**定理10.3.1** (林正炎1989). 假设$\{Xn\}$满足定理10.2.1中的条件(i)和(ii)′存在不减连续函数 $H(x)$，$x>0$，使得

(10.3.1)　对任意的 $\delta>0$，$\sum_{n=1}^{\infty}P\{H(|X_n|)>\delta n\}<\infty$；

(10.3.2)　对任意的 $\beta<1$ 关于 $n$ 一致地有

$$\varlimsup_{K\to\infty}\frac{1}{k}\sum_{j=n+1}^{n+k}E(H(|X_j|))^{\beta}\leqslant M<\infty;$$

(10.3.3)对某个 $\gamma>0$，$x^{-(2+\gamma)}H(x)$是不减的；

(10.3.4)　$\varliminf_{x\to\infty}H(x/2)/H(x)>0$，

(iii)′存在 $l\geqslant 1+4/r$ 使得 $\varphi_n=0(n^{-l})$.

又设$\{a_n\}$是不减的正整数序列，满足

(a)′存在 $a>0$使得 $a(\mathrm{inv}H(n))^d\leqslant a_n\leqslant n$，其中 $d=z(l+1)/(l-1)$.

那么定理10.2.1的结论仍然成立.

**证**　本定理的证明与定理10.2.1的证明类似，我们仅给出差别部分，容易验证：条件(10.3.1)可以用下述条件代替：对某列 $\delta_n\downarrow 0(n\to\infty)$，

(10.3.5)　$$\sum_{n=1}^{\infty}P\{H(|X_n|)>\delta_n n\}<\infty.$$

定义 $Y_n=X_nI(|X_n|\leqslant\mathrm{inv}H(\delta_n n))$，$Y'_n=Y_n-EY_n$，$T_n=\sum_{k=1}^{n}Y'_k$. 由(10.3.5)我们有

(10.3.6)　$$P\{X_n\neq Y_n,i,o.\}=0.$$

不失一般性，可取 $\delta_n$ 使之对任意的 $\varepsilon>0$满足 $n^{-\varepsilon}=0(\delta_n)$. 由此并利用条件 $EX_j=0$，(10.3.2)和(10.3.3)，对充分大的 $n$ 我们有

(10.3.7)　$$\begin{aligned}\sum_{j=n+1}^{n+k}|EY_j|&\leqslant c\sum_{j=n+1}^{n+k}\int_{\{H(|X_j|)>\delta_j j\}}(H(|X_j|))^{\frac{1}{2+\gamma}}dP\\&\leqslant c\sum_{j=n+1}^{n+k}(\delta_j j)^{-\frac{2+\gamma}{4+\gamma}}E(H(|X_j|))^{\frac{1}{2+\gamma}+\frac{2+\gamma}{4+\gamma}}\\&\leqslant c\sum_{j=n+1}^{n-k}j^{-\frac{3+\gamma}{6+\gamma}}E(H(|X_j|))^{\frac{\gamma^2+5\gamma+8}{\gamma^2+6\gamma+8}}\end{aligned}$$

$$\leqslant ck^{-\frac{3+\gamma}{6+\gamma}}\left\{\sum_{j=n+1}^{n+k} E(H(|X_j|))^{\frac{\gamma^2+5.5\gamma+8}{\gamma^2+6\gamma+8}}\right\}^{\frac{\gamma^2+5\gamma+8}{\gamma^2+5.5\gamma+8}}$$

$$\leqslant ck^{\frac{3}{6+\gamma}}.$$

因此(10.2.8)也等价于(10.2.12). 记 $h=1/(l+1)$, $p_n=[n^h]$, $q_n=[\alpha_n p_n]$, 其中 $\alpha_n$ 是足够慢地趋于零的正数列. 利用条件(iii)′和(a)′, 我们得

$$\varphi_{q_{a_N}} P_{a_N}(\mathrm{inv}H(\delta_N N))^2 \leqslant c\alpha_{a_N}^{-l}\alpha_N^{-h(l-1)}(\mathrm{inv}H(\delta_n N))^2$$

$$\leqslant c\alpha_{a_N}^{-l}(\mathrm{inv}H(\delta_N N)/\mathrm{inv}H(N))^2.$$

由条件(10.3.4)可推出 $\mathrm{inv}H(\delta_N N)/\mathrm{inv}H(N)\to 0$(见林正炎和陆传荣1992,(2.3.12)). 因此只要 $\alpha_N$ 趋于零足够慢我们就有

$$\varphi_{q_{a_N}} P_{a_N}(\mathrm{inv}H(\delta_N N))^2 \to 0, N\to\infty.$$

此外引理10.1.2中的其它条件也是被满足的. 于是我们也有(10.2.13), 下面的证明与定理10.2.1的类似, 不再赘述了.

**注 10.3.1** 当条件(iii)′中的 $l$ 充分大也即 $\varphi_n$ 趋于零足够快时, $a_n$ 可以是 $O((\mathrm{inv}H(n))^{2+\epsilon})$, 其中 $\epsilon$ 是任意给定的正数. 对于一个独立序列, 相应的要求是 $a_n\geqslant c(\mathrm{inv}H(n))^2/\log n$(见林正炎1987).

# 第十一章　混合随机场的强逼近

设$\{X_j, j\in \mathbb{N}^d\}$是实值混合相依平稳随机场，$EX_j=0$. 各种混合性的定义及记号$\mathscr{A}\subset\epsilon\subset\mathscr{B}^d, C_j, \mathscr{A}_\delta, A(\delta)$等与第六章中相同。定义部分和过程

(11.0.1)　$S_n(A) = \sum_j |nA \cap C_j| X_j \qquad A \in \mathscr{A},$

其中$\boldsymbol{n}=(n_1,\cdots,n_d)\in \mathbb{N}^d$, $\boldsymbol{n}A=\{(n_1x_1,\cdots,n_dx_d), \boldsymbol{x}=(x_1,\cdots,x_d)\in A\}$.

集指标过程强逼近的中心问题就是研究在$\mathscr{A}$及$\{X_j\}$上附设一定条件下，构作一新的概率空间，在其上有一独立同分布中心化的 Gauss 随机场$\{Y_j, j\in \mathbb{N}^d\}$，其方差等于$\sigma^2=EX_1^2+2\sum_j EX_{j+1}X_1$，且不改变$\{X_j, j\in \mathbb{N}^d\}$的分布，可在新的概率空间中重新定义$\{X_j\}$使得

(11.0.2)　$$D_n = \sup_{A\in\mathscr{A}} \Big\{\sum_{j\in \mathbb{N}^d} |nA\cap C_j|(X_j - Y_j)\Big\}$$
$$= O(n^{d/2-\epsilon}) \text{ 或 } O(n^{d/2}(\log n)^{-a}) \quad (a,\epsilon>0).$$

对$0<\beta\leqslant 1$，记

(11.0.3)　$|\boldsymbol{n}| = n_1n_2\cdots n_d, \qquad$ 若 $\boldsymbol{n}=(n_1,\cdots,n_d)$,

$$G_\beta = \Big\{\boldsymbol{n}=(n_1,\cdots,n_d), n_i \geqslant \prod_{k\neq i} n_k^\beta, i=1,2,\cdots,d\Big\}.$$

Berkes 和 Morrow(1981)首先对$\alpha$混合平稳随机场讨论了强逼近. 假设$EX_1=0$，对某个$\delta>0$，$E|X_1|^{2+\delta}<\infty$且$\alpha(n)=O(n^{-d(1+\epsilon)/(1+2/\delta)})$. 他们证明对任一$\boldsymbol{n}\in G_\beta$，有某$\lambda>0$使

(11.0.4)　$$\sup_{1\leqslant j\leqslant n}\Big|\sum_{i\leqslant j}(X_i - Y_i)\Big| = O(|\boldsymbol{n}|^{1/2-\lambda}) \quad \text{a.s.}$$

Strittmatter(1990)改进了这一结果，去掉了“$\boldsymbol{n}\in G_\beta$”的限制，证明着下述结果：若

(11.0.5) $N(\epsilon):=\text{Card}(\mathscr{A}(\epsilon))\leqslant c\epsilon^{-u}, u<\frac{\delta s-4-2\delta}{d(4+\delta)}-2,$

(11.0.6) $b(\epsilon):=\sup\{n^{-d}|((nA)^{n\epsilon}\cap(nA^c)^{n\epsilon})|:A\in\bigcap_{\eta>0}\mathscr{A}(\eta),$

$n\geqslant 1/\epsilon\}\leqslant C\epsilon^h,$

(11.0.7) $\alpha(n)=O(n^{-s}),$ 某 $s>1+\left(\frac{17}{2}\right)^{d-1}/(1\wedge\delta)$

其中$0<h\leqslant 1$.那么对某 $\gamma>0$成立着

(11.0.8) $\sup_{A\in\mathscr{A}}\sum_{j\in N^d}|nA\cap C_j|(X_j-Y_j)=O(n^{d/2-\gamma})$ a.s.

这里满足 $N(\epsilon)\leqslant c\epsilon^{-u}(u>0)$的集类 $\mathscr{A}$ 都是较小的集类.苏中根(1992)对 $H(\epsilon)\leqslant c\epsilon^{-r}$(某$0<r<1$),$n\in G_\beta$ 的强平稳 $\varphi$混合随机场给出强逼近结果.我们将在§11.1中介绍他的结果,Strittmatter 的结果将在§11.2中介绍.

## §11.1 $\varphi$混合随机场的强逼近

为叙述简单计,我们仅给出 $d=2$时的结果.

**定理11.1.1**(苏中根1992) 设$\{X_j,j\in N^2\}$是强平稳 $\varphi$混合随机场,$EX_j=0$,对某 $\delta>0$,$E|X_j|^{2+\delta}<\infty$.设 $\mathscr{B}^2$的子集类 $\mathscr{A}$ 满足(11.0.6)及熵条件:对某$0<r<\delta/(4(1+\delta))$

(11.1.1) $H(\epsilon)\leqslant c\epsilon^{-r}.$

又设

(11.1.2) $\varphi(x)=O(x^{-q})$,某 $q>\frac{4(2+\delta)}{\delta-r(4+\delta)}-2.$

那么

(11.1.3) $\sigma^2=EX_1^2+2\sum_{v\neq 1}EX_1X_v=O(1).$

进而可在一较大的概率空间上,不改变$\{X_j,j\in N^2\}$的分布而重新定义$\{X_j,j\in N^2\}$,同时在此概率空间上有独立正态随机场$\{Y_j,j\in N^2\}$,$EY_j=0$,$EY_j^2=\sigma^2$,使对任一 $n\in G_\beta$,$\delta/(2(2+\delta))<\beta\leqslant 1$有

(11.1.4) $\sup_{A\in\mathscr{A}}\sum_{j\in N^2}|nA\cap C_j|(X_j-Y_j)=O(|n|(\log|n|)^{-\alpha_1})$a.s.

其中 $\sigma_1=\sigma_1(r,q,\delta)>0$.

首先,我们来证关于 $\varphi$ 混合随机场的 Bernstein 不等式.

**引理 11.1.1** 设 $\{X_j, j\in N^d\}$ 是 $\varphi$ 混合随机场, $EX_j=0$, $|X_j|\leqslant\Delta_n$ a.s. $1\leqslant j\leqslant n$. 记 $\sigma_n=\max\limits_{1\leqslant j\leqslant n}\|X_j\|_2$. 若 $\sum\limits_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$, 那么对任一 $A\in\mathscr{B}^d$ 有

(11.1.5)
$$P\left\{\left|\sum_j|nA\cap C_j|X_j\right|\geqslant x_n\right\}$$
$$\leqslant 2\exp\left\{\frac{c|n|\varphi(m)}{m^d}-\alpha x_n+c\alpha^2|nA|\sigma_n^2\right\},$$

其中 $x_n>0$, $m\leqslant\min\limits_{1\leqslant i\leqslant d}n_i$ 且 $2^d\alpha m^d\Delta_n\leqslant 1/2$.

特别地,若 $\alpha=1/(2^{d+1}m^d\Delta_n)$, 那么
$$P\left\{\left|\sum_j|nA\cap C_j|X_j\right|\geqslant x_n\right\}$$
$$\leqslant 2\exp\left\{\frac{c|n|\varphi(m)}{m^d}-\frac{x_n}{2^{d+1}m^d\Delta_n}+\frac{c|nA|\sigma_n^2}{(2^{d+1}m^d\Delta_n)^2}\right\}.$$

**证** 令 $N=(N_1,\cdots,N_d)$ 满足 $2m(N_i-1)\leqslant n_i\leqslant 2mN_i$, $i=1,\cdots,d$. 记 $V=\{(v_1,\cdots,v_d);v_i=1或2,i=1,\cdots,d\}$. 定义
$$I_{v,k}=[l_1(1),l_2(1)]\times\cdots\times[l_1(d),l_2(d)],v\in V,1\leqslant k\leqslant N,$$
其中 $l_1(i)=\{(2k_i+v_i-3)m+1\}\wedge n_i$, $l_2(i)=\{l_1(i)+m-1\}\wedge n_i$, $i=1,\cdots,d$. 其次,记
$$I_v=\sum_{1\leqslant k\leqslant N}I_{v,k};\qquad A_{v,k}=\sum_j|nA\cap I_{v,k}\cap C_j|X_j;$$
$$B_{v,N}=\sum_{1\leqslant k\leqslant N}A_{v,k},\qquad S=\sum_{v\in V}B_{v,N}.$$

注意到当 $|x|\leqslant 1/2$ 时, $e^x\leqslant 1+x+x^2$; 又对任何 $x$, $1+x\leqslant e^x$. 所以当 $2^d\alpha m^d\Delta_n\leqslant 1/2$ 时,我们有
$$E\exp(2^d\alpha A_{v,k})\leqslant\exp\{(2^d\alpha)^2EA_{v,k}^2\}.$$

由引理1.2.10和引理6.2.3有

(11.1.6) $E\exp(2^d\alpha B_{v,N})=E\exp\left(2^d\alpha\sum\limits_{k\neq N}A_{v,k}\right)\exp(2^d\alpha A_{v,N})$
$$\leqslant E\exp\left(2^d\alpha\sum_{k\neq N}A_{v,k}\right)E\exp(2^d\alpha A_{v,N})$$

$$+ 2\varphi(m)\|\exp(2^d\alpha A_{v,N})\|_\infty E\exp\Big(2^d\alpha\sum_{k\neq N}A_{v,k}\Big)$$

$$\leqslant (1+2e^{1/2}\varphi(m))\exp(c(2^d\alpha)^2|\boldsymbol{n}A\cap I_{v,N}|\sigma_n^2)$$

$$E\exp\Big(2^d\alpha\sum_{k\neq N}A_{v,k}\Big)$$

$$\leqslant (1+2e^{1/2}\varphi(m))^{|N|}\exp(c(2^d\alpha)^2|\boldsymbol{n}A\cap I_v|\sigma_n^2),$$

其中$\|f(x)\|_\infty$表示$f(x)$的上确界尾数. 从指数函数的凸性及(11.1.6)式即得

$$E\exp(\alpha S)\leqslant\frac{1}{2^d}\sum_{v\in V}E\exp(2^d\alpha B_{v,N}) \tag{11.1.7}$$

$$\leqslant(1+2e^{1/2}\varphi(m))^{|N|}\exp(c(2^d\alpha)^2|\boldsymbol{n}A|\sigma_n^2)..$$

由于$|N|\leqslant(3/2)^d|\boldsymbol{n}|/m^d$及(11.1.7)式对$E\exp(-\alpha S)$也成立，从 Markov 不等式得证引理11.1.1.

定理11.1.1的证明.

首先，在定理11.1.1的条件下，容易看到(11.1.3)式成立，而且有

$$\lim_{\substack{|\boldsymbol{n}|\to\infty\\ \boldsymbol{n}\in G_\beta}}E\Big(\sum_{1\leqslant j\leqslant \boldsymbol{n}}X_j\Big)^2/|\boldsymbol{n}|=\sigma^2<\infty. \tag{11.1.8}$$

为证明对任一$\boldsymbol{n}\in G_\beta$, $\delta/(2(2+\delta))<\beta\leqslant1$, (11.1.4)成立，我们将通过下述诸引理，运用截尾，拆项，分块诸方法证明之.

**1. 截尾** 取$\tau$充分小(在下面确定)，令

$$X'_j=X_jI(|X_j|\leqslant|\boldsymbol{j}|^{(1+\tau)/(2+\delta)}),$$

$$S'_n(A)=\sum_j|\boldsymbol{n}A\cap C_j|X'_j.$$

**引理11.1.2** 我们有

$$\sup_{A\in\mathscr{A}}|S_n(A)-S'_n(A)|=O(1)\quad \text{a. s.} \tag{11.1.9}$$

$$\sum_{j\leqslant n1}E|X_j-X'_j|=O(n^{d/(2+\delta)})\quad\text{a. s.} \tag{11.1.10}$$

**证** 因为

$$P\{X_j\neq X'_j\}=P\{|X_j|\geqslant|\boldsymbol{j}|^{(1+\tau)/(2+\delta)}\}\leqslant c|\boldsymbol{j}|^{-(1+\tau)},$$

由 Borel-Cantelli 引理有$P\{X_j-X'_j\neq0\ i.o.\}=0$，所以对每一$n\in N$有

$$\sup_A |S_n(A) - S'_n(A)| \leqslant \sum_{j \in n} |X_j - X'_j| \leqslant c < \infty \text{ a.s.}$$

和

$$\sum_{j \leqslant n} E|X_j - X'_j| \leqslant \sum_{j \leqslant n} (|j|^{-(1+r)(1+\delta)/(2+\delta)}) = O(n^{d/(2+\delta)}).$$

**2. 拆项** 对任给的 $\boldsymbol{n}=(n_1,n_2)\in G_\beta$，$\delta/(2(2+\delta))<\beta\leqslant 1$，设 $a,b$ 分别是使

(11.1.11) $\qquad 2^a \geqslant (\log|\boldsymbol{n}|)^{1/r'}, 2^b \geqslant |\boldsymbol{n}|^{\frac{1}{2}(\tau+\frac{4+\delta}{2+\delta})},$

的最小正整数，其中 $r'>r$ 待定.

**引理11.1.3** 我们有

$$\sup_A |S'_n(A) - S'_n(A(2^{-b}))| = O\left(|\boldsymbol{n}|^{\frac{1}{2}(1-\frac{\tau\delta}{2+\delta})}\right) \text{a.s.}$$

**证** 对任给的 $A\in\mathscr{A}$，由条件(11.0.6)，(11.1.1)和(11.1.11)有

$$\begin{aligned}
&|S'_n(A) - S'_n(A(2^{-b}))| \\
&\leqslant \sum_j \big| |\boldsymbol{n}A \cap C_j| - |\boldsymbol{n}A(2^{-b}) \cap C_j| \big| |X'_j| \\
&\leqslant c|\boldsymbol{n}| |A\Delta A(2^{-b})| \cdot |\boldsymbol{n}|^{\frac{1+\tau}{2+\delta}} = O\left(|\boldsymbol{n}|^{\frac{1}{2}(1-\frac{\tau\delta}{2+\delta})}\right).
\end{aligned}$$

**引理11.1.4** 对每一 $\boldsymbol{n}\in G_\beta$ 有

$$\sup_{A\in\mathscr{A}} |S'_n(A(2^{-b})) - S'_n(A(2^{-a}))| = O\left(|\boldsymbol{n}|^{\frac{1}{2}}(\log|\boldsymbol{n}|)^{-\sigma_1}\right) \text{ a.s.}$$

**证** 选取 $\tau, r'$ 和 $\tau'$ 充分小使得

(11.1.12) $\left(\dfrac{\delta-2\tau}{2(2+\delta)} - \dfrac{1}{2}\tau'\left(\tau+\dfrac{4+\delta}{2+\delta}\right)\right)(q+1)>1, \tau'-\tau>r'>r.$

记 $m_i=[|\boldsymbol{n}|^{\frac{\delta-2\tau}{4(2+\delta)}}2^{-\frac{i\tau'}{2}}]$，由于 $\boldsymbol{n}\in G_\beta$，显然有 $m_i\leqslant n_1\wedge n_2$，$i=1,2$. 另一方面我们有

$$|X'_j - EX'_j| \leqslant 2|\boldsymbol{n}|^{(1+\tau)/(2+\delta)}, j \leqslant \boldsymbol{n}.$$

在引理11.1.1中取 $\Delta\boldsymbol{n}=2|\boldsymbol{n}|^{(1+\tau)/(2+\delta)}$，我们得

(11.1.13) $\quad P\{|S'_n(A(2^{-(i+1)})) - S'_n(A(2^{-i}))| \geqslant |\boldsymbol{n}|^{1/2}2^{-i\tau}\}$

$$\leqslant P\left\{\left|\sum_j |\boldsymbol{n}(A(2^{-(i+1)})\backslash A(2^{-i}))|(X'_j - EX'_j)\right| > \frac{1}{4}|\boldsymbol{n}|^{1/2}2^{-i\tau}\right\} + P\left\{\left|\sum_j |\boldsymbol{n}(A(2^{-i})\backslash A(2^{-(i+1)}))\right.\right.$$

$$|(X_j' - EX_j')| > \frac{1}{4}|\boldsymbol{n}|^{1/2}2^{-ir}\Big\} \leqslant c\exp(-c2^{i(r'-r)})$$

所以从(11.1.1)即得

$$\sum_{n\in G_\beta} P\{\sup_{A\in\mathscr{A}} S_n'(A(2^{-b})) - S_n'(A(2^{-a}))| \geqslant |\boldsymbol{n}|^{1/2}(\log|\boldsymbol{n}|)^{-r/r'}\}$$

$$\leqslant \sum_{n\in G_\beta}\sum_{i=a}^{b}\sum_{A\in\mathscr{A}} P\{|S_n'(A(2^{-(i+1)})) - S_n'(A(2^{-i}))| \geqslant |\boldsymbol{n}|^{1/2}2^{-ir}\}$$

$$\leqslant c\sum_{n\in G_\beta}\sum_{i=a}^{b}\exp(H(2^{-i}) + H(2^{-(i+1)}) - c2^{i(r'-r)}) < \infty,$$

由 Borel－Cantelli 引理即得引理11.1.4成立.

设 $\boldsymbol{k}=(k(1),k(2))\geqslant 0$,定义 $\boldsymbol{t}_k=(t_k(1),t_k(2))$ 如下:

$$t_k(1) = [\exp(k(1))^{s/4}], t_k(2) = [\exp(k(2))^{s/4}], 0 < s < 1/2.$$

显然,对 $n\in G_\beta$ 充分大,由关系式 $\boldsymbol{t}_k+1\leqslant \boldsymbol{n}\leqslant \boldsymbol{t}_{k+1}$ 所确定的 $\boldsymbol{t}_k$ 仍然有 $\boldsymbol{t}_k\in G_\beta$. 反之也成立.

**引理11.1.5** 对 $\boldsymbol{t}_k\in G_\beta$,当 $|\boldsymbol{k}|\to\infty$ 时我们有

$$\sup_{A\in\mathscr{A}}\max_{t_k<n\leqslant t_{k+1}}|S_n(A(2^{-a})) - S_{t_k}'(A(2^{-a}))|$$
$$=O(|\boldsymbol{t}_k|^{1/2}|\log|\boldsymbol{t}_k||^{-1/4}) \text{ a.s.}$$

**证** 注意到当 $\boldsymbol{t}_k<\boldsymbol{n}\leqslant \boldsymbol{t}_{k+1}, A\subset[0,1]^2$ 时有

$$|\boldsymbol{n}A\Delta \boldsymbol{t}_k A| \leqslant c|\boldsymbol{n}|(\varepsilon_1 + \varepsilon_2),$$

其中

(11.1.14) $\qquad \varepsilon_i=(t_{k+1}(i)-t_k(i))/n_i, i=1,2.$

令 $m_k=[|\boldsymbol{t}_k|^{\frac{\delta-2r}{4(2+\delta)}}|\boldsymbol{k}|^{-\frac{s}{4}}]$. 当 $\boldsymbol{t}_k\in G_\beta$ 时,有 $m_k\leqslant t_k(1)\wedge t_k(2)$. 在引理11.1.1中令 $\Delta_n=2|\boldsymbol{t}_k|^{(1+r)/(2+\delta)}$,我们得

(11.1.15) $P\{|S_n'(A(2^{-a})) - S_{t_k}'(A(2^{-a}))| \geqslant |\boldsymbol{t}_k|^{1/2}$

$$(k(1)^{-s/16} + k(2)^{-s/16})\}$$

$$\leqslant P\{|\sum_j|(\boldsymbol{n}A(2^{-a})\backslash \boldsymbol{t}_k A(2^{-a}))\cap$$

$$C_j|(X_j' - EX_j')| \geqslant \frac{1}{4}|\boldsymbol{t}_k|^{\frac{1}{2}}(k(1)^{-\frac{s}{16}} + k(2)^{-\frac{s}{16}})\Big\}$$

$$+P\left\{\left|\sum_j |t_kA(2^{-a})\backslash nA(2^{-a})|\cap C_j|(X'_j-\right.\right.$$

$$\left.\left.EX'_j)\right|\geqslant\frac{1}{4}|t_k|^{\frac{1}{2}}\left(k(1)^{-\frac{s}{16}}+k(2)^{-\frac{s}{16}}\right)\right\}$$

$$\leqslant c\exp(-c(k(1)^{s/2}+k(2)^{s/2})).$$

注意到当 $t_k\in G_\beta$ 有

$$k(1)^{-s/16}+k(2)^{-s/16}\leqslant c(\log|t_k|)^{-1/4},$$

由此即得

$$\sum_{t_k\in G_\beta}P\{\sup_{A\in\mathscr{A}}\sup_{t_k<n\leqslant t_{k+1}}|S'_n(A(2^{-a}))-S'_{t_k}(A(2^{-a}))|$$

$$\geqslant|t_k|^{\frac{1}{2}}(\log|t_k|)^{-\frac{1}{4}}\}$$

$$\leqslant c\sum_{t_k\in G_\beta}\exp\{2^{(\log|t_k|)^{1/r'}}\}|t_{k+1}-t_k|$$

$$\exp(-c(k(1)^{s/2}+k(2)^{s/2})<\infty.$$

由 Borel—Cantelli 引理，得证引理11.1.5成立.

设 $R_l=[t_l(1),t_{l+1}(1)]\times[t_l(2),t_{l+1}(2)]$. 对任给 $A\in n\bigcup_{\delta>0}\mathscr{A}_\delta$，我们定义

$$A^*=\bigcup\{R_l:R_l\cap A\neq\emptyset\}\quad A_*=\bigcup\{R_l:R_l\subset A\}.$$

显然，$A_*\subset A\subset A^*$ 且当 $t_k<n\leqslant t_{k+1}$时，有 $|A^*\backslash A_*|\leqslant c|n|(\epsilon_1+\epsilon_2)$，其中 $\epsilon_i$ 由(11.1.14)定义. 类似于引理11.1.5，我们有

**引理11.1.6** 对 $t_k\in G_\beta$，当 $|k|\to\infty$时

$$\sup_{A\in\mathscr{A}}\max_{t_k<n\leqslant t_{k+1}}\left|\sum_j|(t_kA(2^{-a})\backslash(t_kA(2^{-a}))_*)\cap C_j|X_j\right|$$

$$=O(|t_k|^{1/2}(\log|t_k|)^{-1/4})\quad\text{a.s.}$$

**3. 分块** 余下部分证明的关键是估计

$$\sup_{A\in A}\max_{t_k<n\leqslant t_{k+1}}\left|\sum_j|(t_kA(2^{-a}))_*\cap C_j|X_j\right|.$$

设 $u=(u(1),u(2))\geqslant\mathbf{0}$，$H_u=\{v\in N^2,t_u+\mathbf{1}<t_{u+1}\}$，$0<\rho<\beta$. 记

$$L=\{u:H_u\subset G_\rho\},H=\bigcup_{u\in L}H_u,$$

$$\Delta_u=\bigcup_{i=1,2}\{v\in H_u:t_{u+1}(i)-[\exp(u(i)^{s/4}/4)]\leqslant v_i\leqslant t_{u+1}(i)\}.$$

对 $t_k\in G_\beta$，定义 $t_k^{(p)}=(t_k^{(p)}(1),t_k^{(p)}(2))$，$p=1,2$，如下：

(11.1.16) $\quad t_k^{(p)}(m)=\delta_{m,p}\min_{\substack{v\in H\\ \text{若}l\neq m,v_l=n_l}} v_m+(1-\delta_{m,p})n_m, m=1,2,$

其中 $\delta_{m,p}=1$，若 $m=p$；$\delta_{m,p}=0$，若 $m\neq p$. 令

$$I_1=[0,t_k^{(1)}]\cup[0,t_k^{(2)}],$$

$$I_2=\bigcup_{(t_k^{(1)}(1),t_k^{(2)}(2))\leqslant u\leqslant k} H_u\backslash\Delta_u, I_3=\bigcup_{u\in L,u<k}\Delta_u.$$

我们有

$$\begin{aligned}&\Big|\sum_j\big|(t_kA(2^{-a}))_*\cap C_j\big|(X_j-Y_j)\Big|\\ \leqslant&\Big|\sum_j\big|(t_kA(2^{-a}))_*\cap I_1\cap C_j\big|X_j\Big|\\ &+\Big|\sum_j\big|(t_kA(2^{-a}))_*\cap I_1\cap C_j\big|Y_j\Big|\\ &+\Big|\sum_j\big|(t_kA(2^{-a}))_*\cap I_2\cap C_j\big|(X_j-Y_j)\Big|\\ &+\Big|\sum_j\big|(t_kA(2^{-a}))_*\cap I_3\cap C_j\big|X_j\Big|\\ &+\Big|\sum_j\big|(t_kA(2^{-a}))_*\cap I_3\cap C_j\big|Y_j\Big|\end{aligned}$$

其中 $Y_j$ 将在后面的第4小段中定义.

**引理11.1.7** 对 $t_k\in G_\beta$，当 $|k|\to\infty$时有

$$\sup_{A\in\mathscr{A}}\max_{t_k<n\leqslant t_{k+1}}\Big|\sum_j\big|(t_kA(2^{-a}))_*\cap I_1\cap C_j\big|X_j\Big|$$

$$=O\left(|t_k|^{\frac{1}{2}}(\log|t_k|)^{-a_1}\right)\text{ a.s.}$$

**证** 容易看出

$$\sup_{A\in\mathscr{A}}\max_{t_k<n\leqslant t_{k+1}}\Big|\sum_j\big|(t_kA(2^{-a}))_*\cap I_1\cap C_j\big|X_j\Big|$$

$$\leqslant\sum_{l<k}\Big|\sum_j\big|R_l\cap I_1\cap C_j\big|X_j\Big|$$

由于 $t_k^{(1)}(1)\leqslant ct_k(2)^\rho, t_k^{(2)}(2)\leqslant ct_k(1)^\rho$，所以对 $t_k\in G_\beta$，我们有

$$|t_k^{(1)}|/|t_k|\leqslant c|t_k|^{-\beta(\beta-\rho)/2};\ |t_k^{(2)}|/|t_k|\leqslant c|t_k|^{-\beta(\beta-\rho)/2}.$$

由 Markov 不等式及引理6.2.3即得

(11.1.17) $\quad P\Big\{\sum_{l<k}\Big|\sum_j\big|R_l\cap I_l\cap C_j\big|X_j\Big|\geqslant|t_k|^{1/2}(\log|t_k|)^{-a_1}\Big\}$

$$\leqslant \sum_{l<k} P\left\{\left|\sum_j |R_l \cap I_1 \cap C_j| X_j\right| \geqslant |t_k|^{1/2}(\log|t_k|)^{-\sigma 1}|k|^{-1}\right\}$$

$$\leqslant c\sum_{l<k} |R_l \cap I_1|^{1+\delta/2}/(|t_k|^{1+\delta/2}((\log|t_k|)^{-\sigma 1}|k|^{-1})^{2+\delta})$$

$$\leqslant c|t_k|^{-\frac{\beta(\beta\rho)}{2}(1+\frac{\delta}{2})}((\log|t_k|)^{\sigma 1}|k|)^{2+\delta}$$

由 Borel—Cantelli 引理即得引理11.1.7成立.

**引理11.1.8** 对 $t_k \in G_\beta$，当 $|k| \to \infty$ 时有

$$(11.1.18)\quad \sup_{A\in\mathscr{A}} \max_{t_k<n\leqslant t_{k-1}} \left|\sum_j |(t_k A(2^{-a}))_* \cap I_3 \cap C_j| X_j\right| = O\left(|t_k|^{\frac{1}{2}}(\log|t_k|)^{-\sigma 1}\right) \text{a. s.}$$

**证** 类似于引理11.1.7的证明，我们有

$$\sup_{A\in\mathscr{A}} \max_{t_k<n\leqslant t_{k+1}} \left|\sum_j |(t_k A(2^{-a}))_* \cap I_3 \cap C_j| X_j\right| \leqslant \sum_{l<k}\left|\sum_j |R_l \cap I_3 \cap C_j| X_j\right|.$$

记 $\Delta_u = \Delta_u(1) \cup \Delta_u(2)$，其中 $\Delta_u(1)$ 和 $\Delta_u(2)$ 是互不相交的矩形，

$$|\Delta_u(1)| = [\exp(u(1)^{s/4}/4)]([\exp((u(2)+1)^{s/4})] - [\exp((u(2))^{s/4})]),$$

$$|\Delta_u(2)| = [\exp(u(2)^{s/4}/4)]([\exp((u(1)+1)^{s/4})] - [\exp((u(1))^{s/4})]).$$

显然

$$\sum_{\substack{u\in L\\ u<k}} |\Delta_u(1)| \leqslant ck(1)^{1-s/4} t_k(1)^{1/4} t_k(2),$$

$$\sum_{\substack{u\in L\\ u<k}} |\Delta_u(2)| \leqslant ck(2)^{1-s/4} t_k(2)^{1/4} t_k(1).$$

那么类似于引理11.1.7中的(11.1.17)式有

$$(11.1.19)\quad P\left\{\sum_{l<k}\left|\sum_j |R_l \cap I_3 \cap C_j| X_j\right| \geqslant |t_k|^{\frac{1}{2}}(\log|t_k|)^{-\sigma 1}\right\}$$

$$\leqslant c\sum_{l<k} |R_l \cap I_3|^{1+\delta/2}/(|t_k|^{1+\delta/2}((\log|t_k|)^{-\sigma 1}|k|^{-1})^{2+\delta})$$

$$\leqslant c|t_k|^{-(3\beta/8)(1+\delta/2)}((\log|t_k|)^{\sigma 1}|k|)^{2+\delta}\cdot (k(1)^{1-s/4} + k(2)^{1-s/4})^{1-\delta/2}$$

由 Borel—Cantelli 引理即得引理11.1.8成立.

4. $\{X_j\}$和$\{Y_j\}$的构造

为构造随机场$\{X_j\}$和$\{Y_j\}$，我们不加证明地写出下述命题. 记

$$h_u = \text{Card}((H_u \backslash \Delta_u) \cap N^2), \bar{X}_u = \sum_{j \in H_u \backslash \Delta_u} X_j / h_u^{1/2}.$$

**命题11.1.1**(Berkes 和 Morrow1981)　在定理11.1.1的条件下，存在$t$，$0<t<1$使当$|x|\leqslant h_u^t$时有

(11.1.20)
$$\left| E\exp(ix\bar{X}_u) - \exp\left(\frac{1}{2}\sigma^2 x^2\right) \right| \leqslant c h_u^{-t}.$$

**命题11.1.2**(Berkes 和 Philipp1979)　设$\{X_k, k\geqslant 1\}$是取值于$R^{d_k}$的随机变量序列，设$\{\mathscr{F}_k, k\geqslant 1\}$是非降的$\sigma$域序列，使得$X_k$是$\mathscr{F}_k$可测的. $R^{d_k}$上的概率分布列$\{G_k, k\geqslant 1\}$有特征函数$g_k(u)$，$u\in R^{d_k}$. 假设对某非负数$\lambda_k$，$\delta_k$和$T_k\geqslant 10^8 d_k$，使对所有$|u|\leqslant T_k$有

(11.1.21)　$E|E\{\exp(i\langle u, X_k\rangle)|\mathscr{F}_k\} - g_k(u)| \leqslant \lambda_k$；

(11.1.22)
$$G_k\{u: |u| > \frac{1}{4}T_k\} \leqslant \delta_k.$$

那么不改变分布，可在一较大的概率空间上重新定义序列$\{X_k, k\geqslant 1\}$，而在这个概率空间上有一独立序列$\{Y_k, k\geqslant 1\}$，其分布为$G_k$，且满足

(11.1.23)　$P\{|X_k - Y_k| \geqslant \alpha_k\} \leqslant \alpha_k \quad k=1,2\cdots,$

其中$\alpha_1 = 1$，

(11.1.24)　$\alpha_k = 16 d_k T_k^{-1}\log T_k + 4\lambda_k^{1/2} T_k^{d_k} + \delta_k, k=2,3,\cdots,$

**命题11.1.3**(Berkes 和 Philipp1979)　设$S_i(i=1,2,3)$是可分Banach 空间，$F$是$S_1\times S_2$上的分布，$G$是$S_2\times S_3$上的分布，使得$F$的第二边际分布与$G$的第一边际分布相等. 那么存在一概率空间和其上的三个随机变量$Z_i$，$i=1,2,3$，使得$(Z_1,Z_2)$的联合分布为$F$，$(Z_2,Z_3)$的联合分布为$G$.

设　$(t_{k_0}^{(1)}(1), t_{k_0}^{(2)}(2))$充分大，使对任一$u\in L_0 := \{u\in L, u\geqslant (t_{k_0}^{(1)}(1), t_{k_0}^{(2)}(2))\}$，存在$\rho'>0$满足

$t_{n+1}(1) - t_n(1) - [\exp(u(1)^{\varepsilon/4}/4)] \geqslant (t_{n+1}(2) - t_n(2) - [\exp(u(2)^{\varepsilon/4}/4)])^{\rho'}$，

$t_{n+1}(2)-t_n(2)-[\exp(u(2)^{s/4}/4)]\geqslant(t_{n+1}(1)-t_n(1)-[\exp(u(1)^{s/4}/4)])^r$.

令 $\psi$ 是 $\{1,2,\cdots\}$ 到 $L_0$ 上一一对应，记 $\psi(l)=(\psi_1(l),\psi_2(l))$. 由命题11.1.1，存在 $t$，$0<t<1$，使对任一 $|x|\leqslant h'_{\psi(l)}$ 有

(11.1.25) $$|E\exp(ix\overline{X}_{\psi(l)})-\exp(-\frac{1}{2}\sigma^2x^2)|\leqslant ch_{\psi(l)}^{-t}.$$

另一方面，由于 $h_{\psi(l)}=O(|t_{\psi(l)}|)$，由 $\varphi$ 混合性对任一 $|x|\leqslant h'_{\psi(l)}$ 有

$$\begin{aligned}\lambda_l(x):&=E|E\{\exp(ix\overline{X}_{\psi(l)})|\overline{X}_{\psi(1)},\cdots,\overline{X}_{\psi(l-1)}\}-\exp(-\sigma^2x^2/2)|\\&\leqslant E|E\{\exp(ix\overline{X}_{\psi(l)})|\overline{X}_{\psi(1)},\cdots,\overline{X}_{\psi(l-1)}\}-E\exp(ix\overline{X}_{\psi(l)})|\\&\quad+|E\exp(ix\overline{X}_{\psi(l)})-\exp(-\sigma^2x^2/2)|\\&\leqslant ch_{\psi(l)}^{-t}+2\pi\psi([\exp(\psi_1(l)^{s/4}/4)]\wedge[\exp(\psi_2(l)^{s/4}/4)])\leqslant c|t_{\psi(l)}|^{-t_0},\end{aligned}$$

其中 $t_0=t\wedge(\rho q/8)$.

令 $t'<t_0/2$，$T_l=|t_{\psi(l)}|^{t'}$，$\lambda_l=|t_{\psi(l)}|^{-t_0}$，$\delta_l=N(0,\sigma^2)\{x:|x|>T_l/4\}$，$\alpha_l=16T_l^{-1}\log T_l+4\lambda_l^{1/2}T_l+\delta_l$. 由命题11.1.2，不改变分布，可在一较大的概率空间上重新定义序列 $\{\overline{X}_{\psi(l)}\}$，而在这一概率空间上存在一独立正态序列 $\{\overline{Y}_{\psi(l)}\}$ 使得 $E\overline{Y}_{\psi(l)}^2=\sigma^2$ 且满足

(11.1.26) $$P\{|\overline{X}_{\psi(l)}-\overline{Y}_{\psi(l)}|\geqslant\alpha_l\}\leqslant\alpha_l.$$

此外，由命题11.1.3，在一较大的概率空间上存在两个随机场 $\{x_j,j\in N^2\}$ 和 $\{Y_j,j\in N^2\}$，$\{X_j\}$ 的分布不改变，而 $\{Y_j\}$ 是独立中心化正态随机变量序列，$EY_j^2=\sigma^2$，使得

(11.1.27) $$P\{h_{\psi(l)}^{-1/2}|\sum_{j\in H_{\psi(l)}\setminus\Delta_{\psi(l)}}(X_j-Y_j)|\geqslant\alpha_l\}\leqslant\alpha_l.$$

显然有 $\sum_l\alpha_l<\infty$. 由 Borel-Cantelli 引理即得

(11.1.28) $$\Big|\sum_{j\in H_{\psi(l)}\setminus\Delta_{\psi(l)}}(X_j-Y_j)\Big|=O(\alpha_lh_{\psi(l)}^{1/2})\text{a. s.}$$

由于 $I_2=\bigcup_{(t_k^{(1)}(1),t_k^{(2)}(2))\leqslant u<k}H_u\setminus\Delta_u$，若我们取 $t_k\in G_\beta$，使得 $t_k^{(1)}(1)\geqslant t_{k_0}^{(1)}(1)$，$t_k^{(2)}(2)\geqslant t_{k_0}^{(2)}(2)$，那么对每一 $A\in\mathscr{A}$ 和某个 $t'>0$ 有

(11.1.29) $$\Big|\sum_j I(t_kA(2^{-a}))_{\circ}\cap I_2\cap C_j\}(X_j-Y_j)\Big|$$

$$\leqslant \sum_{(t_k^{(1)}(1),\, t_k^{(2)}(2)) \leqslant u < k} \Big| \sum_{j \in H_u \setminus \Delta_u} (X_j - Y_j) \Big|$$

$$\leqslant \sum_{\phi(l) < k,\, \phi_i(l) \geqslant t_k^{(i)}(i),\, i=1,2} a_l h_{\phi(l)}^{1/2}$$

$$\leqslant c \left| t_k \right|^{1/2-\varepsilon'} \quad \text{a. s.}$$

对在(11.1.27)中定义的随机场$\{X_j\}$和$\{Y_j\}$,类似于$X'_j$可定义$Y'_j$.那么对$t_k < n \leqslant t_{k+1}$,$A \in \mathscr{A}$有

$$\Big| \sum_j |nA \cap C_j| (X_j - Y_j) \Big| \leqslant \Big| \sum_j |nA \cap C_j| (X_j - X'_j) \Big|$$
$$+ \Big| \sum_j |nA \cap C_j| (X'_j - Y'_j) \Big| + \Big| \sum_j |nA \cap C_j| (Y'_j - Y_j) \Big|.$$

其中

$$(11.1.30) \qquad \Big| \sum_j |nA \cap C_j| (X'_j - Y'_j) \Big|$$
$$\leqslant \Big| \sum_j \big| |nA \cap C_j| - |nA(2^{-b}) \cap C_j| \big| (X'_j - Y'_j) \Big|$$
$$+ \Big| \sum_j \big| |nA(2^{-b}) \cap C_j| - |nA(2^{-a}) \cap C_j| \big| (X'_j - Y'_j) \Big|$$
$$+ \Big| \sum_j \big| |nA(2^{-a}) \cap C_j| - |t_k A(2^{-a}) \cap C_j| \big| (X'_j - Y'_j) \Big|$$
$$+ \Big| \sum_j |(t_k A(2^{-a}) \setminus (t_k A(2^{-a}))_* \cap C_j| (X'_j - Y'_j) \Big|$$
$$+ \Big| \sum_j |(t_k A(2^{-a}))_* \cap C_j| (X_j - Y_j) \Big|$$
$$+ \sum_j |X_j - X'_j| + \sum_j |(Y_j - Y'_j)|$$

而且

$$(11.1.31) \qquad \Big| \sum_j |(t_k A(2^{-a}))_* \cap C_j| (X_j - Y_j) \Big|$$
$$\leqslant \Big| \sum_j |(t_k A(2^{-a}))_* \cap I_1 \cap C_j| (X_j - Y_j) \Big|$$
$$+ \Big| \sum_j |(t_k A(2^{-a}))_* \cap I_2 \cap C_j| (X_j - Y_j) \Big|$$
$$+ \Big| \sum_j |(t_k A(2^{-a}))_* \cap I_3 \cap C_j| (X_j - Y_j) \Big|.$$

显然对独立正态随机场$\{Y_j, j \in \mathbb{N}^2\}$,引理11.1.2—11.1.8均成立,因此从这些引理和(11.1.29)—(11.1.31)式得证定理11.1.1

成立.

## §11.2 $\alpha$ 混合随机场的强逼近

对 $\alpha$ 混合随机场，Strittmatter(1990)去掉了条件“$\boldsymbol{n}\in G_\beta$”，对较小的集指标类，即 $N(\varepsilon)\leqslant c\varepsilon^{-u}$(某 $u>0$)，给出了强逼近结果.

**定理11.2.1** 设$\{x_j,\boldsymbol{j}\in N^d\}$是弱平稳 $R^q$ 值 $\alpha$ 混合随机场 $EX_j=0$. 记$\|X_j\|^2=(X_{j_1}^2+\cdots+X_{j_q}^2)$. 假设存在正常数 $C_i,i=1,\cdots,4$，使得

(11.2.1) 对某 $\delta>0$，每一 $\boldsymbol{j}\in N^d$，$E\|X_j\|^{2+\delta}\leqslant C_1$，

(11.2.2) 对某 $S>1+2\left(\frac{17}{2}\right)^{d-1}/(1\wedge\delta)$，$\alpha(t)\leqslant C_2t^{-s}$，

(11.2.3) 对某 $u<\frac{\delta s\cdot(4-2\delta)}{d(4+\delta)}-2$，$N(\varepsilon)\leqslant C_3\varepsilon^{-u}$，

(11.2.4)
$$
\begin{aligned}
b(\varepsilon)=\sup\{&n^{-d}\mu((nA)^{n\varepsilon}\cap(nA^C)^{n\varepsilon})\\
&:A\in\bigcup_{\eta>0}A_\eta,n\geqslant1/\varepsilon\}\\
\leqslant{}&C_4\varepsilon^h\qquad\qquad 某\quad 0<h\leqslant1.
\end{aligned}
$$

那么级数

$$
T=\sum_{j\in Z^d}r(\boldsymbol{j})\quad,r(\boldsymbol{j})=\mathrm{Cov}(X_j,X_j),
$$

绝对收敛且 $T$ 是非负定矩阵. 进一步，我们可重新定义序列$\{X_k,\boldsymbol{k}\in N^d\}$在一较大的概率空间上，而不改变其分布在这新的概率空间上有一 $i.i.d.$ 的中心化 Gauss 序列$\{Y_k,\boldsymbol{k}\in N^d\}$，$Y_k$ 具有协方差阵 $T$ 且对某 $\gamma=\gamma(u,s,d,h,\delta,N(\varepsilon))$有

(11.2.5)
$$
\sup_{A\in\mathscr{A}}\left\|\sum_{j\in I^d}|nA\cap C_j|(X_j-Y_j)\right\|=O(n^{d/2-\gamma})\qquad\text{a.s.}
$$

**引理11.2.1** 设$\{x_j,\boldsymbol{j}\in N^d\}$是 $\alpha$ 混合随机场，$Ex_j=0$，$E\|X_j\|^{2+\delta}\leqslant C_1<\infty$且

(11.2.6)
$$
C_0:=\sum_{r=1}^{\infty}r^{d-1}\alpha(r)^{\delta/(2+\delta)}<\infty.
$$

那么对$0\leqslant d_j\leqslant1$，$\boldsymbol{j}\in N^d$，其中只有有限个 $\boldsymbol{j}$ 使 $d_j\neq0$，我们有

$$E\left\|\sum_{j\in N^d}d_jX_j\right\|^2\leqslant C\sum_{j\in N^d}d_j,$$

其中 $C=(1+15d3^dC_0)C_1^{2/(2+\delta)}$.

**证** 由 $\alpha$ 混合的性质(引理1.2.4)有

$$\|EX_jX_k\|\leqslant 10\alpha(d(j,k))^{\delta/(2+\delta)}\|X_j\|_{2+\delta}\|X_k\|_{2+\delta}$$
$$=10\alpha(d(j,k))^{\delta/(2+\delta)}C_1^{2/(2+\delta)},$$

其中 $d(j,k)=\max\limits_{1\leqslant i\leqslant d}|j_i-k_i|$. 证毕.

**引理11.2.2** 设$\{x_j,j\in N^d\}$是弱平稳 $\alpha$ 混合随机场,$EX_j=0$,$|X_j|\leqslant M<\infty$且满足(11.2.6). 假设立方体 $D_1,D_2\cdots,D_q$ 两两不相交,其中 $D_k=(n_k-2p\mathbf{1},n_k]$,某固定的 $p,n_k\in 2pN^d$,$k=1,\cdots,q$. 设 $O\leqslant d_j\leqslant 1,j\in N^d$. 令

$$D=\bigcup_{k=1}^q D_k\quad F_k=\sum_{j\in D_k\cap N^d}d_j,F=\sum_{j\in D}d_j.$$

那么对每一 $K>0$有

$$P\left\{\left|\sum_{j\in D}d_jX_j\right|>2^{d+1}K\right\}$$
$$\leqslant c(K^{-2}M^2F^2\alpha(p)+p^{2d}C^{-2}M^4\alpha(p)$$
$$+\exp(-K^2/(8CF))+\exp(-2^{-d-2}K/Mp^d),$$

其中 $C$ 是引理11.2.1中给定的.

这个引理是 Philipp(1984)定理4的一个推广,证明的路线相仿,从略.

定理11.2.1的证明.

从(11.2.3)即得存在正数 $\tau$ 和 $\zeta$ 使

$$\zeta<\frac{1}{2}-\frac{1+\tau}{2+\delta},\quad u<\frac{\zeta s-1}{d\left(\frac{1}{2}+\frac{1+\tau}{2+\delta}\right)}-2 \tag{11.2.7}$$

对 $j\in N^d$,定义

$$X_j'=X_jI(\|X_j\|\leqslant|j|^{(1+\tau)/(2+\delta)}),$$
$$\overline{X}_j=X_j'-EX_j',\quad X_j''=X_j-X_j'.$$

对常数 $\beta$(待定),定义

$$\psi(m)=\sum_{k=1}^m k^\beta\quad m\in N. \tag{11.2.8}$$

假设 $m$ 和 $n$ 满足

(11.2.9) $$\psi(m)<n\leqslant\psi(m+1).$$

对 $r=(r_1,\cdots,r_d)\in\mathbb{N}^d$,写

$$R_r=\{(v_1,\cdots,v_d)\in R_+^d:\psi(r_i)<v_i\leqslant\psi(r_i+1),i=1,\cdots,d\}.$$

对 $A\subset R_+^d\cap\mathscr{B}^d$,设

$$A_*=\bigcup_{R_r\subset A}R_r.$$

且对 $A\in\mathscr{B}^d\cap[0,1]^d$,设

$$S_n(A)=\sum_j|nA\cap C_j|X_j,$$

$$\overline{S}_n(A)=\sum_j|n\boldsymbol{A}\cap C_j|X_j,$$

$$\overline{V}_n(A)=\sum_j|(n\boldsymbol{A}\backslash(nA)_*)\cap C_j|\overline{X}_j.$$

我们不加证明地引入 Berkes 和 Morrow(1981)的下述结果.

**命题11.2.1** 设$\{\xi_j,j\in\mathbb{N}^d\}$是弱平稳 $\alpha$ 混合随机场,$E\xi_j=0,E\|\xi_j\|^{2+\delta}<\infty,\alpha(t)=O(t^{-d(1+\varepsilon)(1+2/\delta)})$.记

$$G_\theta=\bigcap_{k=1}^d\{j\in\mathbb{N}_+^d:j_k\geqslant\prod_{l\neq k}j_l^\theta\},\qquad 0<\theta<1.$$

那么级数

$$\sigma^2=E\xi_1^2+2\sum_j E\xi_1\xi_j$$

绝对收敛.不失一般性可设 $\sigma^2=1$.进一步,不改变$\{\xi_j\}$的分布,可重新定义$\{\xi_j,j\in\mathbb{N}^d\}$在一较大的概率空间上,在其上有一 $i.i.d.$ 中心化 Gauss 随机场$\{\eta_j,j\in\mathbb{N}^d\}$,$E\eta_j^2=1$,使对任一 $\boldsymbol{n}\in G_\theta$ 有

(11.2.10) $$\sup_{1\leqslant\boldsymbol{m}\leqslant\boldsymbol{n}}\left\|\sum_{k\leqslant m}\xi_k-\sum_{k\leqslant m}r/\boldsymbol{k}\right\|=O(\ \boldsymbol{n}|^{1/2-\lambda})\quad \text{a.s.}$$

其中常数 $\lambda>0$仅依赖于随机场$\{\xi_j\}$.

现在我们取 $\theta=1/(8d-1)$,记 $G=G_{1/(8d-1)}$,

$$L=\{\boldsymbol{r}\in\mathbb{N}^d:(\psi(r_1),\cdots,\psi(r_d))\in G\}.$$

由命题11.2.1,我们有一 i.i.d. 中心化Gauss随机场$\{Y_j,j\in\mathbb{N}^d\}$,$EY_j^2=1$,使得对某个 $\gamma_1>0$,

$$(11.2.11)\quad \sum_{\substack{i\in L\\ \psi(\gamma_i)\leqslant n}} \Big\| \sum_{j\in R_i}(X_j - Y_j) \Big\| = O(n^{d/2-\gamma_1}) \quad \text{a.s.}$$

记

$$Y'_j = y_j I(\|Y_j\| \leqslant |j|^{(1+\tau)/(2+\delta)}),$$

$$T_n(A) = \sum_j |nA \cap C_j| Y_j,$$

$$T'_n(A) = \sum_j |nA \cap C_j| Y'_j,$$

$$U'_n(A) = \sum_j |(nA \backslash (nA)_*) \cap C_j| Y'_j.$$

那么对 $\kappa>0$(待定)我们有

$$(11.2.12)\quad \Big\| \sum_j |nA \cap C_j|(X_j - Y_j) \Big\| = \|S_n(A) - T_n(A)\|$$

$$\leqslant \|S_n(A) - \bar{S}_n(A)\| + \|\bar{S}_n(A) - \bar{S}_n(A(n^{-\kappa}))\|$$
$$+ \|\bar{V}_n(A(n^{-\kappa}))\|$$
$$+ \|T_n(A) - T'_n(A)\| + \|T'_n(A) - T'_n(A(n^{-\kappa}))\|$$
$$+ \|U'_n(A(n^{-\kappa}))\|$$
$$+ \sum_{1\leqslant j\leqslant n1}(\|X_j - \bar{X}_j\| + \|Y_j - Y'_j\|)$$
$$+ \Big\| \sum_j |(nA(n^{-\kappa}))_* \cap C_j|(X_j - Y_j) \Big\|$$
$$=: I_{11} + I_{12} + I_{13} + I_{21} + I_{22} + I_{23} + I_3 + I_4.$$

由引理11.1.2即得

$$(11.2.13)\quad I_{11} \vee I_{21} \vee I_3 = O(n^{d/(2+\delta)}).$$

又由引理11.1.3得

$$(11.2.14)\quad I_{12} \vee I_{22} = O(n^{d/2-\gamma_2}),$$

其中 $\gamma_2 = \kappa - d\,\dfrac{1+\tau}{2+\delta} - \dfrac{d}{2} > 0$.

为估计 $I_{13}$, $I_{23}$和 $I_4$,我们需要下述引理.

**引理11.2.3** 存在 $d\left(\dfrac{1}{2} + \dfrac{1+\tau}{2+\delta}\right) < \kappa \leqslant d, \gamma, \gamma'' > 0$使对任一 $n \in N$有

$$P\{\sup(|\bar{V}_n(A(n^{-\kappa}))| : A(n^{-\kappa}) \in \mathscr{A}(n^{-\kappa})) > cn^{d/2-\gamma}\} \leqslant cn^{-1-\gamma''}.$$

**证** 令 $p=[n^{\zeta}]$，对给定的 $A\in\mathscr{B}^d\cap[0,1]^d$，以边长为 $2p$ 的 $R_+^d$ 中形如引理11.2.2中的矩形复盖 $(nA)\backslash(nA)_*$，以 $D$ 记这些矩形的併．对 $j\in\mathbb{N}^d$，记 $d_j=|((nA)\backslash(nA)_*)\cap C_j|$．由(11.2.4)有

$$F=\sum_{j\in D}d_j=|(nA)\backslash(nA)_*| \tag{11.2.15}$$

$$\leqslant n^d b((\psi(m+1)-\psi(m))/n)$$

$$\leqslant cn^{d-h}(\psi(m+1)-\psi(m))^h$$

$$\leqslant cn^{d-h}n^{h\beta/(\beta+1)}=cn^{d-\gamma},$$

其中 $\gamma=h/(\beta+1)$．存在 $\gamma'>0$ 使 $\gamma'<\gamma/2$ 且

$$\gamma'<d\left(\frac{1}{2}-\frac{1+\tau}{2+\delta}-\zeta\right). \tag{11.2.16}$$

那么由引理11.2.1，(11.2.15)和引理11.2.2我们有

$$P\{|\bar{V}_n(A)|>cn^{d/2-\gamma'}\}$$

$$\leqslant c\Big(n^{-d-2\gamma'}M^2F^2\alpha(p)+p^{2d}M^4\alpha(p)$$

$$+\exp\left(-\frac{n^{d-2\gamma'}}{cF}\right)+\exp\left(\frac{cn^{d/2-\gamma'}}{Mp^d}\right)\Big).$$

由于此时 $M=2n^{d(1+\tau)/(2+\delta)}$，$p\sim n^{\zeta}$，我们有

$$P\{|\bar{V}_n(A)|>cn^{d/2-\gamma'}\}\leqslant c\Big(n^{-d+2\gamma'+2d\frac{1-\tau}{2+\delta}+2d-2\gamma-\zeta s} \tag{11.2.17}$$

$$+n^{2d\zeta+4d\frac{1+\tau}{2+\delta}-\zeta s}+\exp(-cn^{-2\gamma'+\gamma})$$

$$+\exp\left(-cn^{d/2-\gamma'-d\frac{1+\tau}{2+\delta}-d\zeta}\right)\Big)$$

$$\leqslant cn^{-\zeta s+2\left(1+d\frac{1-\tau}{2+\delta}\right)}.$$

存在 $\theta$ 和 $\kappa$，$0<\theta<\zeta s-1-d\left(1+\frac{2(1+\tau)}{2+\delta}\right)$，$d\left(\frac{1}{2}+\frac{1+\tau}{2+\delta}\right)<\kappa\leqslant d$ 且 $\theta/\kappa=u$．对(11.2.17)两边求和，由 $N(n^{-\kappa})\leqslant cn^{\kappa u}=cn^{\theta}$ 得

$$P\Big\{\sup_{A(n^{-\kappa})}|\bar{V}_n(A(n^{-\kappa}))|>cn^{d/2-\gamma'}\Big\}$$

$$\leqslant cn^{\theta-\zeta s+2\left(1+d\frac{1-\tau}{2+\delta}\right)}$$

$$=cn^{-1-\gamma''}.$$

引理11.2.3证毕．

**引理11.2.4** 若在(11.2.8)中 $\beta\geqslant 6d$，那么有

(11.2.18) $$P\Big\{\sum_{r\in L,\psi(r_i)\leqslant n}\Big\|\sum_{j\in R_r}X_j\Big\|>n^{d/2-1/16}\Big\}\leqslant cm^{-2}.$$

其中 $m$ 与 $n$ 间的关系由(11.2.9)决定.因此由 Borel-Cantelli 引理

(11.2.19) $$\sum_{r\in L,\psi(r_i)\leqslant n}\Big|\sum_{j\in R_r}X_j\Big|=O(n^{d/2-1/16})\quad \text{a.s.}$$

**证** 当 $r\in L$ 且 $\psi(r_i)\leqslant n,i=1,\cdots,d$ 时,由 $L$ 的定义对某 $i=1,\cdots,d,\psi(r_i)^{8d}<\prod_{i=1}^{d}\psi(r_i)\leqslant n^d$.因此 $\psi(r_i)\leqslant n^{1/8}$.所以

$$\mathrm{Card}(R_r\cap \mathbb{N}^d)\leqslant n^{1/8}\cdot n^{d-1}=n^{d-7/8}.$$

此外,我们有

$$\mathrm{Card}\{t:t\in\mathbb{N},\psi(t)\leqslant n\}\leqslant \mathrm{Card}\{t:t\in\mathbb{N},ct^{\beta+1}\leqslant n\}\leqslant cn^{1/(\beta+1)},$$

$$\mathrm{Card}\{r:r\in\mathbb{N}^d,\psi(r_i)\leqslant n,i=1,\cdots,d\}\leqslant cn^{d/(\beta+1)}.$$

由引理11.2.1和 Chebyshev 不等式得

$$P\Big\{\sum_{r\in L,\psi(r_i)\leqslant n}\Big|\sum_{j\in R_r}X_j\Big|>n^{d/2-1/16}\Big\}$$

$$\leqslant cn^{d/(\beta+1)}\max_{r\in L,\psi(r_i)\leqslant n}P\Big\{\Big|\sum_{j\in R_r}X_j\Big|\geqslant cn^{\frac{d}{2}-\frac{1}{16}-\frac{d}{\beta+1}}\Big\}$$

$$\leqslant cn^{\frac{d}{\beta+1}}\cdot n^{d-\frac{7}{8}-d-\frac{1}{8}+\frac{2d}{\beta+1}}$$

$$=cn^{-\frac{3}{4}+\frac{3d}{\beta+1}}$$

$$\leqslant cm^{-2}$$

得证引理11.2.4成立.

由引理11.2.3即得 $I_{22}VI_{23}=O(n^{d/2-\gamma})$.最后,由于

$$I_4\leqslant\sum_{r\in L,\psi(r_i)\leqslant n}\Big|\sum_{j\in R_r}(X_j-Y_j)\Big|+\sum_{r\in L,\psi(r_i)\leqslant n}\Big|\sum_{j\in R_r}(X_j-Y_j)\Big|.$$

由引理11.2.4和(11.2.11)我们有 $I_\psi=0(n^{d/2-\gamma})$a.s.这样就完成了定理11.2.1的证明.

# 第Ⅳ部分　相依样本的统计量

60年代以来，由相依样本产生的各类统计量的极限性质已被众多学者研究. 在这一部分我们将介绍有关成果. 首先，我们将在第十二章中介绍混合相依样本的经验过程的弱收敛和强逼近. 有关混合相依样本的 $U$ 统计量、线性模型中的误差方差估计和密度函数估计等统计量的极限性质将在第十三章中给出. 而在第十四章中，我们将介绍其它类型的弱相依序列(如缺项三角级数、Gauss 序列、Markov 过程的可加泛函等)的某些渐近性质.

# 第十二章　经验过程

设$\{X_n, n\geqslant 1\}$是随机变量序列，具有共同的分布函数$F(x)$. $X_1,\cdots,X_n$的经验分布$F_n(t)$定义为

$$F_n(t) = n^{-1}\sum_{i=1}^{n} I(X_i \leqslant t) \quad -\infty < t < \infty,$$

相应的经验过程$\beta_n(t)$定义为

(12.0.1)　$\beta_n(t) = \sqrt{n}\,(F_n(t) - F(t)) \quad -\infty < t < \infty.$

如果$F$是连续的，则对一切$n\geqslant 1, U_n := F(X_n)$是$[0,1]$上的均匀分布，相应的经验过程是

(12.0.2)　$\alpha_n(t) = \sqrt{n}\,(E_n(t) - t) \quad 0\leqslant t\leqslant 1,$

其中经验分布函数

(12.0.3)　$E_n(t) = F_n(\text{inv } F(t)) = \dfrac{1}{n}\sum_{i=1}^{n} I(U_i \leqslant t),\ 0\leqslant t\leqslant 1.$

因此对于任何连续的分布函数$F$，由$U_i = F(X_i), i=1,\cdots,n$，可有

(12.0.4)　$\{\beta_n(\text{inv } F(t)), 0\leqslant t\leqslant 1\} = \{\alpha_n(t), 0\leqslant t\leqslant 1\}, n = 1,2,\cdots.$

于是，对所有关于$\alpha_n$成立的定理关于$\beta_n$也正确，只要在(12.0.4)中令$y=F(x)$即可. 所以在这一章中我们将只限于讨论均匀的经验过程$\alpha_n(\cdot)$.

在独立情况，早已证明过.

(12.0.5)　$\alpha_n \Rightarrow B \qquad n\to\infty,$

其中$B$是 Brown 桥. $\{\alpha_n(\cdot)\}$用一列 Brown 桥强逼近的结果曾由 Komlos-Major-Tusnády (1975) 得出. 他们证明：不改变$\{\alpha_n(\cdot), n\geqslant 1\}$的分布，可在一较大的概率空间上与一列独立的 Brown 桥一起重新定义$\{\alpha_n(\cdot)\}$，使得

(12.0.6)　$\sup\limits_{0\leqslant t\leqslant 1} |\alpha_n(t) - B_n(t)| = O(n^{-1/2}\log n) \quad \text{a.s.}$

对于一个平稳序列，保证(12.0.5)或(12.0.6)成立的条件也曾被不少学者讨论过. 本章的目的即是介绍混合样本的有关成果. 在§12.1中，我们将给出经验过程弱收敛方面迄今为止的最好的结果. 对于$\alpha$混合和$\rho$混合样本的经验过程的加权弱收敛性将在§12.2中讨论. 经验过程用Gauss过程强逼近的某些结果在§12.3中介绍. 而在§12.4中我们将建立经验过程连续模方面的若干定理.

## §12.1 弱收敛

设$\{U_n, n\geqslant 1\}$是$[0,1]$上均匀分布的强平稳随机变量序列，记

$$
\begin{aligned}
(12.1.1)\qquad R(s,t) = {} & s\wedge t - st \\
& + \sum_{k=2}^{\infty} E\{[I(U_k\leqslant s)-s][I(U_1\leqslant t)-t] \\
& + [I(U_1\leqslant s)-s][I(U_k\leqslant t)-t]\}.
\end{aligned}
$$

当$\{U_n\}$是$\alpha$混合且满足$\sum_{n=1}^{\infty}\alpha(n)<\infty$时，由引理1.2.1，(12.1.1)中的级数是绝对收敛的.

$\{U_n\}$的经验过程$\{\alpha_n(t), 0\leqslant t\leqslant 1\}$的弱收敛性已被若干学者研究过，这一课题的研究梗概可参见Billingsley(1968)，Sen(1971, 1974)，Yoshihara(1975, 1978)和邵启满(1986)等，迄今为止最佳结果是邵启满(1986)给出的.

12.1.1 $\varphi$混合样本经验过程的弱收敛性.

**定理12.1.1** 设$\{U_n, n\geqslant 1\}$是$[0,1]$上均匀分布的强平稳$\varphi$混合的随机变量序列，它的经验过程如(12.0.2)中定义. 如果(12.1.1)中的级数绝对收敛，而且

$$(12.1.2)\qquad \sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty.$$

则有

$$(12.1.3)\qquad \alpha_n\Rightarrow Y,$$

其中 $Y=\{Y(t),0\leqslant t\leqslant 1\}$ 是 Gauss 过程，$EY(t)=0$ 且 $EY(s)Y(t)=R(s,t)$.

**证** 对任给的 $t\in[0,1]$,

$$\alpha_n(t)=\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(I(X_k\leqslant t)-t).$$

由推论 4.1.1，$\alpha_n(t)$ 依分布收敛于 $Y(t)$. 进一步，容易验证 $\alpha_n$ 的有限维分布收敛于 $Y$ 的相应的有限维分布.

为了证明 $\{\alpha_n\}$ 的胎紧性，只需验证：对任意的 $\varepsilon>0,\eta>0$，存在 $0<\delta<1$ 使得对任何 $s,0\leqslant s\leqslant 1-\delta$，和充分大的 $n$

(12.1.4) $$P\{\sup_{s\leqslant t\leqslant s+\delta}|\alpha_n(t)-\alpha_n(s)|\geqslant 4\varepsilon\}<\eta\delta.$$

对 $0\leqslant s<t\leqslant 1$，记 $\xi_i=(I(U_i\leqslant t)-t)-(I(U_i\leqslant s)-s)=I(s<U_i\leqslant t)-(t-s)$. 我们有

$$E\xi_i=0,|\xi_i|\leqslant 1,E\xi_i^2=t-s-(t-s)^2\leqslant t-s.$$

从引理 2.2.2 和 2.2.8 可得：存在 $C>0$,

(12.1.5) $$\begin{aligned}E|\alpha_n(t)-\alpha_n(s)|^4&\leqslant\frac{1}{n^2}E\Big(\sum_{j=1}^{n}\xi_j\Big)^4\\&\leqslant C(t-s)^2.\end{aligned}$$

令 $p$ 是一个满足 $\varepsilon/n\leqslant p$ 的正数. 考虑随机变量 $\alpha_n(s+ip)-\alpha_n(s+(i-1)p),i=1,2,\cdots,m$. 由 Billingsley(1968)的定理 12.2，对任意的 $\lambda>0$ 有

(12.1.6) $$P\{\max_{0\leqslant i\leqslant m}|\alpha_n(s+ip)-\alpha_n(s)|\geqslant\lambda\}\leqslant\frac{K}{\varepsilon\lambda^4}m^2p^2,$$

其中常数 $K$ 仅与 $\varphi(\cdot)$ 有关.

对 $s\leqslant t\leqslant s+p$，我们有

(12.1.7) $$|\alpha_n(t)-\alpha_n(s)|\leqslant|\alpha_n(s+p)-\alpha_n(s)|+p\sqrt{n},$$

由此推出

(12.1.8) $$\begin{aligned}&\sup_{s\leqslant t\leqslant s+pm}|\alpha_n(t)-\alpha_n(s)|\\&\leqslant\sup_{s\leqslant t\leqslant s+pm}\{|\alpha_n(t)-\alpha_n(s+ip)|+|\alpha_n(s+ip)-\alpha_n(s)|\}\\&\leqslant 3\max_{0\leqslant i\leqslant m}|\alpha_n(s+ip)-\alpha_n(s)|+p\sqrt{n}.\end{aligned}$$

如果 $\varepsilon/n\leqslant p<\varepsilon/\sqrt{n}$，从(12.1.6)和(12.1.8)可得

(12.1.9) $$P\left\{\sup_{s\leqslant t\leqslant s+pm}|\alpha_n(t)-\alpha_n(s)|\geqslant 4\varepsilon\right\}\leqslant\frac{K}{\varepsilon^5}m^2p^2.$$

取 $\delta$ 使得 $K\delta/\varepsilon^5<\eta$. 对充分大的 $n$ 存在 $m$ 使得 $(\delta/\varepsilon)\sqrt{n}<m\leqslant(\delta/\varepsilon)n$ 且 $mp=\delta$. 从(12.1.8)和(12.1.9)得到(12.1.4)，定理 12.1.1 证毕.

**推论 12.1.1** 设 $\{U_n,n\geqslant1\}$ 是 $[0,1]$ 上均匀分布的强平稳 $\rho$ 混合的随机变量序列. 如果(12.1.1)中的级数绝对收敛且 $\sum_{n=1}^{\infty}\rho(2^n)<\infty$，则有 $\alpha_n\Rightarrow Y$.

**注 12.1.1** 在 $p$ 维样本($p\geqslant2$)情形，定理 2.1.1 的结论也成立.

**注 12.1.2** 混合序列的 Glivenko-Gantelli 定理，也即经验过程的 a.s. 收敛性是 §8.3—§8.5 中的结果的直接推论. 例如，从推论 8.3.1 我们有

**命题 12.1.1** 设 $\{X_n,n\geqslant1\}$ 是 $\varphi$ 混合随机变量序列，具有共同的连续分布函数 $F(\cdot)$. 那么对任意的 $\theta>0,\varepsilon>0$,

$$\sum_{n=1}^{\infty}P\{|F_n(x)-F(x)|\geqslant\varepsilon n^{-1/2+\theta}\}<\infty,$$

也就是说给出了 Glivenko-Cantelli 定理中的收敛速度：

$$|F_n(x)-F(x)|=O(n^{-1/2+\theta})\quad \text{a.s.}$$

当样本是 $\rho$-mixing 或 $\alpha$-mixing 时在一定的条件下有类似的结果.

12.1.2 $\alpha$ 混合样本经验过程的弱收敛性.

**定理 12.1.2** 设 $\{U_n,n\geqslant1\}$ 是 $[0,1]$ 上均匀分布的强平稳 $\alpha$ 混合的随机变量序列且设

(12.1.10) $$\alpha(n)=O(n^{-r}),\text{某个 } r>2.$$

则有 $\alpha_n\Rightarrow Y$.

利用下面给出的引理，仿照定理 12.1.1 的证明即可推出定理 12.1.2.

**引理 12.1.1** 设 $\{\xi_n,n\geqslant1\}$ 是强平稳 $\alpha$ 混合随机变量序列，

$|\xi_1|\leqslant 1$　a. s. , $E\xi_1=0$, $E\xi_1^2=\tau$, $E|\xi_1|=2\tau$ 且(12.1.10)被满足．令 $S_n=\sum_{j=1}^{n}\xi_j$. 假设 $r>2$ 不是整数且 $m=2k$，其中 $k$ 是整数，满足 $r-1<m<r+1$. 那么我们有

$$E|S_n|^m\leqslant c\sum_{i=0}^{(m-2)/2}\left(n^{i+1}\tau^{i\theta_1+\theta_{m-1-2i}}\right),\tag{12.1.11}$$

其中 $[r]-1<r\theta_1<r-1$, $\theta_k=\theta_1-(k-1)/r$, $k=1,\cdots,[r]$.

**证**　由(12.1.10)和 $\theta_k$ 的定义，我们有

$$\sum_{i=0}^{\infty}(i+1)^{k-1}\alpha^{1-\theta_k}(i)<\infty\quad k=1,\cdots,[r].\tag{12.1.12}$$

设 $\sum_{n(k)}$ 是对 $i_1,\cdots,i_k\geqslant 0$, $i_1+\cdots+i_k\leqslant n$ 求和，$\sum_{n(k)}^{(l)}$ 是对 $i_1,\cdots,i_k\geqslant 0$, $i_1+\cdots+i_k\leqslant n$ 且 $i_l=\max\limits_{1\leqslant j\leqslant k}i_j$ 求和，其中 $l=1,\cdots,k$; $k=1,\cdots,[r]$. 记

$$\sum\nolimits_{n(k)}^{(l)}=\sum\nolimits_{n(k)}^{(l)}|E\xi_0\xi_{i_1}\xi_{i_1+i_2}\cdots\xi_{i_1+\cdots+i_k}|.$$

注意到对任一偶数 $d$，由平稳性我们有

$$E|S_n|^d\leqslant d\,!\,n\sum\nolimits_{n(d-1)}|E\xi_0\xi_{i_1}\cdots\xi_{i_1+\cdots+i_{d-1}}|.\tag{12.1.13}$$

从引理 1.2.5 和(12.1.12)我们有

$$\sum\nolimits_{n(1)}|E\xi_0\xi_{i_1}|\leqslant 6\sum_{i_1=0}^{n}\alpha^{1-\theta_1}(i_1)\tau^{\theta_1}\leqslant c\tau^{\theta_1},\tag{12.1.14}$$

$$\begin{aligned}&\sum\nolimits_{n(2)}|E\xi_0\xi_{i_1}\xi_{i_1+i_2}|\\&\leqslant\left\{\sum\nolimits_{n(2)}^{(1)}+\sum\nolimits_{n(2)}^{(2)}\right\}|E\xi_0\xi_{i_1}\xi_{i_1+i_2}|\\&\leqslant 6\sum_{n(2)}^{(1)}\alpha^{1-\theta_2}(i_1)\tau^{\theta_2}+6\sum_{n(2)}^{(2)}\alpha^{1-\theta_2}(i_2)\tau^{\theta_2}\\&\leqslant 6\sum_{i_1=0}^{n}(i_1+1)\alpha^{1-\theta_2}(i_1)\tau^{\theta_2}\\&\quad+6\sum_{i_2=0}^{n}(i_2+1)\alpha^{1-\theta_2}(i_2)\tau^{\theta_2}\leqslant c\tau^{\theta_2}.\end{aligned}\tag{12.1.15}$$

其次，用归纳法来证明：对 $k=1,2,\cdots,[r]$ 有

$$(12.1.16)\qquad \sum\nolimits_{n(k)} \leqslant c \sum_{i=0}^{[(k-1)/2]} n^i \tau^{i\theta_1+\theta_{k-2i}}.$$

从(12.1.14)和(12.1.15)即得对 $k=1,2$,(12.1.16)成立. 假设(12.1.16)对 $k-1, 2\leqslant k-1<[r]$成立. 我们来证(12.1.16)对 $k$ 也成立. 注意到

$$\sum\nolimits_{n(k)} \leqslant \sum\nolimits_{n(k)}^{(1)} + \cdots + \sum\nolimits_{n(k)}^{(k)},$$

$$\sum\nolimits_{n(k)}^{(1)} \leqslant 6\sum\nolimits_{n(k)}^{(1)} \alpha^{1-\theta_k}(i_1)\tau^{\theta_k},$$

$$\leqslant 6\sum_{i_1=0}^{n}(i_1+1)^{k-1}\alpha^{1-\theta_k}(i_1)\tau^{\theta_k} \leqslant c\tau^{\theta_k}.$$

类似地, $\sum\nolimits_{n(k)}^{(k)} \leqslant c\tau^{\theta_k}$. 当 $2\leqslant l\leqslant k-1$,

$$\sum\nolimits_{n(k)}^{(l)} \leqslant 6\sum\nolimits_{n(k)}^{(l)} \alpha^{1-\theta_k}(i_l)\tau^{\theta_k}$$

$$+ \sum\nolimits_{n(k)}^{(l)} |E\xi_0\xi_{i_1}\cdots\xi_{i_1+\cdots+i_{l-1}}|$$

$$\cdot |E\xi_{i_1+\cdots+i_l}\cdots\xi_{i_1+\cdots+i_k}| =: I_1 + I_2.$$

我们有 $I_1\leqslant c\tau^{\theta_k}$. 由归纳假设和平稳性有

$$I_2 \leqslant cn\Big(\sum_{i=0}^{[(l-2)/2]} n^i\tau^{i\theta_1+\theta_{l-1-2i}}\Big)\Big(\sum_{j=0}^{[(k-l-1)/2]} n^j\tau^{j\theta_1+\theta_{k-l-2j}}\Big)$$

$$\leqslant c \sum_{0\leqslant i\leqslant\left[\frac{l-2}{2}\right],\,0\leqslant j\leqslant\left[\frac{k-l-1}{2}\right]} n^{i+j+1}\tau^{(i+j+1)\theta_1+\theta_{k-2(i+j+1)}}$$

$$\leqslant c\sum_{i=1}^{[(k-1)/2]} n^i\tau^{i\theta_1+\theta_{k-2i}}.$$

这样(12.1.16)对 $k=1,2,\cdots,[r]$成立. 从(12.1.13),(12.1.16)并注意到 $m-1\leqslant[r]$,得证引理 12.1.1 成立.

**引理 12.1.2** 若对 $2<r<3$ 满足引理 12.1.1 的条件,那么

$$(12.1.17)\qquad ES_n^4\leqslant c(n^{4-r}+n^2\tau^{2\theta_1}).$$

**证** 从引理 1.2.5 和(12.1.14)我们有

$$\sum\nolimits_{n(3)}^{(1)} \leqslant 6\sum\nolimits_{n(3)}^{(1)}\alpha(i_1) \leqslant 6\sum_{i_1=0}^{n}(i_1+1)^2\alpha(i_1) \leqslant cn^{3-r}.$$

类似地 $\sum\nolimits_{n(3)}^{(3)} \leqslant cn^{3-r}$. 又由(12.1.16) 我们有

$$\sum\nolimits_{n(3)}^{(2)} \leqslant 6\sum\nolimits_{n(3)}^{(2)} \alpha(i_2) + \sum\nolimits_{n(3)}^{(2)} |E\xi_0\xi_{i_1}| \cdot |E\xi_{i_1+i_2}\xi_{i_1+i_2+i_3}|$$
$$\leqslant c(n^{3-r} + n\tau^{2\theta_1}).$$

把这些不等式与(2.1.13)相结合得(12.1.17).

现在定理 12.1.2 可依定理 12.1.1 的证明思路证明之，只需用引理 12.1.2 代替引理 2.2.2 和引理 2.2.8.

**注 12.1.3** 在 $p$ 维($p\geqslant 2$)样本情形，利用引理 12.1.1，定理 12.1.2 的结果也成立，假如

$$r > \begin{cases} p-1 & \text{当 } p \text{ 是偶数}, \\ p^2/(p-1) & \text{当 } p \text{ 是奇数}. \end{cases}$$

**注 12.1.4** 具有随机指标的部分和过程及经验过程的弱收敛性也已被一些作者所讨论. 关于这一课题可参见 Rényi(1958, 1960)，Billingsley(1968, § 17)，Aldous(1978)和 Lu(1984)等. 在此我们不加证明地列出具有随机指标经验过程弱收敛的一个结果.

设$\{\alpha_n, n\geqslant 1\}$是定理 12.1.2 中的经验过程序列，又设$\{\tau_n, n\geqslant 1\}$是同一概率空间上的正整值随机变量序列. 假设 $\alpha_n \Rightarrow Y$，其中 $Y$ 是 Gauss 过程且$\{\tau_n\}$满足

$$\tau_n/n \xrightarrow{p.} \tau,$$

其中 $\tau$ 是正随机变量，那么我们有

$$\alpha_{\tau_n} \Rightarrow Y.$$

## § 12.2 加权弱收敛

设 $q$ 是(0,1)上的一个权函数，它是正的，也就是说对任意的 $0<\delta<1/2$，$\inf\limits_{\delta\leqslant t\leqslant 1-\delta} q(t)>0$. 定义加权均匀经验过程为$\{\alpha_n(t)/q(t), 0<t<1\}$. 当$\{U_n, n\geqslant 1\}$是独立的[0,1]上均匀分布的随机变量序列时，经验过程的加权弱收敛性已在近年来有了深入的研究. 值得提及的工作有 M. Csörgo，S. Csörgö，Horváth 和 Mason(1986a)，Shorack 和 Wellner(1986)以及 Csörgö和 Horváth(1993)等. 下面我们重述 M. Csörgö，S. Csörgö，Horváth 和 Mason(1986a)的一个

定理. 在 Csörgo和 Horváth(1993)的第四章中给出了该定理的一个较简洁的证明.

**定理 12.2.1** 假设 $q$ 在(0,1)上是正且连续的,在 0 的一个邻域内是不减的,在 1 的一个邻域内是不增的,那么,对一切 $\lambda>0$,成立

$$(12.2.1)\quad I(q,\lambda):=\int_0^1\frac{1}{t(1-t)}\exp(-\lambda q^2(t)/(t(1-t)))dt<\infty$$

的充要条件是当 $n\to\infty$ 时

$$(12.2.2)\qquad \alpha_n(\cdot)/q(\cdot)\Rightarrow B(\cdot)/q(\cdot).$$

对于相依序列经验过程的加权弱收敛的研究还并不多. 这时,由于协方差项的出现,极限过程就可能不再是(加权)Brown 桥的形式了(参见 § 12.1 中的结果).

邵启满和于浩(1995)在混合相依或相伴(assθciated)相依的假设下研究了强平稳序列经验过程的加权弱收敛性. 这里我们将只介绍混合情形时的有关结果. 首先,给出下列基本定理.

**定理 12.2.2** 设$\{U_n,n\geqslant1\}$是$[0,1]$上均匀分布的强平稳随机变量序列. 假设对任意的 $0\leqslant s,t\leqslant1$ 和 $n\geqslant1$ 有

(A1) 存在 $c_1>0,p>2,p_1>1,0\leqslant r_1\leqslant1$ 和 $p_2>1-r_1$ 使得 $E|\alpha_n(t)-\alpha_n(s)|^p\leqslant c_1(|t-s|^{p_1}+n^{-p_2/2}|t-s|^{r_1})$;

(A2) 存在 $c_2>0$ 和 $0<r_2\leqslant1$ 使得 $E(\alpha_n(t)-\alpha_n(s))^2\leqslant c_2|t-s|^{r_2}$.

又设

$$(12.2.3)\qquad \alpha_n\Rightarrow Y,$$

其中 $Y$ 是 § 12.1 中定义的 Gauss 过程. 那么

$$(12.2.4)\qquad \alpha_n/q\Rightarrow Y/q,$$

此处权函数 $q$ 具有下列性质;存在 $c>0$ 和 $\beta>\frac{1}{2}$,对一切 $0<t<1$,

$$(12.2.5)\qquad q(t)\geqslant c(t(1-t))^{\mu}(\log 1/(t(1-t)))^{\beta},$$

其中

(12.2.6) $$\mu=\min\left\{\frac{p_1}{p},\frac{r_1+p_2}{p+p_2},\frac{r_2}{2}\right\}.$$

**注 12.2.1** 通过常规程序(参见 Billingsley)(1968)),定理 12.2 和(22.18))容易验证:在条件(A1)下,$\{\alpha_n(t),0\leqslant t\leqslant 1\}$是胎紧的.因此,为使(12.2.3)成立,只需证明:$\{\alpha_n(t),0\leqslant t\leqslant 1\}$的任何有限维分布收敛于$\{Y(t),0\leqslant t\leqslant 1\}$相应的有限维分布而且(12.1.1)中的级数绝对收敛.

**注 12.2.2** 定理 12.2.1 中采用的权函数 $q$ 通常称为 Chibisov-O'Reilly 权函数.若写 $q(t)=(t(1-t))\log\log(1/(t(1-t)))^{1/2}g(t)$,则当 $t\to 0$ 或 $t\to 1$ 时必有 $g(t)\to\infty$.因此(12.2.5)中的权函数 $q$ 可与 Chibisov-O'Reilly 权函数相比较,如果我们适当选取 $p,p_1,p_2,r_1$ 和 $r_2$ 使得 $\mu$ 接近或等于 1/2.事实上,下面的定理 12.2.3 和 12.2.4 将说明:在混合情形,取 $\mu=1/(2+\varepsilon)$(某个 $\varepsilon>0$)是可能的.特别地,在较强的混合条件之下,对 $\rho$ 混合序列可得 $\mu=1/2$ 的最佳速度.然而在大多数情况,$\mu<1/2$.顺便指出,如果对一般的权函数 $q$,满足 $\int_0^1 1/q^2(t)dt<\infty$,那么我们也有(12.2.1)(任意 $\lambda>0$),也就是说 $q$ 一定是 Chibisov-O'Reilly 权函数.

**注 12.2.3** 如果在(12.2.6)中 $\mu=(r_1+p_2)/(p+p_2)<\min(p_1/p,r_2/2)$,那么从定理 12.2.2 的证明,对 $\beta$ 的限制可从 $\beta>1/2$ 放宽到 $\beta>1/(p+p_2)=(1-\mu)/(p-r_1)$.此外,在 $\mu\geqslant 1/(p+1-r_1)$时我们可用一个简单的充分条件 $\int_0^1 1/q^{1/\mu}(t)dt<\infty$ 代替(12.2.5).

定理 12.2.2 的一个直接应用是对 $\alpha_n$ 的积分泛函建立弱收敛性.例如,考虑积分泛函

$$\Delta_n(t)=\int_0^t\alpha_n(s)dQ(s)=\int_0^t\beta_n(\mathrm{inv}F(s))dQ(s),0\leqslant t<1,$$

和对应的 Gauss 过程

(12.2.7) $$\Delta(t)=\int_0^t Y(s)dQ(s),0\leqslant t\leqslant 1,$$

其中 $Q(s)=\mathrm{inv}F(x)$是 $X$ 的分布函数 $F$ 的分位点函数(回顾(12.

0.4)).函数 $\Delta(t)$ 在理论上和实用上都有重要的作用(参见 M. Csörgö, S. Csörgö, Horvath 和 Mason(1986b).

**推论 12.2.1** 如果定理 12.2.2 的条件被满足,且

(12.2.8) $\int_0^1 (t(1-t))^{\mu}(\log 1/(t(1-t)))^{\beta} dQ(t) < \infty,$

则在 $D[0,1]$ 中有

$$\Delta_n \Rightarrow \Delta.$$

**注 12.2.4** 设 $F$ 是随机变量 $X$ 的分布函数,条件(12.2.8)稍强于 $X$ 的 $1/\mu$ 阶矩的存在性.相反的结论虽未必成立,但从 $E|X|^{1/\mu}(\log(1+|X|))^{(1-\beta)/\mu+\delta}<\infty$(存在 $\delta>0$)可推出(12.2.8).

定理 12.2.2 使我们能够对平稳混合序列的经验过程建立加权弱收敛性.

**定理 12.2.3** 设 $\{U_n, n\geqslant 1\}$ 是 $[0,1]$ 上均匀分布的强平稳 $\alpha$ 混合随机变量序列.若存在 $\theta\geqslant 1+\sqrt{2}$ 和 $\varepsilon>0$ 使得

(12.2.9) $$\alpha(n)=O(n^{-\theta-\varepsilon})$$

那么对满足 $q(t)\geqslant c(t(1-t))^{(1-1/\theta)/2}$(某个 $c>0$)的权函数 $q$ 成立着

$$\alpha_n/q \Rightarrow Y/q.$$

**定理 12.2.4** 设 $\{U_n, n\geqslant 1\}$ 是 $[0,1]$ 上均匀分布的强平稳 $\rho$ 混合随机变量序列.假设(12.1.1)中的级数绝对收敛,且

(12.2.10) $$\sum_{n=1}^{\infty}\rho(2^n)<\infty.$$

那么

$$\alpha_n/q \Rightarrow Y/q,$$

其中 $q$ 满足:对任意的 $\varepsilon>0$,存在 $c>0$ 使得 $q(t)\geqslant c(t(1-t))^{1/(2+\varepsilon)}$.

如果以条件:对某个 $p>2$,

(12.2.11) $$\sum_{n=1}^{\infty}\rho^{2/p}(2^n)<\infty$$

代替(12.2.10),那么在 $D[0,1]$ 中

$$\alpha_n/q \Rightarrow Y/q,$$

其中 $q$ 满足：存在 $c>0$ 和 $\beta>1/2$，使得 $q(t)\geqslant c(t(1-t))^{1/2}(\log 1/(t(1-t)))^{\beta}$.

**推论 12.2.2** 假设满足定理 12.2.3 的条件且

$$\int_{-\infty}^{\infty}|x|^{2/(1-1/(2\beta))}dF(x)<\infty,$$

那么

$$\Delta_n\Rightarrow\Delta.$$

**推论 12.2.3** 设 $\{U_n,n\geqslant 1\}$ 是 $[0,1]$ 上均匀分布的强平稳 $\rho$ 混合序列，如果 (12.2.10) 成立且对任意的 $\varepsilon>0$

$$\int_{-\infty}^{\infty}|x|^{2+\varepsilon}dF(x)<\infty,$$

那么在 $D[0,1]$ 中

$$\Delta_n\Rightarrow\Delta.$$

如果以条件 (12.2.11) 代替 (12.2.10) 且

$$\int_0^1(t(1-t))^{1/2}(\log 1/(t(1-t)))^{\beta}dQ(t)<\infty,$$

那么在 $D[0,1]$ 中也有

$$\Delta_n\Rightarrow\Delta.$$

为了证明这些定理，我们需要下列引理

**引理 12.2.1** 设 $\{\xi_i,i\geqslant 1\}$ 是一随机变量序列，$\mathscr{F}_i=\sigma(\xi_j,j\leqslant i)$. 那么对任意的 $p\geqslant 2$，存在常数 $D=D(p)$ 使得

$$E\Big|\sum_{i=1}^{n}\xi_i\Big|^p\leqslant D\Big(\Big(\sum_{i=1}^{n}E\xi_i^2\Big)^{p/2}+\sum_{i=1}^{n}E|\xi_i|^p+n^{p-1}\sum_{i=1}^{n}E|E(\xi_i|\mathscr{F}_{i-1})|^p+n^{p/2-1}\sum_{i=1}^{n}E|E(\xi_i^2|\mathscr{F}_{i-1})-E\xi_i^2|^{p/2}\Big).\tag{12.2.12}$$

**证** 记 $\eta_i=\xi_i-E(\xi_i|\mathscr{F}_{i-1})$. $\{\eta_i,\mathscr{F}_{i-1},i\geqslant 1\}$ 构成一鞅差序列. 由熟知的 Burkholder(1973) 不等式，存在 $D=D(p)<\infty$，使得

$$E\Big|\sum_{i=1}^{n}\eta_i\Big|^p\leqslant D\Big(E\Big|\sum_{i=1}^{n}E(\eta_i^2|\mathscr{F}_{i-1})\Big|^{p/2}+\sum_{i=1}^{n}E|\eta_i|^p\Big)\tag{12.2.13}$$

$$\leqslant 2^p D\Big(\Big(\sum_{i=1}^n E\xi_i^2\Big)^{p/2}+\sum_{i=1}^n E|\xi_i|^p$$

$$+E\Big(\sum_{i=1}^n\big|E(\xi_i^2|\mathscr{F}_{i-1})-E\xi_i^2\big|\Big)^{p/2}\Big)$$

$$\leqslant 2^{2p} D\Big(\Big(\sum_{i=1}^n E\xi_i^2\Big)^{p/2}+\sum_{i=1}^n E|\xi_i|^p$$

$$+n^{p/2-1}\sum_{i=1}^n E\big|E(\xi_i^2|\mathscr{F}_{i-1})-E\xi_i^2\big|^{p/2}\Big).$$

另一方面，容易看出

$$E\Big|\sum_{i=1}^n\xi_i\Big|^p\leqslant 2^p\Big(E\Big|\sum_{i=1}^n\eta_i\Big|^p+n^{p-1}\sum_{i=1}^n E|E(\xi_i|\mathscr{F}_{i-1})|^p\Big).$$

由这两个不等式即得(12.2.12).

下面我们来建立一个关于 $\alpha$ 混合序列的 Rosenthal 型不等式.

**引理 12.2.2** 设 $2<p<r\leqslant\infty$，$2<v\leqslant r$，$\{X_n,n\geqslant 1\}$是 $\alpha$ 混合随机变量序列，$EX_n=0$ 且 $\|X_n\|_r<\infty$. 假设存在 $c>0$ 和 $\theta>0$

(12.2.14) $$\alpha(n)<cn^{-\theta}.$$

那么对任意的 $\epsilon>0$，存在 $K=K(\epsilon,r,p,v,\theta,c)<\infty$，使得

(12.2.15) $$E|S_n|^p\leqslant K\Big((nC_n)^{p/2}\max_{1\leqslant i\leqslant n}\|X\|_v^p$$

$$+n^{((p-(r-p)\theta/r)\vee 1)+\epsilon}\max_{1\leqslant i\leqslant n}\|X_i\|_r^p\Big),$$

其中 $c_n=\Big(\sum_{i=0}^n(i+1)^{2/(v-2)}\alpha(i)\Big)^{(v-2)/v}$. 特别地，对任意 $\epsilon>0$. 如果 $\theta>v(v-2)$ 且 $\theta\geqslant(p-1)r/(r-p)$，则有

(12.2.16) $$E|S_n|^p\leqslant K\big(n^{p/2}\max_{1\leqslant i\leqslant n}\|X_i\|_v^p+n^{1-\epsilon}\max_{1\leqslant i\leqslant n}\|X_i\|_r^p\big);$$

如果 $\theta\geqslant pr/(2(r-p))$，则有

(12.2.17) $$E|S_n|^p\leqslant Kn^{p/2}\max_{1\leqslant i\leqslant n}\|X_i\|_r^p.$$

**证** 为简单计，假设$\{X,X_n,n\geqslant 1\}$是强平稳的 $\alpha$ 混合序列. 由 Rio(1993)的一个结果，存在 $D_1=D_1(v)$使得

(12.2.18) $$ES_n^2\leqslant D_1 nC_n\|X\|_v^2.$$

我们将通过对 $n$ 使用归纳法来证明(12.2.15). 假设对每一 $1\leqslant k<n$，

(12.2.19) $$E|S_k|^p \leqslant K((kc_k)^{p/2}\|X\|_v^p + k^{(p-(r-p)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p).$$

现在来证明上式对 $k=n$ 仍成立。令 $0<a<1/2$（待定）又令 $m=[an]+1$. 对 $1\leqslant i\leqslant k_n:=[n/(2m)]+1$，定义

$$\xi_i=\sum_{j=2(i-1)m+1}^{n\wedge(2i-1)m}X_j,\ \eta_i=\sum_{j=(2i-1)m+1}^{n\wedge 2im}X_j.$$

易知

$$E|S_n|^p\leqslant 2^{p-1}\Big(E\Big|\sum_{i=1}^{k_n}\xi_i\Big|^p+E\Big|\sum_{i=1}^{k_n}\eta_i\Big|^p\Big)=:2^{p-1}(I_1+I_2).$$

记 $\mathscr{F}_i=\sigma(\xi_j,j\leqslant i)$. 由引理 12.2.1，存在常数 $D_2\geqslant(2D_1)^{p/2}$，使得

(12.2.20) $$\begin{aligned}I_1\leqslant D_2\Big(&\sum_{i=1}^{k_n}E|\xi_i|^p+\Big(\sum_{i=1}^{k_n}E\xi_i^2\Big)^{p/2}\\&+k_n^{p-1}\sum_{i=1}^{k_n}E|E(\xi_i|\mathscr{F}_{i-1})|^p\\&+k_n^{p/2-1}\sum_{i=1}^{k_n}E\big|E(\xi_i^2|\mathscr{F}_{i-1})-E\xi_i^2\big|^{p/2}\Big)\\=:&D_2\Big(\sum_{i=1}^{k_n}E|\xi_i|^p+I_{11}+I_{12}+I_{13}\Big).\end{aligned}$$

由(12.2.18)，有

(12.2.21) $$\begin{aligned}I_{11}&\leqslant(D_1k_nmc_m\|X\|_v^2)^{p/2}\\&\leqslant(2D_1nc_n)^{p/2}\|X\|_v^p\leqslant D_2(nc_n)^{p/2}\|X\|_v^p.\end{aligned}$$

为了估计 $I_{13}$，记

$$Y_i=E(\xi_i^2|\mathscr{F}_{i-1})-E\xi_i^2.$$

于是由引理 1.2.4

$$\begin{aligned}E|Y_i|^{p/2}&=E|Y_i|^{p/2-1}\mathrm{sgn}(Y_i)Y_i\\&=E(|Y_i|^{p/2-1}\mathrm{sgn}(Y_i)(\xi_i^2-E\xi_i^2))\\&=\sum_{2(i-1)m<j,l\leqslant n\wedge(2i-1)m}E|Y_i|^{p/2-1}\\&\qquad\mathrm{sgn}(Y_i)(X_jX_l-EX_jX_l)\end{aligned}$$

$$\leqslant 12 \sum_{2(i-1)m \leqslant j, l \leqslant n \wedge (2i-1)m} \alpha^{1-\frac{p-2}{p} \cdot \frac{2}{r}}(m)$$

$$(E|Y_i|^{p/2})^{(p-2)/p} \| X_j X_l \|_{r/2}$$

$$\leqslant 12 m^2 \alpha^{\frac{2}{p}-\frac{2}{r}}(m)(E|Y_i|^{p/2})^{(p-2)/p} \| X \|_r^2,$$

因此

$$(12.2.22) \qquad E|Y_i|^{p/2} \leqslant 12^{p/2} m^p \alpha^{1-p/r}(m) \| X \|_r^p.$$

它与(12.2.14)一起推出

$$(12.2.23) \qquad I_{13} \leqslant k_n^{p/2} 12^p m^p \alpha^{1-p/r}(m) \| X \|_r^p$$

$$\leqslant c 24^p n^p m^{(p-r)\theta/r} \| X \|_r^p$$

$$\leqslant c 24^p a^{(p-r)\theta/r} n^{p+(p-r)\theta/r} \| X \|_r^p$$

$$\leqslant c 24^p a^{(p-r)\theta/r} n^{(p+(p-r)\theta/r) \vee (1+\varepsilon)} \| X \|_r^p.$$

类似于(12.2.22),我们有

$$(12.2.24) \qquad E|E(\xi_i | \mathscr{F}_{i-1})|^p \leqslant 12^p m^p \alpha^{1-p/r}(m) \| X \|_r^p.$$

因此

$$(12.2.25) \qquad I_{12} \leqslant k_n^p 12^p m^p \alpha^{1-p/r}(m) \| X \|_r^p$$

$$\leqslant c 24^p a^{(p-r)\theta/r} n^{(p+(p-r)\theta/r) \vee (1+\varepsilon)} \| X \|_r^p.$$

综合这些不等式即得

$$I_1 \leqslant D_2 \Big\{ \sum_{i=1}^{k_n} E|\xi_i|^p + D_2 (nc_n)^{p/2} \| X \|_v^p$$

$$+ 2c 24^p a^{(p-r)\theta/r} n^{(p+(p-r)\theta/r) \vee (1+\varepsilon)} \| X \|_r^p \Big\}.$$

类似地

$$I_2 \leqslant D_2 \Big\{ \sum_{i=1}^{k_n} E|\eta_i|^p + D_2 (nC_n)^{p/2} \| X \|_v^p$$

$$+ 2c 24^p a^{(p-r)\theta/r} n^{(p+(p-r)\theta/r) \vee (1+\varepsilon)} \| X \|_r^p \Big\}.$$

因此我们得

$$(12.2.26) \quad E|S_n|^p \leqslant 2^{p-1} D_2 \Big\{ \sum_{i=1}^{k_n} (E|\xi_i|^p + E|\eta_i|^p)$$

$$+ 2D_2 (nC_n)^{p/2} \| X \|_v^p + 4C 24^p a^{(p-r)\theta/r}$$

$$n^{(p+(p-r)\theta/r) \vee (1+\varepsilon)} \| X \|_r^p \Big\}.$$

记

$$a = (2^{p+4}D_2)^{-1/\varepsilon - p/(p-2)},$$
$$K = 2^{p+1}D_2(D_2 + 2C24^p a^{(p-r)\theta/r}).$$

从(12.2.26)和归纳假设(12.2.19)即得

$$E|S_n|^p \leqslant 2^p D_2(k_n K((mc_m)^{p/2}\|X\|_v^p + m^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p)$$
$$+ D_2(nC_n)^{p/2}\|X\|_v^p + 2C24^p a^{(p-r)\theta/r} n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p)$$
$$\leqslant 2^p D_2(n/m)K((mC_n)^{p/2}\|X\|_v^p + m^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p)$$
$$+ 2^p D_2(D_2 + 2C24^p a^{(p-r)\theta/r})((nc_n)^{p/2}\|X\|_v^p$$
$$+ n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p)$$
$$\leqslant 2^p D_2 K(a^{(p-2)/2}(nC_n)^{p/2}\|X\|_v^p + a^{\varepsilon} n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p)$$
$$+ (K/2)((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p)$$
$$\leqslant (K/2)((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p)$$
$$+ (K/2)((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p)$$
$$= K((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p).$$

因此(12.2.19)对 $k=n$ 也成立. 引理证毕.

**定理 12.2.2 的证明.** 由 Billingsley(1968)的定理 4.2,只需证明:对任意的 $\varepsilon>0$

(12.2.27) $\quad \lim\limits_{\theta\to 0}\ \lim\limits_{n\to\infty}\sup P\{\sup\limits_{0<t\leqslant\theta}|\alpha_n(t)/q(t)|\geqslant\varepsilon\}=0,$

(12.2.28) $\quad \lim\limits_{\theta\to 0}\ \lim\limits_{n\to\infty}\sup P\{\sup\limits_{1-\theta\leqslant t<1}|\alpha_n(t)/q(t)|\geqslant\varepsilon\}=0,$

(12.2.29) $\quad \lim\limits_{\theta\to 0} P\{\sup\limits_{0<t\leqslant\theta}|Y(t)/q(t)|\geqslant\varepsilon\}=0,$

(12.2.30) $\quad \lim\limits_{\theta\to 0} P\{\sup\limits_{1-\theta\leqslant t<1}|Y(t)/q(t)|\geqslant\varepsilon\}=0.$

注意到

$$P\{\sup_{0<t\leqslant\theta}|\alpha_n(t)/q(t)|\geqslant\varepsilon\}$$
$$\leqslant\sum_{j=1}^{\infty}P\{\sup_{\theta 2^{-j}<t\leqslant\theta 2^{-j-1}}|\alpha_n(t)/q(t)|\geqslant\varepsilon\}$$
$$\leqslant\sum_{j=1}^{\infty}P\{\sup_{\theta 2^{-j}<t\leqslant\theta 2^{-j-1}}|\alpha_n(t)|\geqslant\varepsilon q(\theta 2^{-j})\}.$$

(12.2.27)可以改写为

(12.2.31) $\quad \lim\limits_{\theta\to 0}\ \lim\limits_{n\to\infty}\sup P\{\sup\limits_{0<t\leqslant\theta}|\alpha_n(t)/q(t)|\geqslant\varepsilon\}$

$$\leqslant \limsup_{\theta\to 0} \limsup_{n\to\infty}\sum_{j=1}^{\infty} B_{jn},$$

其中 $B_{jn}=P\{\sup_{0\leqslant t\leqslant\theta 2^{-j-1}}|\alpha_n(t)|\geqslant\varepsilon q(\theta 2^{-j})\}$.

记

$$\varepsilon_j=\varepsilon q(\theta 2^{-j}), G_n=\{j: n^{1/2}\theta 2^{-j+1}\leqslant\varepsilon_j/2\},$$

$$H_n=\{j: n^{1/2}\theta 2^{-j+1}>\varepsilon_j/2\}.$$

不难验证:对任意的 $0\leqslant s\leqslant t\leqslant s+h\leqslant 1$,

$$|\alpha_n(t)-\alpha_n(s)|\leqslant|\alpha_n(s+h)-\alpha_n(s)|+n^{1/2}h. \tag{12.2.32}$$

因此由(A2),(12.2.5)和(12.2.6),对 $j\in G_n$ 我们有

$$\begin{aligned}B_{jn}&\leqslant P\{|\alpha_n(\theta 2^{-j+1})|+n^{1/2}\theta 2^{-j+1}\geqslant\varepsilon_j\}\\&\leqslant P\{|\alpha_n(\theta 2^{-j+1})|\geqslant\varepsilon_j/2\}\\&\leqslant c\varepsilon_j^{-2}(\theta 2^{-j})^{r_2}\\&\leqslant c\varepsilon^{-2}(\log(2^j/\theta))^{-2\beta}.\end{aligned}\tag{12.2.33}$$

由此即可推出

$$\limsup_{\theta\to 0}\limsup_{n\to\infty}\sum_{j\in G_n}B_{jn}=0. \tag{12.2.34}$$

在 $\mu<r_2/2$ 情形,(12.2.34)对 $\beta=0$ 也成立.

写

$$\Delta:=\Delta_{jn}=\frac{1}{4}\frac{\varepsilon_j}{n^{1/2}}=\frac{\varepsilon}{4}\frac{q(\theta 2^{-j})}{n^{1/2}}. \tag{12.2.35}$$

当 $j\in H_n$ 时,再次应用(12.2.32)我们得

$$\begin{aligned}B_{jn}\leqslant{}&P\{\max_{1\leqslant i\leqslant\theta 2^{-j+1}/\Delta}|\alpha_n(i\Delta)|\geqslant\varepsilon_j/2\}\\&+P\{\max_{0\leqslant i\leqslant\theta 2^{-j+1}/\Delta}\sup_{i\Delta<t\leqslant(i+1)\Delta}|\alpha_n(t)-\alpha_n(i\Delta)|\geqslant\varepsilon_j/2\}\\\leqslant{}&P\{\max_{1\leqslant i\leqslant\theta 2^{-j+1}/\Delta}\alpha_n(i\Delta)|\geqslant\varepsilon_j/2\}\\&+P\{\max_{0\leqslant i\leqslant\theta 2^{-j+1}/\Delta}|\alpha_n((i+1)\Delta)-\alpha_n(i\Delta)|+\Delta n^{1/2}\geqslant\varepsilon_j/2\}\\\leqslant{}&P\{\max_{1\leqslant i\leqslant\theta 2^{-j-1}/\Delta}\alpha_n(i\Delta)|\geqslant\varepsilon_j/2\}\\&+P\{\max_{0\leqslant i\leqslant\theta 2^{-j+1}/\Delta}|\alpha_n((i+1)\Delta)-\alpha_n(i\Delta)|\geqslant\varepsilon_j/4\}\\\leqslant{}&3P\{\max_{1\leqslant i\leqslant\theta 2^{-j+2}/\Delta}|\alpha_n(i\Delta)|\geqslant\varepsilon_j/\delta\}.\end{aligned}\tag{12.2.36}$$

从条件(A1)可知：对任意的 $0\leqslant i<k\leqslant \theta 2^{-j+2}/\Delta$，

$$E|\alpha_n(k\Delta)-\alpha_n(i\Delta)|^p\leqslant c_1(((k-i)\Delta)^{p_1}+n^{-p_2/2}((k-i)\Delta)^{r_1})$$
$$\leqslant c_1(((k-i)\Delta)^{p_1}+n^{-p_2/2}(k-i)\Delta^{r_1}).$$

因此，由 Móricz 定理(引理 4.1.2)，存在只与 $c_1$ 和 $p_1$ 有关的常数 $c$ 使得

$$(12.2.37)\qquad E\max_{0\leqslant i\leqslant \theta 2^{-j+2}/\Delta}|\alpha_n(i\Delta)|^p$$
$$\leqslant C((\theta 2^{-j+2}/\Delta)^{p_1}\Delta^{p_1}$$
$$+n^{-p_2/2}(\theta 2^{-j+2}/\Delta)\Delta^{r_1}\log^p(\theta 2^{-j+2}/\Delta))$$
$$\leqslant C4^{p_1}((\theta 2^{-j})^{p_1}+n^{-p_2/2}\theta 2^{-j}$$
$$\Delta^{r_1-1}\log^p(\theta 2^{-j+2}/\Delta)).$$

因为(A1)中的 $p_2>1-r_1$，所以对一切 $x\geqslant 1$，$l(x):=(\log x)^p/x^{-1+r_1+p_2}\leqslant c$. 这样，从(12.2.5)，(12.2.6)，(12.2.35)，(12.2.37)和关系 $\theta 2^{-j+4}n^{1/2}/\varepsilon_j\geqslant 8$，对 $j\in H_n$ 成立着

$$P\{\max_{0\leqslant i\leqslant \theta 2^{-j+2}/\Delta}|\alpha_n(i\Delta)|\geqslant \varepsilon_j/8\}$$
$$\leqslant c\varepsilon_j^{-p}((\theta 2^{-j})^{p_1}+n^{-p_2/2}\theta 2^{-j}\Delta^{r_1-1}\log^p(\theta 2^{-j+2}/\Delta))$$
$$\leqslant c\varepsilon_j^{-p}((\theta 2^{-j})^{p_1}+\varepsilon_j^{r_1-1}(\theta 2^{-j})n^{(1-r_1-p_2)/2}$$
$$\cdot\log^p(\theta 2^{-j+4}n^{1/2}/\varepsilon_j))$$
$$\leqslant c\varepsilon_j^{-p}((\theta 2^{-j})^{p_1}+\varepsilon_j^{-p_2}(\theta 2^{-j})^{r_1+p_2}$$
$$\cdot l(\theta 2^{-j+4}n^{1/2}/\varepsilon_j))$$
$$\leqslant c(\varepsilon_j^{-p}(\theta 2^{-j})^{p_1}+\varepsilon_j^{-p-p_2}(\theta 2^{-j})^{r_1+p_2})$$
$$\leqslant c\varepsilon^{-p-p_2}(\log(2^j/\theta))^{-p\beta}.$$

这个不等式与(12.2.36)一起就证明了

$$(12.2.38)\qquad \limsup_{\theta\to 0}\limsup_{n\to\infty}\sum_{j\in H_n}B_{jn}=0.$$

当 $\mu<p_1/p$ 时，(12.2.38)对于 $\beta>1/(p+p_2)$ 是成立的. 至此，由(12.2.31)，(12.2.34)和(12.2.38)，(12.2.27)的证明就完成了. 类似地我们能够证明(12.2.28).

由(12.2.3)，对任意的 $0\leqslant s,t\leqslant 1$ 我们有

$$\alpha_n(t)-\alpha_n(s)\Rightarrow Y(t)-Y(s).$$

于是由$(A2)$和 Billingsley(1968)的定理5.3，

$$E(Y(t)-Y(s))^2\leqslant\liminf_{n\to\infty}E(\alpha_n(t)-\alpha_n(s))^2\leqslant c_2|t-s|^{r_2}. \tag{12.2.39}$$

由此并注意到$\{Y(t),0\leqslant t\leqslant 1\}$是 Gauss 过程这一事实，对所有的$0\leqslant s,t\leqslant 1$，我们有

$$E(Y(t)-Y(s))^4\leqslant cE^2(Y(t)-Y(s))^2\leqslant c|t-s|^{2r_2}.$$

直接应用 Billingsley(1968)的定理12.2，我们立即得到(12.2.29)和(12.2.30)。定理12.2.2证毕.

**推论12.2.1的证明**. 首先我们来验证：$\{\Delta_n(t),n\geqslant 1\}$和$\Delta(t)$在$[0,1]$上都是有定义的. 由 Schwarz 不等式，$(A2)$，(12.2.6)和(12.2.8)，对$0\leqslant t\leqslant 1$，

$$\begin{aligned}E\Delta_n^2(t)&=\int_0^t\int_0^t E\alpha_n(u)\,\alpha_n(v)dQ(u)dQ(v)\\&\leqslant\left(\int_0^1 E^{1/2}(\alpha_n(t))^2dQ(t)\right)^2\\&\leqslant 2^{r_2}c_2\left(\int_0^1(t(1-t))^{r_2/2}dQ(t)\right)^2<\infty,\end{aligned}$$

最后的不等号在于以下事实：$\alpha_n(0)=\alpha_n(1)=0$；对$0\leqslant t\leqslant 1/2$，$E(\alpha_n(t))^2\leqslant c_2t^{r_2}$；对$1/2\leqslant t\leqslant 1$，$E(\alpha_n(t))^2\leqslant c_2(1-t)^{r_2}$. 类似地，利用(12.2.39)和$(A2)$我们有

$$\begin{aligned}E\Delta_n^2(t)&=\int_0^t\int_0^t EY(u)Y(v)dQ(u)Q(v)\\&\leqslant\left(\int_0^1 E^{1/2}(Y(t))^2dQ(t)\right)^2\\&\leqslant 2^{r_2}c_2\left(\int_0^1(t(1-t))^{r_2/2}dQ(t)\right)^2<\infty.\end{aligned}$$

这就证明了$\{\Delta_n(t),0\leqslant t\leqslant 1;n\geqslant 1\}$和$\{\Delta(t),0\leqslant t\leqslant 1\}$都是平方可积的. 现在对任意的$\theta>0$我们有

$$\sup_{0<t\leqslant\theta}|\Delta_n(t)|\leqslant\sup_{0<t\leqslant\theta}|\alpha_n(t)/q^*(t)|\int_0^1 q^*(t)dQ(t),$$

和

$$\sup_{1-\theta\leqslant t<1}|\Delta_n(1)-\Delta_n(t)|\leqslant\sup_{1-\theta\leqslant t<1}|\alpha_n(t)/q^*(t)|\int_0^1 q^*(t)dQ(t),$$

其中 $q^*(t)=(t(1-t))^\alpha(\log 1/(t(1-t)))^\beta$. 因此(12.2.27),(12.2.28)和(12.2.8)推出对任意的 $\epsilon>0$

(12.2.40)
$$\lim_{\theta\to 0}\limsup_{n\to\infty}P\{\sup_{0<t\leqslant\theta}|\Delta_n(t)|\geqslant\epsilon\}$$
$$=\lim_{\theta\to 0}\limsup_{n\to\infty}P\{\sup_{1-\theta\leqslant t<1}|\Delta_n(1)-\Delta_n(t)|\geqslant\epsilon\}=0.$$

类似地,从(12.2.29),(12.2.30)和(12.2.8)推出:对任意的 $\epsilon>0$

(12.2.41)
$$\lim_{\theta\to 0}P\{\sup_{0<t\leqslant\theta}|\Delta(t)|\geqslant\epsilon\}$$
$$=\lim_{\theta\to 0}P\{\sup_{1-\theta\leqslant t<1}|\Delta(1)-\Delta(t)|\geqslant\epsilon\}=0.$$

因此推论12.2.1从定理12.2.2和 Billingsley(1968)的定理4.2得到.

**定理12.2.3的证明.** 令 $\theta$ 和 $\epsilon$ 如(12.2.9)中定义.因为 $\theta\geqslant 1+\sqrt{2}$,我们可取引理12.2.2中的 $r=\infty$,$v$ 和 $p$ 使得

(12.2.42)
$$\frac{2(\theta+\epsilon)}{\theta+\epsilon-1}<v<\frac{2\theta}{\theta-1},v<\theta+1<p<\theta-1+\epsilon.$$

因此,由(12.2.16)和(12.2.18),对 $0<\eta<(p-1-\theta)/\theta$,存在 $K<\infty$,使对任意的 $0\leqslant s,t\leqslant 1$

$$E\left|\sum_{i=1}^{n}(I(U_i\leqslant t)-I(U_i\leqslant s)-(t-s))\right|^p$$
$$\leqslant K(n^{p/2}|t-s|^{p/v}+n^{1+\eta/2})$$

和

$$E\left|\sum_{i=1}^{n}(I(U_i\leqslant t)-I(U_i\leqslant s)-(t-s))\right|^2\leqslant Kn|t-s|^{2/v}.$$

它们推出

$$E|\alpha_n(t)-\alpha_n(s)|^p\leqslant K(|t-s|^{p/v}+n^{-(p-2-\eta)/2})$$

和

$$E|\alpha_n(t)-\alpha_n(s)|^2\leqslant K|t-s|^{2/v}.$$

因此条件(A1)和(A2)对于 $p_1=p/v>1$,$p_2=p-2-\eta>1$,$r_1=0$和 $r_2=2/v$ 成立.注意到 $0<\eta<(p-1-\theta)/\theta$,从(12.2.42)容易看出

(12.2.43)
$$\min\left(\frac{p_1}{p},\frac{r_1+p_2}{p+p_2},\frac{r_2}{2}\right)\geqslant\min\left(\frac{1}{v},\frac{p-2-\eta}{2p-2}\right)$$

$$>\frac{1}{2}\left(1-\frac{1}{\theta}\right).$$

由定理12.1.2,(12.2.3)成立.从定理12.2.2得证定理12.2.3.

**定理12.2.4的证明.** 从推论12.1.1可知(12.2.3)成立.由邵启满(1995)的定理1.1,对 $p\geqslant 2$我们有

$$\begin{aligned}
&E\left|\sum_{i=1}^{n}(I(U_i\leqslant t)-I(U_i\leqslant s)-(t-s))\right|^p\\
&\leqslant c\Big\{n^{p/2}\exp\Big\{c\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}(E(I(U_1\leqslant t)\\
&\quad -I(U_1\leqslant s)-(t-s))^2)^{p/2}\\
&\quad +n\exp\Big(k\sum_{i=0}^{[\log n]}\rho^{2/p}(2^i)\Big)E\{I(U_1\leqslant t)\\
&\quad -I(U_1\leqslant s)-(t-s)|^p\Big)\\
&\leqslant c\Big\{n^{p/2}\exp\Big\{c\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}|t-s|^{p/2}\\
&\quad +n\exp\Big(c\sum_{i=0}^{[\log n]}\rho^{2/p}(2^i)\Big)|t-s|\Big\},
\end{aligned}$$

显然,在条件(12.2.10)之下,对 $p\geqslant 2$和任意的$0<\delta<\min\{\varepsilon(p-1)/(2(2+\varepsilon)),(p-2)/2\}$,我们有

$$\begin{aligned}
&E\left|\sum_{i=1}^{n}(I(U_i\leqslant t)-I(U_i\leqslant s)-(t-s))\right|^p\\
&\leqslant c\Big\{n^{p/2}|t-s|^{p/2}+n\exp\Big(c\sum_{i=0}^{[\log n]}\rho^{2/p}(2^i)\Big)|t-s|\Big\}\\
&\leqslant c(n^{p/2}|t-s|^{p/2}+n^{1+\delta}|t-s|),
\end{aligned}$$

这里用到了 $\exp\Big(c\sum_{i=0}^{[\log n]}\rho^{2/p}(2^i)\Big)$为 $n$ 的缓变函数这一事实.于是,对于 $p>2$

$$E|\alpha_n(t)-\alpha_n(s)|^p\leqslant c(|t-s|^{p/2}+n^{-(p-2-2\delta)/2}|t-s|);$$

对于 $p=2$

$$E|\alpha_n(t)-\alpha_n(s)|^2\leqslant c|t-s|.$$

因此条件$(A1)$和$(A2)$对于 $p_1=p/2,p_2=p-2-2\delta>0$和 $r_1=r_2=1$是被满足的.从而由(12.2.6)

(12.2.44)　$\mu=(1+p-2-2\delta)/(p+p-2-2\delta)>1/(2+\varepsilon)$.

另一方面，在条件(12.2.11)下，对 $p\geqslant 2$ 有

$$E\left|\sum_{i=1}^{n}(I(U_i\leqslant t)-I(U_i\leqslant s)-(t-s))\right|^p$$
$$\leqslant c(n^{p/2}|t-s|^{p/2}+n|t-s|).$$

因而

$$E|\alpha_n(t)-\alpha_n(s)|^p\leqslant c(|t-s|^{p/2}+n^{-(p-2)/2}|t-s|).$$

所以条件(A1)和(A2)对于 $p_1=p/2$，$p_2=p-2$ 和 $r_1=r_2=1$ 是被满足的. 显然 $\mu=1/2$. 定理 12.2.4证毕.

## §12.3　强逼近

这一节中我们将介绍有关由相依样本产生的经验过程的强逼近理论. 着重介绍 philipp(1982)关于 $\alpha$ 混合样本的两个结果. Strittmatter(1990)曾对绝对正则样本研究过这类问题，限于篇幅，不准备提及了.

为了讨论相依样本产生的经验过程的强逼近，我们首先给出一个取值于 Banach 空间的随机元序列的强逼近结果. 设 $\{x_j, j\geqslant 1\}$ 是一个定义在概率空间 $(\Omega,\sum,P)$ 上、取值于可测空间 $(A,\mathscr{A})$ 的强平稳 $\alpha$ 混合随机元序列，又设 $(S,\|\cdot\|)$ 是一 Banach 空间，$h$ 是一个 $A\to S$ 的映射. 我们称 $\{X_j:=h(x_j), j\geqslant 1\}$ 是一个 $S$ 值的强平稳 $\alpha$ 混合的随机元序列.

**定理12.3.1**　设 $\{X_j, j\geqslant 1\}$ 是一个如上定义的 $S$ 值强平稳 $\alpha$ 混合的随机元序列. 存在正随机变量 $\xi$，$E\xi^{2+\delta}<\infty$（某个 $0<\delta\leqslant 1$），满足 $\|x_1\|\leqslant\xi$. 又设存在 $\rho>0$，使得

(12.3.1)　　$\alpha(n)\leqslant cn^{-(2+\rho)(1+2/\delta)}$.

再设对每一 $m\geqslant 1$，存在线性映射 $\Lambda_m:S\to S$，具有下列性质

(12.3.2)　　$\sup\limits_{m\geqslant 1}\|\Lambda_m\|<\infty$,

(12.3.3)　　存在 $c>0$，$\dim\Lambda_m S\leqslant cm$.

又设存在 $\theta>0$，使得对每一 $m\geqslant 1$，有 $n_0(m)\leqslant c\exp(m^{1-\theta})$ 和不增

函数 $g(m)\geqslant m^{-1/2}$ 使得对任一 $n\geqslant n_0$ 成立着

(12.3.4) $$P^*\left\{n^{-1/2}\left\|\sum_{j\leq n}(X_j-\wedge_m X_j)\right\|\geq g(m)\right\}\leqslant m^{-1-\vartheta}.$$

其中 $P^*$ 是 $P$ 的外测度. 此外,对每一 $m\geqslant 1$,映射 $\wedge_m\circ h$ 是从 $(A,\mathscr{A})$ 到 $L_mS$(由 $\wedge_m S$ 生成的线性空间)的一个可测函数,而且

(12.3.5) $$E\wedge_m X_1=0,\ E\|\wedge_m X_1\|^{2+\delta}<\infty.$$

令 $T$ 是 $\bigcup_{m\geq 1}\wedge_m S$ 生成的线性空间的完备化,因此它是一可分的 Banach 空间. 那么,存在一个定义在 $\left(\Omega,\sum,P\right)$ 上的 $i.i.d.$ 的 $T$ 值 Gauss 随机变量序列 $\{Y_j,j\geqslant 1\}$, $EY_1=0$,而且对任意的 $s,t\in T'$($T$ 上的有界线性泛函空间),$Y_1$ 的协方差结构如下:

(12.3.6) $$\begin{aligned}Es(Y_1)t(Y_1)=&\lim_{m\to\infty}E\{s(\wedge_m X_1)t(\wedge_m X_1)\}\\&+\lim_{m\to\infty}\sum_{j\geq 2}E\{s(\wedge_m X_1)t(\wedge_m X_j)\}\\&+\lim_{m\to\infty}\sum_{j\geq 2}E\{s(\wedge_m X_j)t(\wedge_m X_1)\},\end{aligned}$$

其中的极限存在且级数绝对收敛. 进一步,存在随机变量 $V_n$ 和正常数 $a$,对任意的 $1-\theta/4\leqslant\beta<1$,成立着如下的逼近界:

(12.3.7) $$\begin{aligned}\left\|\sum_{j\leq n}(X_j-Y_j)\right\|\leqslant V_n=&O(n^{1/2}\{(\log n)^{-(1-\beta)/4}\\&+(\log n)^{(1-\beta)\cdot\beta}g(a(\log n)^{1-\theta/2})\})\quad \text{a.s.}\end{aligned}$$

这个定理的证明不在这里给出了.

**定理12.3.2** 设 $\mathscr{C}\subset\mathscr{A}$ 是一个可测集类,满足

(12.3.8) $$N(\delta,\mathscr{C})\leqslant c\delta^{-\tau}\qquad 某个\ \tau>1/6.$$

又设 $\{x_n,n\geqslant 1\}$ 是强平稳 $\alpha$ 混合 $A$-值随机元序列,满足

(12.3.9) $$\alpha(n)\leqslant cn^{-2\tau-4}.$$

记 $g_n(C)=I(x_n\in C)-P(C),C\in\mathscr{C}$. 那么存在定义在同一概率空间 $\left(\Omega,\sum,p\right)$ 上、指标为 $C\in\mathscr{C}$ 的 $i.i.d.$ Gauss 过程序列 $\{Y_j,j\geqslant 1\}$,满足 $EY_1(C)=0$,

(12.3.10) $$\begin{aligned}E(Y_1(C)Y_1(D))=&Eg_1(C)g_1(D)+\sum_{n\geq 2}Eg_1(C)g_n(D)\\&+\sum_{n\geq 2}Eg_n(C)g_1(D),C,D\in\mathscr{C}\end{aligned}$$

使得对某个可测的 $V_n$ 和 $\lambda>0$，以概率1成立

(12.3.11) $\sup_{C\in\mathscr{C}}\left|\sum_{j\leqslant n}(I(x_j\in C)-P(C)-Y_j(C))\right|\leqslant V_n=O(n^{1/2}$

$(\log n)^{-\lambda})$.

为了证明定理12.3.2，先进行一系列准备工作.

设 $\{\xi_n\}$ 是强平稳 $\alpha$ 混合的实随机变量序列，$E\xi_1=0$，$|\xi_1|\leqslant 1$ 且 $\alpha(n)\leqslant cn^{-\rho}$，其中 $c>0$，$\rho>2$. 我们来建立一个 $\{\xi_n\}$ 的部分和的指数界. 记 $p=[n^{1/2-\kappa}]$（某个 $0<\kappa<1/2$），$q=[n/(2p)]+1$. 定义长度为 $p$ 的整数区组 $H_j$ 和 $I_j$，$1\leqslant j<q$，又定义 $H_q$ 由 $n-2p(q-1)$ 个整数组成（因此 $\mathrm{Card}H_q\leqslant 2p$），要求这些区组的顺序是 $H_1,I_1,\cdots,H_{q-1},I_{q-1},H_q$，且相继区组之间不留间隙. 记

$$y_j=\sum_{i\in H_j}\xi_i,\ z_j=\sum_{i\in I_j}\xi_i.$$

容易看出

$$\sigma^2:=E\xi_1^2+2\sum_{n=2}^{\infty}E\xi_1\xi_n<\infty.$$

且对充分大的 $n$

(12.3.12) $$Ey_j^2<\begin{cases}2\sigma^2p, & 1\leqslant j<q,\\ 5\sigma^2p, & j=q.\end{cases}$$

记 $\mathscr{L}_j=\sigma(y_1,\cdots,y_j)$，写

$$y_j=Y_j+v_j,\ 1\leqslant j\leqslant q,$$

其中 $v_j=E(y_j|\mathscr{L}_{j-1})$. 显然 $(Y_j,\mathscr{L}_j,1\leqslant j\leqslant q)$ 是鞅差序列且由引理1.2.1，

(12.3.13) $\|v_j\|_2\leqslant 2\|y_j\|_\infty\alpha^{1/2}(p)\leqslant cp^{1-\rho/2}$.

**引理12.3.1** 令 $A\geqslant\sigma^2$. 我们有

$$P\Big\{\sum_{j\leqslant q}E(Y_j^2|\mathscr{L}_{j-1})\geqslant 4An\Big\}\leqslant cA^{-2}p^{2-\rho}.$$

**证** 由 Hölder 不等式有

(12.3.14) $E(Y_j^2|\mathscr{L}_{j-1})\leqslant E(Y_j^2|\mathscr{L}_{j-1})$.

利用引理1.2.1可得

$$\|E(y_j^2|\mathscr{L}_{j-1})-Ey_j^2\|_2\leqslant 4\|y_j\|_\infty^2\alpha^{1/2}(p)\leqslant 4p^{2-\rho/2}.$$

因此由 Minkowski 不等式和 Chebyshev 不等式，我们有

$$P\Big\{\sum_{j\leqslant q}(E(Y_j^2|\mathscr{L}_{j-1})-EY_j^2)\geqslant An\Big\}$$
$$\leqslant 16n^{-2}A^{-2}(qp^{2-\rho/2})^2\leqslant cA^{-2}p^{2-\rho}.$$

从(12.3.12)和(12.3.14)，引理中的概率不超过

$$P\Big\{\sum_{j\leqslant q}E(Y_j^2|\mathscr{L}_{j-1})\geqslant\sum_{j\leqslant q}Ey_j^2+An\Big\}\leqslant cA^{-2}p^{2-\rho}.$$

证毕.

**引理12.3.2** 对任意的 $R>0$，

$$P\Big\{\Big|\sum_{j\leqslant q}Y_j\Big|>5Rn^{1/2}\Big\}\leqslant c\{\exp(-R^2/A)+A^{-2}p^{2-\rho}\}.$$

**证** 令 $M$ 是 $H_j$ 或 $I_j$ 中包含 $n$ 的那个足标 $j$. 定义

$$U_k=\begin{cases}\sum_{j=1}^{k}Y_j, & 若\ k\leqslant M,\\ U_M, & 若\ k>M;\end{cases}$$

$$s_k^2=\begin{cases}\sum_{j=1}^{k}E(Y_j^2|\mathscr{L}_{j-1}), & 若\ k\leqslant M,\\ s_M^2, & 若\ k>M.\end{cases}$$

显然 $U_j-U_{j-1}=Y_j\leqslant 4p=:C$. 如果 $R\leqslant An^{1/2}/p$，就记 $\lambda=R/(4An^{1/2})$，$K=20An$. 因此 $\lambda C\leqslant 1$. 又记

$$T_k=\exp\Big\{\lambda U_k-\frac{1}{2}\lambda^2\Big(1+\frac{1}{2}\lambda C\Big)s_k^2\Big\}.$$

由 Stout(1974)的引理5.4.1和推论5.4.1，再应用引理12.3.1，我们有

$$P\{\sup_{k\geqslant 0}U_k>5Rn^{1/2}\}$$
$$\leqslant P\Big\{\sup_{k\geqslant 0}T_k>\exp\Big(\lambda^2K-\frac{1}{2}\lambda^2\Big(1+\frac{1}{2}\lambda C\Big)s_q^2\Big)\Big\}$$
$$\leqslant c\{\exp(-R^2/A)+A^{-2}p^{2-\rho}\}.$$

如果 $R>An^{1/2}/p$，记 $\lambda=1/(4p)$，$K=20Rpn^{1/2}$. 则有 $\lambda C=1$ 且如刚才同样的理由又有

$$P\{\sup_{k\geqslant 0}U_k>5Rn^{1/2}\}$$
$$\leqslant c\exp\Big(\frac{-20Rpn^{1/2}+3An}{16p^2}\Big)+A^{-2}p^{2-\rho}$$

$$\leqslant cA^{-2}p^{2-\rho}.$$

这就完成了引理12.3.2的证明.

**引理12.3.3** 成立着

$$P\left\{\left|\sum_{j=1}^{q}y_j\right| \geqslant 7Rn^{1/2}\right\} \leqslant c(\exp(-R^2/A) + n^{1+\rho\kappa-\rho/2}(A^{-2}+R^{-2})).$$

**证** 由 Chebyshev 不等式和 Minkowshi 不等式,从(12.3.13)得

$$P\left\{\left|\sum_{j=1}^{q}v_j\right| \geqslant Rn^{1/2}\right\} \leqslant cR^{-2}n^{-1}(qp^{1-\rho/2})^2 \leqslant cR^{-2}np^{-\rho}.$$

将它与引理12.3.2相结合并注意到

$$\left|\sum_{j=1}^{q}y_j\right| \leqslant \left|\sum_{j=1}^{q}Y_j\right| - \left|\sum_{j=1}^{q}v_j\right|$$

即得引理的结论.

如果用 $z_j$ 代替 $y_j$, $1\leqslant j<q$,并记 $z_q=0$,那么引理12.3.3仍然成立.于是我们得到$\{\xi_n, n\geqslant 1\}$的部分和的概率的指数界:

(12.3.15)
$$P\left\{\left|\sum_{k=1}^{n}\xi_k\right| > 14Rn^{1/2}\right\} \leqslant c(\exp(-R^2/A) + n^{1+\rho\kappa-\rho/2}(A^{-2}+R^{-2})).$$

**引理12.3.4** 如果定理12.3.2的假设被满足,那么对任意给定的 $\varepsilon>0$,存在$0<\delta<c\varepsilon^t$ 和 $n_0\leqslant c\exp(-1/(4\varepsilon))$,使得对一切 $n\geqslant n_0$

$$P^*\{\sup(|\nu_n(C)-\nu_n(D)|: C,D\in\mathscr{C}, P_x(C\Delta D)<\delta)>\varepsilon\} \leqslant c\exp(-1/(8\varepsilon)).$$

其中

$$\nu_n = n^{1/2}(P_n - P_x), P_n(B) = n^{-1}\sum_{j=1}^{n}I(x_j\in B),$$

$$P_x(B) = P(x_1\in B), B\in\mathscr{A}.$$

**证** 令 $r$ 足够大使得

(12.3.16)
$$2^r > \varepsilon^{-6}.$$

记 $\delta_k = 2^{-r-k}(k=0,1,2,\cdots)$, $m_k = N(\delta_k, \mathscr{C}, P_x)$, $d_i = (i+1)^{-2}\varepsilon/$

32. 令 $N(\delta_k,\mathscr{C},P_x)$的定义中的集合为 $A_{k1},\cdots,A_{km(k)}$. 因此,对每一 $c\in\mathscr{C}$ 和 $k=0,1,\cdots$,存在 $r(k)=r(k,c)$和 $s(k)=s(k,c)$使得 $A_{kr(k)}\subset c\subset A_{ks(k)}$且 $P_x(A_{ks(k)})\backslash A_{kr(k)})<\delta_k$. 记

$$B_k := B_k(c) = A_{kr(k)}\backslash A_{k+1,r(k+1)},$$

$$D_k := D_k(c) = AA_{k+1,s(k+1)}\backslash A_{ks(k)}.$$

则有 $P_x(B_k)<\delta_k$ 和 $P_x(D_k)<\delta_{k+1}<\delta_k$.

记 $n_0:=n_0(\varepsilon)=\varepsilon^2/(256\delta_0^2)$. 对每一$n>n_0$存在唯一的 $k=k(n)$使得

$$(12.3.17)\qquad 1/2<8\delta_k n^{1/2}/\varepsilon\leqslant 1.$$

于是对 $n>n_0,k=k(n),\delta=\delta_k$ 和 $c\in\mathscr{C},r=r(k,c),s=s(k,c)$. 我们有

$$(12.3.18)\qquad \nu_n(A_{kr})-\varepsilon/8\leqslant\nu_n(A_{kr})-\delta n^{1/2}\leqslant\nu_n(c)\leqslant\nu_n(A_{ks})+\varepsilon/8.$$

由此即得

$$(12.3.19)\qquad |\nu_n(A_{ks(k)})-\nu_n(A_{0s(0)})|\leqslant\sum_{i=0}^{k-1}|\nu_n(A_{is(i)})-\nu_n(A_{i-1,s(i+1)})|\leqslant\sum_{i=0}^{k-1}(|\nu_n(B_i)|+|\nu_n(D_i)|).$$

令 $\mathscr{B}_i$ 是满足 $P_x(B)<\delta_i$ 的集 $B=A_{is}\backslash A_{i+1,t}$或 $B=A_{i+1,t}\backslash A_{is}$构成的类. 那么对每一 $c\in\mathscr{C},B_i(c)$和 $D_i(c)\in\mathscr{B}_i$. $\mathscr{B}_i$ 中的集合的个数

$$\mathrm{Card}(\mathscr{B}_i)\leqslant 2m(i)m(i+1).$$

先来估计 $P\{|\nu_n(B)|>d_i\}$,其中 $B\in\mathscr{B}_i$. 记 $\xi_n=I(x_n\in B)-P_x(B)$.

因为

$$|E\xi_1\xi_n|\leqslant\|\xi_1\|_4\|\xi_n\|_4(\alpha(n-1))^{1/2}\leqslant P_x^{1/2}(B)n^{-2-\tau},$$

故有

$$\sigma^2=E\xi_1^2+2\sum_{n=2}^{\infty}E\xi_1\xi_n\leqslant cP_x^{1/2}(B).$$

因此我们能够取 $\delta_i^{1/2}\geqslant\sigma^2$. 从(12.3.15)即得

$$P\{|\nu_n(B)|>d_i\}\leqslant c\left(\exp\left\{-\frac{d_i^2}{196}\delta_i^{-1/2}\right\}\right.$$

$$\left.+n^{-\tau-1+\kappa(2\tau+2)}(\delta_i^{-1}+i^2)\right).$$

取 $\kappa=1/(8\tau-16)$并利用(12.3.8)我们有

$$p_i:=P\{|\nu_n(B)|>d_i,\text{某个 } B\in\mathscr{B}_i\}$$

$$\leqslant c\exp\left((-\varepsilon^2/(\delta_i^{1/2}196\cdot 32^2(i+1)^4))\delta_i^{-2\tau}\right.$$

$$\left.+n^{-\tau-3/4}\delta_i^{-1-2\tau}\right).$$

由此并取 $r$ 足够大，再利用(12.3.16)和(12.3.17)，对 $n\geqslant n_0$得到

$$\sum_{i=0}^{k}p_i\leqslant c\left(\exp\left(-\frac{1}{\varepsilon}\right)+n^{-1/4}\varepsilon^{-(1+2\tau)}\right)$$

$$\leqslant c\exp(-1/(8\varepsilon)).$$

记 $Q_n=P(V_n>\varepsilon/8)$，其中

$$V_n=\sup\{|\nu_n(A_{kr})-\nu_n(A_{ks})|:A_{kr}\subset A_{ks},$$

$$P(A_{ks}\backslash A_{kr})<\delta_k,r,s=1,\cdots,m_k\}.$$

从(12.3.15)和(12.3.17)，对 $n\geqslant n_0$有

$$Q_n\leqslant c\delta_k^{-2\tau}\exp\left(-\frac{\varepsilon^2}{64}\delta_k^{-1/2}\right)+n^{-\tau-3/4}\delta_k^{-1-2\tau}$$

$$\leqslant c\exp(-1/(8\varepsilon)).$$

再次利用(12.3.16)又有

$$p_0:=P\{\sup(|\nu_n(A_{0i})-\nu_n(A_{0j})|:P(A_{0i}\Delta A_{0j})<3\delta_0)>\varepsilon/4\}$$

$$\leqslant c\delta_0^{-2\tau}\exp\left(-\frac{\varepsilon^2}{16}\delta_0^{-1/2}\right)+n^{-1-\tau+\rho\kappa}\delta_0^{-1-2\tau}$$

$$\leqslant c\exp(-1/(8\varepsilon)).$$

引理证毕.

**定理12.3.2的证明.**

令 $S$ 是 $\mathscr{C}$ 上的所有有界实值函数组成的空间，对 $f\in S$，记 $\|f\|=\sup\limits_{C\in\mathscr{C}}|f(C)|$. 如果 $x\in A$，记 $h(x)=I(x\in C)-P(C)$. 因此 $h$: $A\to S$. 令 $m\geqslant 1$并记 $\varepsilon=m^{-1/(6\tau)}$，$\delta$ 和 $n_0$根据引理12.3.4确定，于是 $\delta\leqslant cm^{-1/\varepsilon}$而 $n_0\leqslant c\exp(m^{1/(6\tau)}/4)$. 令 $A_1,\cdots,A_d(d=N(\delta))$是这样的一些集合，使得对任意的 $C\in\mathscr{C}$，存在 $A_r$ 满足 $P(A_r\Delta C)<\delta$ 且 $r$

是最小的. 据所设 $N(\delta)\leqslant C\delta^{-r}\leqslant cm$. 如果 $h(x)=I(x\in C)-p(C)$ 且 $p(A_r\Delta C)<\delta$, 我们通过关系 $\Lambda_m h(x)=I(x\in A_r)-p(A_r)$ 定义 $\Lambda_m: S\to S$. 则有 $\dim\Lambda_m S\leqslant cm$. 从引理12.3.4(取 $D=A_r$)可知(12.3.4)对于 $g(m)=m^{-1/(6r)}$ 和 $\theta=1-1/(6r)$ 是成立的. 至此, 通过选取 $\beta(1-\theta/4\leqslant\beta<1)$ 充分接近于1, 从定理12.3.1即得证定理12.3.2.

从这个定理我们立即可以得到下列定理.

**定理12.3.3** 设 $\{X_n, n\geqslant 1\}$ 是强平稳 $\alpha$ 混合的 $d$ 维随机向量序列, 具有共同的分布函数 $F(x)$, 且

(12.3.20) $\qquad \alpha(n)=O(n^{-4-2d}).$

对 $s\in R^d$, 记 $g_n(s)=I(X_n\leqslant s)-F(s)$. 那么下面定义的协方差函数

$$\Gamma(s,s')=Eg_1(s)g_1(s')+\sum_{n=2}^{\infty}E(g_1(s)g_n(s')+g_n(s)g_1(s')),\ s,s'\in R^d,$$

绝对收敛; 而且存在一个定义在同一概率空间 $\{\Omega,\sum,P\}$ 上指标为 $s\in R^d$ 的 $i.i.d.$ Gauss 过程序列 $\{Y_j, j\geqslant 1\}$, 满足

$$EY_1(s)=0,\ EY_1(s)Y_1(s')=\Gamma(s,s'),\ s,s'\in R^d,$$

同时存在一个常数 $\lambda=\lambda(d)>0$, 以概率1成立

$$\sup_{s\in R^d}\left|\sum_{j=1}^{\infty}(I(X_j\leqslant s)-F(s)-Y_j(s))\right|=O(n^{1/2}(\log n)^{-\lambda}).$$

**证** 记 $P$ 是由 $F$ 诱导出的概率测度, $F_i(s_i)$, $1\leqslant i\leqslant d$, 是 $F(s)$, $s=(s_1,\cdots,s_d)$, 的第 $i$ 个边际分布. 给定 $r\geqslant 1$, 定义

$$s_{ij}=\mathrm{inv}F_i(j2^{-r}),\ 1\leqslant i\leqslant d,\ 0\leqslant j\leqslant 2^r.$$

令 $\mathscr{C}$ 是区间 $C=(-\infty,s)$, $s\in R^d$, 组成的集合. 对任意的 $C\in\mathscr{C}$, 存在着形为 $(-\infty,(s_{1j_1},\cdots s_{dj_d}))$ 的两个区间 $A_p$ 和 $A_q$ 使得

$$A_p\subset C\subset A_q \text{ 且 } p(A_q\backslash A_p)\leqslant d2^{-r}.$$

具有上述形式的区间的个数不超过 $2^{dr}$ 个, 也就是说 $N(d2^{-r},\mathscr{C},p)\leqslant 2^{dr}$. 因此当取 $\tau=d$ 且 $c=(2d)^d$ 时, (12.3.8)是被满足的. 这样, 从(12.3.20)即知(12.3.10)成立.

**注12.3.1** Philipp 和 Pinzur(1980)给出过一个多元经验过程

用一个 Kiefer 过程 a.s. 逼近的结果. 设$\{X_n, n\geq 1\}$是一个 $R^q$ 上的强平稳 $\alpha$ 混合随机向量序列，具有连续分布 $F$ 且对某个$0\leqslant\varepsilon\leqslant 1/4$

$$\alpha(n) = O(n^{-4-q(1-\varepsilon)}).$$

定义$\{X_n\}$的经验过程

$$R(s,t) = [t](F_{[t]}(s) - F(s)), t\geqslant 0, s\in R^q.$$

记 $\Gamma(s,s') = E\{g_1(s)g_1(s')\} + \sum_{n=2}^{\infty} E\{g_1(s)g_n(s') + g_n(s)g_1(s')\}$，$s,s'\in R^q$，其中 $g_n(s) = I(X_n\leqslant s) - F(s)$. 那么，不改变原来的分布，在一个较大的概率空间上可以重新定义经验过程$\{R(s,t), s\in R^q, t\geqslant 0\}$，而在这个新的概率空间上存在一个 Kiefer 过程$\{K(s,t), s\in R^q, t\geqslant 0\}$，其协方差函数为$(t\wedge t')\Gamma(s,s')$，又存在一个常数 $\lambda = \lambda(q,\varepsilon)$，使得

(12.3.21) $\quad \sup_{t\leqslant T}\sup_{s\in R^q}|R(s,t) - K(s,t)| = O(T^{1/2}(\log T)^{-\lambda})$ a.s.

当 $q=1$ 且$\{X_n\}$是$[0,1]$上的均匀分布时，(12.3.21)重合于 Yoshihara (1979)的结果，但是现在对于混合速度的要求是放宽了. 由关于 Kiefer 过程的 Strassen 型重对数律(参见 Csörgö 和 Révész 1981，定理1.15.1)和(12.3.21)即可推出关于$\{\eta_s(t) = R(s,t)/\sqrt{2t\log\log t}, 0\leqslant s\leqslant 1\}$的 Strassen 型重对数律.

## §12.4 经验过程的连续模

设$\{X_n, n\geqslant 1\}$是一个具有相同分布 $F(x)$的随机变量序列，$\{\beta_n(t), -\infty<t<\infty\}$，$n=1,2,\cdots$，是它的经验过程序列，经验过程的连续模定义为

$$w_n(a_n) = \sup_{\substack{|t-s|\leqslant a_n \\ -\infty<s<t<\infty}} |\beta_n(t) - \beta_n(s)|.$$

Stute(1982)对 $i.i.d.$ 情形证明了下列定理.

**定理12.4.1** 假设

(i) $0<a_n<1$，$a_n\downarrow$ 且 $na_n\uparrow\infty$，

(ii) $na_n/\log n\to\infty$，

(iii) $\log a_n^{-1}/\log\log n \to 0$.

那么

(12.4.1) $$\lim_{n\to\infty}(a_n \log a_n^{-1})^{-1/2} w_n(a_n) = 1 \quad \text{a.s.}$$

**定义12.4.1** 设$0<\lambda\leqslant 1$，$A\subset R$. 函数 $g(x)$被称为是满足 $A$ 上的一致局部λ阶 Lipschtz（λ-ulL）条件，如果存在$\delta>0$，$M<\infty$使得

(12.4.2) $$\sup_{x\in A}|g(x+z)-g(x)|\leqslant M|z|^{\lambda}, |z|\leqslant\delta.$$

周勇(1994)讨论了样本是$\varphi$混合或$\alpha$混合时的经验过程的连续模，证明了下列定理.

**定理12.4.2** 设$\{X_n, n\geqslant 1\}$是强平稳$\varphi$混合的随机变量序列，具有共同的分布函数 $F(x)$. 假设 $F(x)$满足1-ulL 条件且 $\sum_{i=1}^{\infty}\varphi^{1/2}(2^i)<\infty$. 如果存在一个正整数序列$\{m_n\}$使得

$$1\leqslant m_n\leqslant n, \frac{n}{m_n}\varphi(m_n)\leqslant A \text{ 且} \left(\frac{\log n}{na_n}\right)^{1/2} m_n\leqslant c,$$

其中 $A$ 和 $C$ 是两个常数，那么

(12.4.3) $$w_n(a_n) = O((a_n \log n)^{1/2}) \quad \text{a.s.}$$

**定理12.4.3** 设$\{X_n, n\geqslant 1\}$是强平稳$\alpha$混合的随机变量序列，具有共同的分布函数 $F(x)$. 假设 $F(x)$满足1-ulL 条件且 $\alpha(n)=O(\rho^n)$，其中$0<\rho<1$. 又设 $a_n\to O(n\to\infty)$. 那么对任意的$0<\theta<1$

(12.4.4) $$w_n(a_n) = O(a_n^{\frac{1-\theta}{2}}\log^2 n) \quad \text{a.s.}$$

**注12.4.1** 如果 $F(x)$满足 $A(\subset R)$上的 λ-ulL 条件，那么(12.4.3)和(12.4.4)将分别被改写为

(12.4.5) $$w_n(a_n, A) := \sup_{t,s\in A, |t-s|\leqslant a_n}|\beta_n(t)-\beta_n(s)| = O((a_n^{\lambda}\log n)^{1/2}) \text{a.s.}$$

和

(12.4.6) $$w_n(a_n, A) = O(a_n^{\frac{1-\theta}{2}\lambda}\log^2 n) \quad \text{a.s.}$$

**注12.4.2** 周勇(1994)由于直接应用 Collomb(1984)的引理，所以要求条件 $\sum_{n=1}^{\infty}\varphi(n)<\infty$. 但是 Collomb 的引理可以被改进

(见下面的引理12.4.1),因此只需要 $\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$就够了.

首先给出下面的 Bernstein 型不等式.

**引理12.4.1** 设$\{X_n,n\geqslant 1\}$是 $\varphi$混合序列,$EX_n=0$,$|X_n|\leqslant d$,$EX_n^2\leqslant D$ 且 $\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$. 那么存在 $c_1=c_1(\varphi(\cdot))>0$使得

$$P\left\{\left|\sum_{i=1}^{n}X_i\right|>\varepsilon\right\}$$

$$\leqslant\exp\left\{3\sqrt{e}\,n\frac{\varphi(m)}{m}-\alpha\varepsilon+c_1\alpha^2Dn\right\},$$

其中 $\alpha$ 是实数,$m$ 是正整数,满足 $m\leqslant n$ 且 $\alpha md\leqslant 1/4$.

这个引理的证明可以沿着 Collomb 的证明路线进行,只是其中需要用到引理2.2.2代替 Collomb 的证明中利用的引理1.2.10(参见引理11.1.1).

**引理12.4.2**(Doukhan,Leon 和 Portal 1984). 设$\{X_n,n\geqslant 1\}$是 $\alpha$ 混合序列,$EX_n=0$,$|X_n|\leqslant 1$且 $\alpha(n)\leqslant c\rho^n$. 记$\sigma=\sup\|X_n\|_\gamma$,其中$\gamma=2/(1-\theta)$,$0<\theta<1$. 那么存在 $c_1$和 $c_2$(仅与 $\alpha(\cdot)$有关),使得

$$P\left\{\left|\sum_{i=1}^{n}X_i\right|>\varepsilon\right\}\leqslant c_1\theta^{-1}\exp\{-c_{2n}\varepsilon^{1/2}(n^{1/4}\sigma^{1/2})\},$$

其中

$$c_{2n}=\begin{cases}c_2 & 若\ n^{1/2}\sigma\leqslant 1,\\ c_2n^{1/4}\sigma^{1/2} & 若\ n^{1/2}\sigma>1.\end{cases}$$

**定理12.4.2的证明.**

如同在这一章开头曾经提到过的那样,我们只考虑均匀经验过程. 而由1-$ulL$ 条件,只需证明

$$(12.4.7)\qquad \sup_{0\leqslant t,s\leqslant 1;|t-s|\leqslant Ma_n}|\alpha_n(t)-\alpha_n(s)|=O((a_n\log n)^{1/2})\quad \text{a.s.}$$

就够了,其中 $\alpha_n(\cdot)$是$[0,1]$上均匀分布样本产生的经验过程. 不失一般性可设(12.4.2)中的 $M=1$.

将区间$[0,1]$用点 $t_0=0$,$t_j=j/K_n$,$j=1,\cdots,K_n:=[a_n^{-1}\log n]$,分割成 $K_n$ 个子区间. 记

$$V_n(t) = \sup_{|t-s|\leqslant a_n} |\alpha_n(t) - \alpha_n(s)|.$$

对于任意的 $t,s\in[0,1]$，如果它们满足 $|t-s|\leqslant a_n$，那么有下列两种情况：

(i) 如果 $t$ 和 $s$ 落入同一个子区间，也就是说，存在 $j, 0\leqslant j\leqslant K_n-1$，使得 $t,s\in[t_j,t_{j+1}]$. 于是

$$\begin{aligned}|\alpha_n(t) - \alpha_n(s)| &\leqslant |\alpha_n(t) - \alpha_n(t_j)| + |\alpha_n(t_j) - \alpha_n(s)| \\ &\leqslant 2\max_{0\leqslant j\leqslant K_n} V_n(t_j).\end{aligned}$$

(ii) 如果 $t$ 和 $s$ 落入不同的子区间，那么存在 $j$ 和 $r$, $1\leqslant j+1\leqslant r\leqslant K_n$，使得 $s\in[t_j,t_{j+1}]$, $t\in[t_r,t_{r+1}]$. 因为 $|s-t|\leqslant a_n$，所以 $t_r-t_{j+1}\leqslant a_n$. 于是

$$\begin{aligned}&|\alpha_n(t) - \alpha_n(s)| \\ \leqslant &|\alpha_n(t) - \alpha_n(t_r)| + |\alpha_n(t_r) - \alpha_n(t_{j+1})| \\ &+ |\alpha_n(t_{j+1}) - \alpha_n(s)| \leqslant 3\max_{0\leqslant j\leqslant K_n} V_n(t_j)\end{aligned}$$

因此不论如何都有

$$\sup_{|t-s|\leqslant a_n} |\alpha_n(t)-\alpha_n(s)|\leqslant 3\max_{1\leqslant j\leqslant K_n} V_n(t_j). \tag{12.4.8}$$

对于任意固定的 $j, 1\leqslant j\leqslant K_n$，用点

$$\eta_{jr}=t_j+ra_n/b_n, r= -b_n, -b_n+1, \cdots, b_n-1, b_n,$$

分割区间 $[t_j-a_n, t_j+a_n]$，其中 $b_n=B[(na_n/\log n)^{1/2}]$，常数 $B$ 待定. 记 $\phi_{jr}=n^{-1/2}|\alpha_n(t_j)-\alpha_n(\eta_{jr})|$. 对于任给的 $s\in[t_j-a_n, t_j+a_n]$，存在 $r, -b_n\leqslant r\leqslant b_n$，使得 $s\in[\eta_{jr}, \eta_{j,r+1}]$. 由经验分布 $E_n(t)$ 的单调性，我们有

$$\begin{aligned}n^{-1/2}V_n(t_j)= &\sup_{|t_j-s|\leqslant a_n} |(E_n(t_j)-t_j)-(E_n(s)-s)| \\ \leqslant &\max_{-b_n\leqslant r\leqslant b_n}\ \max_{\eta_{jr}\leqslant s\leqslant \eta_{j,r+1}} |(E_n(t_j)-t_j)-(E_n(s)-s)| \\ \leqslant &\max_{-b_n\leqslant r\leqslant b_n} \{|E_n(t_j)-t_j-(E_n(\eta_{jr})-\eta_{j,r+1})|, \\ &|E_n(t_j)-t_j-(E_n(\eta_{j,r+1})-\eta_{j,r+1})|\} \\ \leqslant &\max_{b_n\leqslant r\leqslant b_n} \{\phi_{jr}, \phi_{j,r+1}\}+|\eta_{j,r+1}-\eta_{jr}| \\ \leqslant &\max_{b_n\leqslant r\leqslant b_n} \{\phi_{jr}\}+a_n/b_n.\end{aligned} \tag{12.4.9}$$

写

$$(12.4.10)\quad \phi_{jr} = \left| \frac{1}{n}\sum_{i=1}^{n}[I(\eta_{jr} < U_i \leqslant t_j) - (t_j - \eta_{jr})] \right| = \left| \sum_{i=1}^{n} Z_i \right|.$$

显然$|Z_i| \leqslant 2/n, EZ_i=0$且$EZ_i^2 \leqslant a_n/n^2$，取$\varepsilon = B(a_n\log n/n)^{1/2}$和$\alpha = (B^{-1}na_n^{-1}\log n)^{1/2}$. 由定理的假设，对充分大的$B$我们有

$$\alpha m_n d = \left(\frac{n\log n}{Ba_n}\right)^{1/2} m_n \frac{2}{n} = 2\left(\frac{\log n}{Bna_n}\right)^{1/2} m_n \leqslant \frac{1}{4}.$$

由引理12.4.1得

$$(12.4.11)\quad \begin{aligned} & P\left\{\left|\sum_{i=1}^{n} Z_i\right| > \varepsilon\right\} \\ & \leqslant c_1 \exp\{-\alpha\varepsilon + c\alpha^2 a_n/n\} \\ & \leqslant c_1 \exp\{-B^{1/2}\log n(1 - c/B^{3/2})\}, \end{aligned}$$

其中$c_1 = \exp(3\sqrt{e}A)$. 取$B$充分大即有

$$P\{\max_{0\leqslant j\leqslant K_n}\max_{-b_n\leqslant r\leqslant b_n} |\phi_{jr}| > B(a_n\log n/n)^{1/2}\}$$
$$\leqslant cK_nb_nn^{-B^{1/2}/2} \leqslant cn^{-2}.$$

由 Borel-Gantelli 引理得证

$$(12.4.12)\quad \max_{0\leqslant j\leqslant K_n}\max_{-b_n\leqslant r\leqslant b_n} |\phi_{jr}| \leqslant B(a_n\log n/n)^{1/2} \quad \text{a. s.}$$

将它与(12.4.8)和(12.4.9)相结合产生(12.4.7). 这就完成了定理12.4.2的证明.

**定理12.4.3的证明.**

可沿着定理12.4.2的证明路线进行，只是在应用引理12.4.1的地方代之以引理12.4.2. 这里仅概述不同之点.

记$0<\theta<1, \gamma = 2/(1-\theta)$. 易知$E|z_i|^\gamma \leqslant 2n^{-\gamma}a_n, i=1,\cdots,n$. 因此$\sigma := \sup\{\|Z_i\|_\gamma : i=1,\cdots,n\} \leqslant 2^{1/\gamma}n^{-1}a_n^{1/\gamma}$且$n^{1/2}\sigma \leqslant 2^{1/\gamma}n^{-1/2}a_n^{1/\gamma} \leqslant 1$. 取引理12.4.2中的$\varepsilon = \varepsilon_n = B(n^{-1/2}a_n^{(1-\theta)/2}\log^2 n)$，可得

$$P\{\phi_{jr} \geqslant \varepsilon n\} \leqslant c_1\exp\{-c_2B(n^{-1}a_n^{1-\theta}\log^4 n)^{1/4}/n^{-1/4}a_n^{1/(2\gamma)}\}$$
$$\leqslant c_1 \exp\{-c_2B\log n\}.$$

因此对于充分大的$B$我们有

$$P\{\max_{0\leqslant j\leqslant K_n}\max_{-b_n\leqslant r\leqslant b_n} \phi_{jr} > \varepsilon_n\} \leqslant c_n^{-2}.$$

# 第十三章　由混合样本产生的某些统计量的收敛性

在数理统计学中，大样本理论是一个十分重要的课题. 通常总假设样本是独立的. 但是在某些实际问题中，观察值常常是相依的. 在这一部分中，我们将对几类重要的统计量，如 $U$-统计量、线性模型中的误差方差估计、密度函数估计等在混合样本下研究它们的某些大样本性质.

## §13.1　$U$ 统计量

设 $\{X_n, n\geqslant 1\}$ 是强平稳序列，它们的公共分布是 $F(\cdot)$，又设 $h: R^m \to R$ 是一个关于它的 $m$ 个自变量对称的函数. $U$-统计量定义为

$$U_n = \binom{n}{m}^{-1} \sum_{1\leqslant i_1 < \cdots < i_m \leqslant n} h(X_{i_1}, \cdots, X_{i_m}), n \geqslant m.$$

这里的 $h$ 称为 $U_n$ 的核函数. 这类统计量最早是由 Hoeffding (1948)作为样本均值的推广而引入的. 不少令人感兴趣的统计量属于这种类型或表成它们的渐近形式.

**注 13.1.1**　与 $U$-统计量紧密相关的另一类称为 von-Mises 的统计量(von-Mises 1947)的定义为

$$V_n = n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} h(X_{i_1}, \cdots, X_{i_m}), n \geqslant 1.$$

这两类统计量有十分相似的极限性质. 所以我们只限于讨论 $U$-统计量.

核 $h$ 称为(关于分布 $F$)是退化的，如果对任意的 $a_i, 1\leqslant i\leqslant m$ 和所有的 $j\in\{1, \cdots, m\}$，

$$Eh(a_1,\cdots,a_{j-1},X_j,a_{j+1},\cdots,a_m)=0;$$

这时相应的 $U$-统计量就称为是退化的.

首先我们介绍一个在研究 $U$-统计量的渐近理论中十分有用的工具:Hoeffding 的投影方法. 记

$$\theta=\int\cdots\int h(x_1,\cdots,x_m)\prod_{k=1}^m dF(x_k), \tag{13.1.1}$$

$$h_r(x_1,\cdots,x_r)=\int\cdots\int h(x_1,\cdots,x_m)\prod_{i=r+1}^m dF(x_i),$$

$$r=1,\cdots,m-1$$

和

$$\tilde{h}_r(x_1,\cdots,x_r)=h_r(x_1,\cdots,x_r)-\theta, r=1,\cdots,m-1.$$

$U_n$ 的投影定义为

$$\hat{U}_n=\frac{m}{n}\sum_{i=1}^n\tilde{h}_1(X_i)+\theta.$$

$U_n-\hat{U}_n$ 本身也可表示为一个 $U$-统计量

$$U_n-\hat{U}_n=\binom{n}{m}^{-1}\sum_{1\le i_1<\cdots<i_m\le n}H(X_{i_1},\cdots,X_{i_m}) \tag{13.1.2}$$

$$=:R_n,$$

其中

$$H(x_1,\cdots,x_m)=h(x_1,\cdots,x_m)-\tilde{h}_1(x_1)-\cdots-\tilde{h}_1(x_m)-\theta$$

是一个退化核. 我们称 $R_n$ 是 $U_n$ 的剩余部分.

首先我们需要对 $\varphi$ 混合性的定义作如下修改:称序列$\{X_n,n\geq 1\}$为 $\varphi^*$ 混合或双向 $\varphi$ 混合的,如果序列本身和它们的反向序列都是 $\varphi$ 混合的,也即当 $n\to\infty$ 时

$$\varphi^*(n):=\sup_{k\in\mathbb{N}}\sup_{A\in\mathscr{F}_1^k,B\in\mathscr{F}_{k+n}^\infty}\max\{|P(A|B)-P(A)|,|P(B|A)-P(B)|\}\to 0.$$

显然,从 $\varphi^*$ 混合性可推出 $\varphi$ 混合性.

在这一节里,我们将对 $\varphi^*$ 混合序列建立弱的和强的收敛性. Denker 和 Keller(1983)曾经给出过它的中心极限定理及相应的收

敛速度、泛函中心极限定理以及用 Wiener 过程强逼近等结果. 将它们与第五和第九章中关于 $\varphi$ 混合序列的弱收敛和强逼近结果相结合，我们能够减弱原来文章中关于矩及 $\varphi^*(\cdot)$ 的假设.

13.1.1 剩余 $R_n$ 的界

设 $\{X_n, n\geq 1\}$ 是一个强平稳 $\varphi^*$ 混合序列，假设

(13.1.3) $\quad s^2 := \sup_{1\leq t_1<\cdots<t_m} E(h(X_{t_1},\cdots,X_{t_m}))^2 < \infty.$

首先我们叙述 Denker 和 Keller(1983)的两个引理. 引理 13.1.1 是引理 1.2.8 当 $p=q=2$ 时的条件化形式. 令 $\mathscr{A}$、$\mathscr{B}$、$\mathscr{B}_1$ 和 $\mathscr{B}_2$ 都是 $\mathscr{F}$ 的子 $\sigma$-域. 对 $\mathscr{G}$ 上的概率 $P$ 和 $Q$，定义当给定 $\mathscr{B}$ 时它们在 $\mathscr{A}$ 上的距离

$$d(P,Q;\mathscr{A}|\mathscr{B}) = \sup_{A\in\mathscr{A},B\in\mathscr{B}} |P(A|B) - Q(A|B)|.$$

又记

$$d(P;\mathscr{A}|\mathscr{B}) = \sup_{A\in\mathscr{A},B\in\mathscr{B}} |P(A|B) - P(A)|.$$

**引理 13.1.1** 设 $f_1$ 和 $f_2$ 分别是关于 $\mathscr{A}\vee\mathscr{B}_1$ 和 $\mathscr{A}\vee\mathscr{B}_2$ 可测的函数，又设 $P$、$Q_1$ 和 $Q_2$ 是三个概率测度，它们在 $\mathscr{A}$ 上相等. 那么

$$\begin{aligned}|E_p(f_1f_2) - E_p[E_{Q_1}(f_1|\mathscr{A})\cdot E_{Q_2}(f_2|\mathscr{A})]| &\leq (4+2\sqrt{2})\\ &\cdot\max\{d^{1/2}(P;\mathscr{B}_1|\mathscr{A}\vee\mathscr{B}_2), d^{1/2}(P,Q_1;\mathscr{B}_1|\mathscr{A}),\\ &\quad d^{1/2}(P,Q_2;\mathscr{B}_2|\mathscr{A})\}\\ &\cdot\{E_p(f_1^2)^{1/2}, E_{Q_1}(f_1^2)^{1/2}\}\cdot\max\{E_p(f_2^2)^{1/2},\\ &\quad E_{Q_2}(f_2^2)^{1/2}\}.\end{aligned}$$

**引理 13.1.2** 设 $f$ 是 $\mathscr{A}\vee\mathscr{B}$ 上的可测函数，$P_n(n\geq 1)$和 $Q$ 是在 $\mathscr{A}$ 上相等的概率测度，如果

$$\lim_{n\to\infty} d(P_n,Q;\mathscr{B}|\mathscr{A}) = 0,$$

则有

$$E_Q|f| \leq \liminf_{n\to\infty} E_{P_n}|f|.$$

**引理 13.1.3** 对给定的 $\epsilon>0$，存在 $c=c_\epsilon>0$，使得

$$ER_n^2 \leqslant cn^{-2+\epsilon}s^2 \quad (n \geqslant m).$$

**证** 由(13.1.2)，只需估计退化 $U$ 统计量的方差就够了. 我们来证明：如果 $h$ 是退化的，那么

$$\binom{n}{m}^2 EU_n^2 \leqslant cn^{2(m-1)+\epsilon}s^2. \tag{13.1.4}$$

对于 $\boldsymbol{a}=(a_1,\cdots,a_m),\boldsymbol{b}=(b_1,\cdots,b_m)\in\mathbb{N}^m$，记

$$W(\boldsymbol{a},\boldsymbol{b}) = \sum h(X_{t_1},\cdots,X_{t_m}), \tag{13.1.5}$$

其中求和是对所有满足 $a_i\leqslant t_i\leqslant b_i$ 且 $t_i\neq t_j, 1\leqslant i\neq j\leqslant m$，的 $t_1,\cdots,t_m$ 进行的. 记 $\mathbf{1}=(1,\cdots,1)\in\mathbb{N}^m$. 于是可写

$$\binom{n}{m}U_n = \frac{1}{m!}W(\mathbf{1},n\mathbf{1}). \tag{13.1.6}$$

为估计 $E(W(\mathbf{1},n\mathbf{1}))^2$，受定理 2.1.2 的证明的启发，我们将 $W(\mathbf{1},n\mathbf{1})$ 分解为若干个在较小的指标集上的和. 令 $(X_n^{(1)})_{n\geqslant1},\cdots,(X_n^{(m)})_{n\geqslant1}$ 是 $(X_n)_{n\geqslant1}$ 的 $m$ 个独立的复制. 对 $q\in\{1,\cdots,m\}^m$，记

$$W(\boldsymbol{a},\boldsymbol{b};q) = \sum h(X_{t_1}^{(q_1)},\cdots,X_{t_m}^{(q_m)}),$$

其中的求和范围与(13.1.5)中的相同. 令

$$\boldsymbol{I}_n = \{(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{N}^{2m}: \text{对所有的 } i,j,$$
$$b_i = a_i + n - 1, a_i = a_j \text{ 或 } |a_i - a_j| \geqslant n\},$$

定义

$$\tau(n) = \sup\{E(W(\boldsymbol{a},\boldsymbol{b};q))^2:(\boldsymbol{a},\boldsymbol{b}) \in \boldsymbol{I}_n, q \in \{1,\cdots,m\}^m\}.$$

考虑固定的非负整数 $k$、$l$、$p$、$n$，它们满足关系 $n=kl+p$. 对 $(\boldsymbol{a},\boldsymbol{b})\in\boldsymbol{I}_n$ 和 $q\in\{1,\cdots,m\}^m$，由 Hölder 不等式和三角不等式，我们有

$$\begin{aligned} &|E((W(\boldsymbol{a},\boldsymbol{b};q))^2 - E(W(\boldsymbol{a},\boldsymbol{b} - p\mathbf{1};q))^2| \\ &\leqslant (\tau(n)^{1/2} + k^m\tau(l)^{1/2})mpn^{m-1}s, \end{aligned} \tag{13.1.7}$$

这里我们已经利用了假设：对一切 $q\in\{1,\cdots,m\}^m$，

$$\sup_{1\leqslant t_1<\cdots<t_m} E(h(X_{t_1}^{(q_1)},\cdots,X_{t_m}^{(q_m)}))^2 \leqslant s^2, \tag{13.1.8}$$

而它可以通过重复地应用引理 13.1.2 去证明.

至此，我们可将 $W(\boldsymbol{a},\boldsymbol{b}-p\mathbf{1};q)^2$ 分解为

$$W(\boldsymbol{a},\boldsymbol{b}-p\mathbf{1};q)^2=\sum_{\boldsymbol{u},\boldsymbol{v}\in\{0,\cdots,k-1\}^m}W(\boldsymbol{a}+l\boldsymbol{u},\boldsymbol{a}+l(\boldsymbol{u}+\mathbf{1})-\mathbf{1};q)$$

$$W(\boldsymbol{a}+l\boldsymbol{v},\boldsymbol{a}+l(\boldsymbol{v}+\mathbf{1})-\mathbf{1};q).$$

每个 $\Gamma(\boldsymbol{u}):=W(\boldsymbol{a}+l\boldsymbol{u},\boldsymbol{a}+l(\boldsymbol{u}+\mathbf{1})-\mathbf{1};q)$ 由 $m$ 个长度为 $l$ 的时间坐标块决定(一块可能算几次). 对固定的 $(\boldsymbol{u},\boldsymbol{v})$ 用 $B_1,\cdots,B_{2m}$ 表示这 $2m$ 块且假设 $\inf B_i\leqslant\inf B_{i+1}(i=1,\cdots,2m-1)$. 在这样的约定之下不难看出

(13.1.9) $\quad\mathrm{Card}\{\boldsymbol{u},\boldsymbol{v}\}:\sup B_1+l\geqslant\inf B_2$

$$\text{且}\inf B_{2m}-l\leqslant\sup B_{2m-1}\}\leqslant c_mk^{2m-2},$$

其中 $c_m$ 是一个与 $m$ 有关的组合数, 对于所有不属于(13.1.9)中确定的集合的 $(\boldsymbol{u},\boldsymbol{v})$, 我们能够以如下方式应用引理 13.1.1: 假设 $\sup B_1+l<\inf B_2$ 且 $B_1$ 是当 $\Gamma(\boldsymbol{u})$ 分解为 $m$ 块中的第一块(其余的三种情形可以同样方式处理.)

取 $f_1=\Gamma(\boldsymbol{u}),f_2=T(\boldsymbol{v}),\mathscr{B}_1=\sigma(X_t,t\in B_1),\mathscr{B}_2$ 是平凡 $\sigma$ 域且

$$\mathscr{A}=\sigma(X_t,t\in B_1\cup\cdots\cup B_{2m}).$$

因为 $h$ 是退化的, 由(13.1.8)我们得

$$|E\Gamma(\boldsymbol{u})\Gamma(\boldsymbol{v})|\leqslant(4+2\sqrt{2})\varphi^*(l)^{1/2}\tau(l).$$

为了应用引理 13.1.1, 我们必须引入原来的过程的一组独立复制, 而且对于上面提到的四种情形中的两种, 我们必须挑出块 $B_{2m}$ (代替 $B_1$). 所以我们需要双向混合性. 将最后的那个估计与(13.1.7)和(13.1.9)相结合得到

$$W(\boldsymbol{a},\boldsymbol{b};q)^2\leqslant c_mk^{2m-2}\tau(h)+k^{2m}(4+2\sqrt{2})\varphi^*(h)^{1/2}\tau(l)$$

$$+(\tau(n)^{1/2}+k^m\tau(l)^{1/2})mpn^{m-1}s.$$

对 $(\boldsymbol{a},\boldsymbol{b})\in\boldsymbol{I}_n$ 和 $q\in\{1,\cdots,m\}^m$ 取上确界就有

(13.1.10) $\quad\tau(n)\leqslant k^{2m-2}\tau(l)(c_m+(4+2\sqrt{2})k^2\varphi^*(l)^{1/2}$

$$+(\tau(n)^{1/2}+k^m\tau(l)^{1/2})mpn^{m-1}s.$$

取 $k$ 足够大使得 $(c_m+(4+2\sqrt{2}))k^{-\epsilon}\leqslant1/4$. 记 $n_0=\min\{s:k^2\varphi^*(s)^{1/2}\leqslant1\}$. 对 $l\geqslant n_0$ 和 $p\leqslant k$, 从(13.1.10)推得

$$\left(\tau(n)^{1/2}-\frac{1}{2}mkn^{m-1}s\right)^2\leqslant\left(\frac{1}{2}k^{m-1+\epsilon/2}\right.$$

$$\tau(l)^{1/2}+k^{1-\varepsilon/2}mkn^{m-1}s\Big)^2.$$

因此

(13.1.11) $\quad \tau(n)^{1/2}\leqslant\frac{1}{2}k^{m-1+\varepsilon/2}\tau(l)^{1/2}+\left(\frac{1}{2}+k^{1-\varepsilon/2}\right)mkn^{m-1}s.$

给定 $n$ 选取 $l_0,l_1,\cdots,l_r$ 使得 $l_0=n,l_{i-1}=kl_i+p_i$(其中 $0\leqslant p_i<k,i=1,\cdots,r$)且 $n_0\leqslant l_r<kn_0$. 应用(13.1.11)于每一对$(l_{i-1},l_i)$我们有

$$\tau(l_{i-1})^{1/2}\leqslant\frac{1}{2}k^{m-1+\varepsilon/2}\tau(l_i)^{1/2}+Al_{i-1}^{m-1}s,$$

其中 $A=\left(\frac{1}{2}+k^{1-\varepsilon/2}\right)mk$. 应用归纳法并注意到 $n\geqslant k^r l_r$ 和 $l_r\leqslant kn_0$,即得

$$\begin{aligned}\tau(n)^{1/2}&\leqslant 2^{-r}k^{r(m-1+\varepsilon/2)}\tau(l_r)^{1/2}+An^{m-1+\varepsilon/2}s\sum_{j=0}^{r-1}2^{-j}\\&\leqslant n^{m-1+\varepsilon/2}\left(\frac{1}{2^r}\frac{l_r^m s}{l_r^{m-1+\varepsilon/2}}+2As\right)\\&\leqslant n^{m-1+\varepsilon/2}((kn_0)^{1-\varepsilon/2}+2A)s.\end{aligned}$$

记 $c=((kn_0)^{1-\varepsilon/2}+2A)^2(m!)^{-2}$,上式可写成

(13.1.12) $\qquad \tau(n)\leqslant cs^2n^{2m-2+\varepsilon}(m!)^2.$

于是从(13.1.6)得到(13.1.4).

**引理 13.1.4** 假设满足条件(13.1.3),那么对任意的 $\varepsilon>0$ 和 $c_N>0$ 成立着

$$R_n=O(n^{-3/4-\varepsilon})\quad \text{a.s.}$$

且

$$P\{\max_{1\leqslant n\leqslant N}n|R_n|\geqslant c_N\}=O(N^{1/2+\varepsilon}c_N^{-2}).$$

**证** 由(13.1.2)易知只需对具有退化的核 $h$ 的 $U$-统计量证明引理就够了. 记

$$Z(p,q)=W(\mathbf{1},(p+q)\mathbf{1})-W(\mathbf{1},p\mathbf{1}),p,q\in N.$$

如果 $2^{r-1}\leqslant n<2^r$ 且 $n=\sum_{i=1}^{r}d_i2^{r-i}$表示 $n$ 的 2 进制展式,

$$W(\mathbf{1},n\mathbf{1})=Z(0,n)=\sum_{k=1}^{r}Z\Big(\sum_{i=1}^{k-1}d_i2^{r-i},d_k2^{r-k}\Big).$$

对 $r, u \in \mathbb{N}$, $l = 1, \cdots, r$ 和 $j = 1, \cdots, 2^l$, 考虑集合

$$E_{j,l}^{r,u} = \{ |Z((j-1)2^{r-l}, 2^{r-l})| \geqslant \alpha_{r,u} \},$$

其中 $\alpha_{r,u}$是待定常数. 我们来证明

(13.1.13)　$E(Z(p,q))^2 = O(q(p+q)^{b-1}), b = 2m-3/2+\varepsilon.$

由 Chebyshev 不等式 $P(E_{j,l}^{r,u}) = O(\alpha_{r,u}^{-2} 2^{b(r-1)} j^{b-1})$. 于是从

$$P\{ \max_{2^{r-1} \leqslant n < 2^r} |Z(0,n)| \geqslant n^{b/2} (\log n)^3 \}$$

$$\leqslant \sum_{l=1}^{r} \sum_{j=1}^{2^l} P(E_{j,l}^{r,1}) + O(r^{-3}),$$

并取 $\alpha_{r,1} = 2^{b(r-1)/2} (r-1)^3/r$, 再利用 Borel-Cantelli 引理即得 $R_n$ 的 $a.s.$ 界.

为了证明极大不等式, 令 $R$ 和 $N$ 是给定的正整数, 满足关系 $2^{R-1} \leqslant N < 2^R$. 对 $r \leqslant R$ 记 $\alpha_{r,N} = 2^{(r-1)(m-1)/r} c_N$, 我们有

$$P\{ \max_{1 \leqslant n \leqslant N} n^{-m+1} |Z(0,n)| \geqslant c_N \}$$

$$\leqslant P\{ \bigcup_{r=1}^{R} \bigcup_{n=2^{r-1}}^{2^r-1} \{ |Z(0,n)| \geqslant n^{m-1} c_N \} \}$$

$$\leqslant \sum_{r=1}^{R} \sum_{l=1}^{r} \sum_{j=1}^{2^l} P(E_{j,l}^{r,N}) = O(N^{1/2+\varepsilon} (\log N)^3 c_N^{-2}).$$

我们还需证明(13.1.13). 如果 $q > p$, 从(13.1.12)可得

$$E(Z(p,q))^2 \leqslant \{ (EZ(0,p)^2)^{1/2} + (EZ(0,p+q)^2)^{1/2} \}^2$$

$$= O((p+q)^{2m-2+\varepsilon}) = O(q(p+q)^{b-3/2}).$$

如果 $p \geqslant q \geqslant p^{1/2}$, 类似地可得

$$E(Z(p+q))^2 = O((p+q)^{2m-2+\varepsilon}) = O(q(p+q)^{b-1}).$$

下面考虑 $q < p^{1/2}$的情形. 记 $\tilde{I}_n = \{ (\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{N}^{2m}$: 或者恰好有 $m-1$ 个坐标 $b_i = a_i + n - 1$, 有一个坐标 $a_i = b_i$; 或者对一切 $i, j$, $|a_i - a_j| \geqslant n \}$, 定义

$$\tilde{\tau}(n) = \sup\{ E(W(\boldsymbol{a}, \boldsymbol{b}; q))^2 : (\boldsymbol{a}, \boldsymbol{b}) \in \tilde{I}_n, q \in \{1, \cdots, m\}^m \}.$$

类似于(13.1.12)可以证明: 存在 $c' > 0$,

$$\tilde{\tau}(n) \leqslant c' n^{2m-3+\varepsilon} s^2.$$

因此我们得到

$$E(Z(p+q))^2 = E\Big(\sum_{k=0}^{q-1} Z(p+k,1)\Big)^2$$
$$\leqslant \Big(\sum_{k=0}^{q-1}(E(Z(p+k,1))^2)^{1/2}\Big)^2$$
$$= O(q^2(p+q)^{2m-3+\varepsilon}) = O(q(p+q)^{\delta-1}).$$

这就完成了引理 13.1.4 的证明.

13.1.2 $U_n$ 的弱不变原理.

记 $\sigma_n^2 = E\Big(\sum_{i=1}^{n}\tilde{h}_1(X_i)\Big)^2$ 和

$$W_n(t) = \frac{nt}{m\sigma_n}(U_{[nt]} - \theta), 0 \leqslant t \leqslant 1.$$

**定理 13.1.1** 设 $h$ 是非退化核. 假设满足条件(13.1.3)且 $\sigma_n^2 \to \infty$. 又假设对任意的 $\varepsilon > 0$.

$$\lim_{n\to\infty}\frac{n}{\sigma_n^2}E\tilde{h}_1(X_1)^2 I(|\tilde{h}_1(X_1)| > \varepsilon\sigma_n) = 0.$$

那么当 $n\to\infty$ 时 $W_n \Rightarrow W$.

**证** 记

$$\hat{W}_n(t) = \frac{nt}{m\sigma_n}(\hat{U}_{[nt]} - \theta), 0 \leqslant t \leqslant 1.$$

由推论 5.1.4, $\hat{W}_n \Rightarrow W$. 从定理 2.1.2 可知, 对任意的 $\varepsilon > 0$, 当 $n \to \infty$ 时 $\sigma_n^2/n^{1-\varepsilon} \to \infty$. 因此利用引理 13.1.4 中的极大不等式得到

$$P\Big\{\sup_{0\leqslant t\leqslant 1}\frac{nt}{m\sigma_n}|R_{[nt]}| \geqslant \varepsilon\Big\}$$
$$\leqslant P\Big\{\sup_{0\leqslant t\leqslant 1} nt|R_{[nt]}| \geqslant \varepsilon m n^{(1-\varepsilon)/2}\Big\}$$
$$= O(n^{-1/2+2\varepsilon}).$$

从(13.1.2)即得待证的结果.

**注 13.1.1** Denker-Keller(1983)和张奕(1989)研究了 $\varphi^*$ 混合样本的 $U$ 统计量的 Berry-Essen 不等式. 张奕的结果是:

设 $\{X_n, n \geqslant 1\}$ 是强平稳 $\varphi^*$ 混合序列, $U_n$ 是以 $h(x_1, x_2)$ 为核函

数的 $U$ 统计量. 记

$$\sigma^2 = Eh_1^2(X_1) - \theta^2 + 2\sum_{k=1}^{\infty}\{Eh_1(X_1)h_1(X_{k+1}) - \theta^2\}.$$

假设存在常数 $c>0$ 和 $\beta>0$ 使得

$$\varphi^*(n) \leqslant ce^{-\beta n}.$$

如果 $\sup_{1\leqslant i<j\leqslant n} E|h(X_i,X_j)|^3<\infty$ 且 $\sigma^2>0$,那么对任意 $\varepsilon>0$,

$$\sup_x\left|P\left\{\frac{U_n-\theta}{\sqrt{\mathrm{Var}U_n}}\leqslant x\right\}-\Phi(x)\right|\leqslant cn^{-1/2+\varepsilon}.$$

### 13.1.3 $U_n$ 用 Wiener 过程强逼近

**定理 13.1.2** 设 $h$ 是非退化核. 假设对某个 $\delta>0$

$$\sup_{1\leqslant t_1<\cdots<t_m} E|h(X_{t_1},\cdots,X_{t_m})|^{2+\delta} < \infty, \tag{13.1.14}$$

且

i) 存在 $c_1>0, \sigma_n^2\geqslant c_1 n$,

ii) 存在 $c_2>0$ 和 $\alpha>0, \varphi^*(n)\leqslant c_n n^{-\alpha}$.

那么,可以在一个定义有 Wiener 过程 $W(\cdot)$的较大的概率空间上重新定义$\{X_n,n\geqslant1\}$而不改变其分布,使得对任意的 $\varepsilon>0$

$$\frac{n}{m}(U_n-\theta) - W(\sigma_n^2) = O(\sigma_n^{2/(2+\delta)}$$

$$(\log\sigma_n)^{1+\varepsilon+(1+\lambda)/(2+\delta)})\text{a.s.}$$

其中 $\lambda=2(\log3)/\log\tau^{-1}, \tau=1-2(\alpha-1)/\alpha(2+\delta)$.

**证** 由引理 13.1.2,从条件(13.1.14)可推出 $E|\tilde{h}_1(X_1)|^{2+\delta}<\infty$. 因此由注 9.1.1 即知可在一定义有 Wiener 过程的新的概率空间上重新定义$\{\tilde{h}_1(X_n),n\geqslant1\}$,使得

$$\sum_{i=1}^{n}\tilde{h}_1(X_i) - W(\sigma_n^2) = O(\sigma_n^{2/(2+\delta)}(\log\sigma_n)^{1+\varepsilon+(1+\lambda)/(2+\delta)})\ \text{a.s.}$$

事实上,我们还不难看出,在这个新的概率空间上可以重新定义$\{X_n\}$本身. 例如,可以通过考虑 $R^2$ 值随机向量的强不变原理. 由引理 13.1.4, $nR_n=O(n^{1/4+\varepsilon})$a.s. 定理得证.

### 13.1.4 $U_n$ 的强大数律.

王启应(1995)对 $\varphi^*$ 混合的样本证明了一个强大数律. 仅考虑 $m=2$ 的情形. 为了证明这个结果, 我们需要下列引理, 它的证明被包含在 Babbel(1989)的定理 3 的证明中.

**引理 13.1.5** 设 $h$ 是退化核. 假设满足条件(13.1.3)且存在 $\delta>0$ 使得

(13.1.15) $$\varphi^*(n) = O(n^{-(4+\delta)}).$$

那么

$$E(\max_{1\leq i\leq n} U_i)^2 \leqslant cn^{-2}s^2.$$

**定理 13.1.3** 假设满足条件(13.1.15)且

(13.1.16) $$\sup_{n\geq 2} E|h(X_1, X_n)| < \infty.$$

那么

$$U_n \to \theta \quad \text{a.s.} \quad n \to \infty,$$

其中 $\theta = \iint h(x_1, x_2) dF(x_1) dF(x_2)$.

**证** 对 $k \in M$, 记

$$h^{(k)}(x_1, x_2) = h(x_1, x_2) I(|h(x_1, x_2)| \leqslant 2^{2k}),$$

$$\theta^{(k)} = \iint h^{(k)}(x_1, x_2) dF(x_1) dF(x_2),$$

$$\bar{h}_1^{(k)}(x) = \int h^{(k)}(x, y) dF(y),$$

$$H^{(k)}(x_1, x_2) = h^{(k)}(x_1, x_2) - \bar{h}_1^{(k)}(x_1) - \bar{h}_1^{(k)}(x_2) + \theta^{(k)}$$

和

$$U_n^{(k)} = \binom{n}{2}^{-1} \sum_{1\leq i<j\leq n} h^{(k)}(X_i, X_j),$$

$$\hat{U}_n^{(k)} = \frac{2}{n} \sum_{j=1}^{n} \bar{h}_n^{(k)}(X_j),$$

$$\Delta_n^{(k)} = \binom{n}{2}^{-1} \sum_{1\leq i<j\leq n} H^{(k)}(X_i, X_j).$$

最后一个是带有退化核 $H^{(k)}$ 的 $U$-统计量. 容易看出

$$U_n^{(k)} = \tilde{U}_n^{(k)} + \Delta_n^{(k)} - \theta^{(k)}.$$

由此我们可写

(13.1.17) $$\limsup_{n\to\infty} |U_n - \theta|$$

$$\leqslant \limsup_{k\to\infty} \max_{2^k\leqslant n<2^{k+1}} \{ |\tilde{U}_n^{(k)} - 2\theta^{(k)}| + |\Delta_n^{(k)}| + |U_n - U_n^{(k)}| + |\theta - \theta^{(k)}| \}.$$

显然,条件(13.1.16)可推出

(13.1.18) $$\theta - \theta^{(k)} \to 0, k \to \infty.$$

应用引理 13.1.5 和条件(13.1.16),对任意 $\varepsilon>0$ 我们有

$$\sum_{k=1}^{\infty} P\{ \max_{2^k\leqslant n<2^{k+1}} |\Delta_n^{(k)}| \geqslant \varepsilon \}$$

$$\leqslant c\varepsilon^{-2} \sum_{k=1}^{\infty} 2^{-2k} \sup_{n\geqslant 2} Eh^2(X_1, X_n) I(|h(X_1, X_n)| \leqslant 2^{2k})$$

$$\leqslant c\varepsilon^{-2} \sup_{n\geqslant 2} \sum_{k=1}^{\infty} 2^{-2k} \sum_{j=1}^{k} Eh^2(X_1, X_n) I(2^{2(j-1)} < |h(X_1, X_n)| \leqslant 2^{2j})$$

$$\leqslant c\varepsilon^{-2} \sup_{n\geqslant 2} \sum_{j=1}^{\infty} Eh^2(X_1, X_n) I(2^{2(j-1)} < |h(X_1, X_n)| \leqslant 2^{2j}) \sum_{k=j}^{\infty} 2^{-2k}$$

$$\leqslant c\varepsilon^{-2} \sup_{n\geqslant 2} E|h^2(X_1, X_n)| < \infty.$$

由此即得

(13.1.19) $$\limsup_{k\to\infty} \max_{2^k\leqslant n<2^{k+1}} |\Delta_n^{(k)}| = 0 \quad \text{a. s.}$$

此外

$$\sum_{k=1}^{\infty} P\{ \bigcup_{n=2^k}^{2^{k+1}-1} \{U_n \neq U_n^{(k)}\} \}$$

$$\leqslant \sum_{k=1}^{\infty} P\{ \bigcup_{n=2^k}^{2^{k+1}-1} \bigcup_{1\leqslant i<j\leqslant n} \{|h(X_i, X_j)| \geqslant 2^{2k}\} \}$$

$$\leqslant \sum_{k=1}^{\infty} 2^{2(k+1)} \sup_{n\geqslant 2} P\{ |h(X_1,X_n)| \geqslant 2^{2k}\}$$

$$\leqslant 4 \sup_{n\geqslant 2} \sum_{k=1}^{\infty} 2^{2k} \sum_{j=k}^{\infty} P\{2^{2j} \leqslant |h(X_1,X_n)| < 2^{2(j+1)}\}$$

$$\leqslant c \sup_{n\geqslant 2} \sum_{j=1}^{\infty} E|h(X_1,X_n)| I(2^{2j} \leqslant |h(X_1,X_n)| < 2^{2(j+1)})$$

$$\leqslant c \sup_{n\geqslant 2} E|h(X_1,X_n)| < \infty.$$

因此又有

$$\limsup_{k\to\infty} \max_{2^k\leqslant n<2^{k+1}} |U_n - U_n^{(k)}| = 0 \quad \text{a.s.} \tag{13.1.20}$$

下面来估计(13.1.17)右边的第一项.记

$$\hat{h}_1^{(k)}(x) = \tilde{h}_1^{(k)}(x) I(|\tilde{h}_1^{(k)}(x)| \leqslant 2^k).$$

写

$$\begin{aligned}|\tilde{U}_n^{(k)} - 2\theta^{(k)}| \leqslant & \left| \frac{2}{n} \sum_{i=1}^{n} (\hat{h}_1^{(k)}(X_i) - E\hat{h}_1^{(k)}(X_i)) \right| \\ & + \left| \frac{2}{n} \sum_{i=1}^{n} (\tilde{h}_1^{(k)}(X_i) - \hat{h}_1^{(k)}(X_i)) \right| + 2|\theta^{(k)} - E\hat{h}_1^{(k)}(X_1)|.\end{aligned} \tag{13.1.21}$$

首先考虑$|\theta^{(k)} - E\hat{h}_1^{(k)}(X_1)|$.我们有

$$\begin{aligned}|\theta^{(k)} - E\hat{h}_1^{(k)}(X_1)| = & |E\tilde{h}_1^{(k)}(X_1) I(|\tilde{h}_1^{(k)}(X_1)| \\ & > 2^k)| \to 0, k \to \infty.\end{aligned} \tag{13.1.22}$$

类似于(13.1.20),我们有

$$\sum_{k=1}^{\infty} P\Big\{ \bigcup_{n=2^k}^{2^{k+1}-1} \Big\{ \sum_{i=1}^{n} \tilde{h}_1^{(k)}(X_i) \neq \sum_{i=1}^{n} \hat{h}_1^{(k)}(X_i) \Big\} \Big\} < \infty,$$

由它推得

$$\limsup_{k\to\infty} \max_{2^k\leqslant n<2^{k+1}} \left| \frac{2}{n} \sum_{i=1}^{n} (\tilde{h}_1^{(k)}(X_i) - \hat{h}_1^{(k)}(X_i)) \right| = 0 \text{ a.s.} \tag{13.1.23}$$

进一步,由于$\{\hat{h}_1^{(k)}(X_n) - E\hat{h}_1^{(k)}(X_n), n\geqslant 1\}$是强平稳$\varphi^*$混合序列,混合系数$\varphi^*(n) = O(n^{-(4+\delta)})$,因此利用引理2.2.10得到

$$\sum_{k=1}^{\infty} P\left\{ \max_{2^k\leqslant n<2^{k+1}} \left| \frac{2}{n} \sum_{i=1}^{n} (\hat{h}_1^{(k)}(X_i) - E\hat{h}_1^{(k)}(X_i)) \right| \geqslant \epsilon \right\}$$

$$\leqslant c \sum_{k=1}^{\infty} 2^{-k} E(\hat{h}_1^{(k)}(X_1))^2$$

$$\leqslant c\sum_{k=1}^{\infty}E|\tilde{h}_1^{(k)}(X_1)| < \infty,$$

由此得到

(13.1.24)
$$\limsup_{k\to\infty}\max_{2^k\leqslant n<2^{k+1}}\left|\frac{2}{n}\sum_{i=1}^{n}(\tilde{h}_1^{(k)}(X_i) - E\tilde{h}_1^{(k)}(X_i))\right| = 0 \quad \text{a.s.}$$

将(13.1.22)—(13.1.24)与(13.1.21)相结合就得

(13.1.25)
$$\limsup_{k\to\infty}\max_{2^k\leqslant n<2^{k+1}}|\tilde{U}_n^{(k)}-2\theta^{(k)}|=0 \quad \text{a.s.}$$

(13.1.17)—(13.1.20)与(13.1.25)一起即可推得定理 13.1.3 的结论.

## §13.2 线性模型中的误差方差估计

考虑线性回归模型

$$Y_i = x_i'\beta + e_i, i = 1,2,\cdots,$$

其中$\{x_i\}$是已知的 $p$ 维设计序列,$\{Y_i\}$是观察值序列,$\beta$ 是未知的 $p$ 维向量,$\{e_i\}$是随机误差序列,它是强平稳的且

(13.2.1) $\qquad Ee_1=0,\sigma^2:=Ee_1^2>0,\nu:=Ee_1^4<\infty.$

基于残差平方和,$\sigma^2$ 的估计是

$$\hat{\sigma}^2(n)=\frac{1}{n-r_n}\left\{\sum_{j=1}^{n}e_j^2-\sum_{i=1}^{r_n}\left(\sum_{j=1}^{n}a_{ij}^{(n)}e_j\right)^2\right\},$$

其中 $r_n$ 是设计矩阵 $X_n=(X_1,\cdots,X_n)'$的秩,且当 $n$ 充分大时稳定于 $r_n\leqslant p$,$(a_{ij}^{(n)})$是 $n$ 阶由矩阵 $X_n$ 决定的正交矩阵.记

$$\tau=\nu-\sigma^4+2\sum_{j=2}^{\infty}E(e_1^2-\sigma^2)(e_j^2-\sigma^2).$$

如果 $0<\tau<\infty$,定义 $C[0,1]$的随机泛函 $Z_n(\cdot)$如下:

$$Z_n(0)=0,Z_n(i/n)=(i-r_i)(\hat{\sigma}_i^2-\sigma^2)/\sqrt{n\tau},i=1,\cdots,n,$$

$$\text{在}\left[\frac{i-1}{n},\frac{i}{n}\right]\text{中 } Z_n(\cdot)\text{是线性的.}$$

又由

$$z(t)=([t]-r_{[t]})(\hat{\sigma}^2([t])-\sigma^2)$$

定义$C(0,\infty)$的随机函数$Z(\cdot)$. 林正炎(1984)和陆传荣(1986)分别研究了$Z_n(\cdot)$的弱不变原理和$Z(\cdot)$的强逼近,我们将集中注意力于$\varphi$混合的误差序列$\{e_i\}$. 类似的方法可以应用于其它类型的混合误差序列.

13.2.1 弱不变原理.

对任意的$q>0$,定义

(13.2.2) $$a(q)=\inf\{a:\sup_{A,P(A)\leqslant \nu/q^4}\sup_i Ee_i^4 I_A\leqslant a\}.$$

显然当$q\uparrow\infty$时$a(q)\downarrow 0$,用$q=q(a)$.表示$a=a(q)$的逆函数,它是不减的. 设$a=d(t)$是方程$a/q(a)^5=t$的解. 记$g(b)=d(2^{1/4}/b)$,易知当$b\to\infty$时$g(b)\downarrow 0$. 设$t_n(\uparrow\infty)$是满足关系$t_n^2 g(t_n^{-1/2}n^{1/4})=0(1)$的最大整数. 容易看出这样的$t_n$是存在的,林正炎(1984)证明:

**定理 13.2.1** 设随机误差序列$\{e_i\}$是强平稳$\varphi$混合的,满足(13.2.1). 假设混合系数$\varphi(n)$满足条件

(i) $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$.

则有$\tau<\infty$. 如果附带假设$\tau>0$且

(ii) $nt_n^{-1}\varphi(t_n)=0(1)$,

那么

$$Z_n\Rightarrow W.$$

为了证明上述弱不变原理,我们需要几个引理.

**引理 13.2.1** 设$\{\alpha_i^{(n)},i=1,\cdots,n\}$是组内独立的随机变量组列,而且各组间也是相互独立的,满足

(13.2.3) $$E\alpha_i^{(n)^4}I(|\alpha_i^{(n)}|\geqslant q)\leqslant a(q),i=1,\cdots,n,$$

其中$a(q)$由(13.2.2)定义. 那么对形为$X=\sum_{i=1}^{\infty}a_i\alpha_i^{(n)}$($a_k$满足$\sum_{k=1}^{\infty}a_k^2\leqslant 1$)的随机变量族$\mathscr{F}$一致地有

(13.2.4) $$b^4P(|X|\geqslant b)=0(g(b)),b\to\infty.$$

**证** 令 $f_i:=f_{in}$ 和 $F_i:=F_{in}$ 分别是 $\alpha_i^{(n)}$ 的特征函数和分布函数，$f_i^{(k)}$ 是 $f_i$ 的 $k$ 阶导数. 首先我们来证明：对 $i=1,\cdots,n$ 一致地成立

(13.2.5) $$f_i(t)=\sum_{k=0}^{4}\frac{f_i^{(k)}(0)}{k!}t^k+o(t^4),t\to 0.$$

为此，写

(13.2.6) $$f_i(t)-\sum_{k=0}^{4}\frac{f_i^{(k)}(0)}{k!}t^k=-\frac{t^4}{4!}\int_{-\infty}^{\infty}(1-e^{it\theta x})x^4dF_i(x),$$

其中 $|\theta|\leqslant 1$. 取 $t=\pm a(q)/q^5$. 于是对 $|x|\leqslant q$

$$|1-e^{it\theta x}|\leqslant|t\theta x|\leqslant a(q)/q^4.$$

因此从(13.2.6)和(13.2.3)推出：对 $i=1,\cdots,n$,

$$|f_i(t)-\sum_{k=0}^{4}\frac{f_i^{(k)}(0)}{k!}t^4|\leqslant\frac{t^4}{24}\left\{\int_{|x|<q}|1-e^{it\theta x}|x^4dF_i(x)\right.$$
$$\left.+\int_{|x|\geqslant q}2x^4dF_i(x)\right\}\leqslant t^4a(q)/8.$$

注意到 $a(q)$ 与 $i$ 无关且当 $q\to\infty$ 时 $a(q)\downarrow 0$,(13.2.5)的一致性得证，(13.2.5)可以改写为

(13.2.7) $$\log f_i(t)=\sum_{j=1}^{4}b_{ij}t^j+g_i(t),i=1,\cdots,n,$$

此处，当 $t\to 0(q\to\infty)$ 时，对 $i=1,\cdots,n$ 一致地有 $g_i(t)/t^4=O(a(q))$. 令 $f$ 和 $F$ 分别是 $X=\sum_{i=1}^{n}a_i\alpha_i^{(n)}$ 的特征函数和分布函数. 从(13.2.7)可得

$$\log f(t)=\sum_{i=1}^{n}\log f_i(a_it)=\sum_{j=1}^{4}\left(\sum_{i=1}^{n}b_{ij}a_i^j\right)t^j+\sum_{i=1}^{n}g_i(a_it),$$

其中

$$\sum_{i=1}^{n}g_i(a_it)\leqslant ca(q)\sum_{i=1}^{n}(a_it)^4\leqslant cd(t)t^4.$$

由 $d(\cdot)$ 的定义，当 $t\to 0$ 时 $t/d(t)\to 0$. 因此

$$f(t)=\sum_{k=0}^{4}\frac{f^{(k)}(0)}{k!}t^k+O(t^5+d(t)t^4)$$
$$=\sum_{k=0}^{4}\frac{f^{(k)}(0)}{k!}t^k+O(d(t)t^4).$$

于是只要$(bt)^4\geq 2$,就对$f$一致地有

$$O(d(t)) = f^{(4)}(0) - \{f(4t) - 4f(2t) + 6f(0) - 4f(-2t) + f(-4t)\}/(2t)^4$$

$$= \int x^4 dF(x) - \int \left(\frac{e^{itx}-e^{-itx}}{2t}\right)^4 dF(x)$$

$$= \int \left(1-\left(\frac{\sin tx}{tx}\right)^4\right) x^4 dF(x)$$

$$\geq \int_{|x|\geq b} \left(1-\left(\frac{\sin tx}{tx}\right)^4\right) x^4 dF(x)$$

$$\geq \frac{1}{2}\int_{|x|\geq b} x^4 dF(x) \geq \frac{1}{2} b^4 P(|x|\geq b).$$

由此即得(13.2.4).

**注 13.2.1** 类似地,如果代替条件(13.2.3)我们假设存在整数$m>0$和$\delta$,$0\leq\delta<1$,当$q\to\infty$时

$$\max_{1\leq i\leq n} E|\alpha_i^{(n)}|^{m+\delta} I(|\alpha_i^{(n)}|>q)\to 0,$$

那么对引理 13.2.1 中定义的$\mathcal{X}$一致地有

$$P\{|X|>b\} = o(b^{-m-\delta}),\ b\to\infty.$$

**引理 13.2.2** 在定理 13.2.1 的条件(i)和(ii)之下,对任意的$\varepsilon>0$,关于形为$X=\sum_{k=1}^{n} a_k e_k\ (\sum_{k=1}^{n} a_k^2\leq 1)$的随机变量一致地有

$$nP\{|X|\geq n^{1/4}\varepsilon\} = o(1).$$

**证** 记$h_n=[n t_n^{-1}/2]$, $\xi_j = \sum_{k=2jt_n+1}^{(2j+1)t_n} a_k e_k$, $\eta_j = \sum_{k=(2j+1)t_n+1}^{2(j+1)t_n} a_k e_k$, $j=0,1,\cdots,h_n-1$, 和 $\xi_{h_n} = \sum_{k=2h_n t_n+1}^{n} a_k e_k$. 对任意的$q>0$,我们来估计$E\xi_j^4 I(|\xi_j|>5^{1/4} t_n^{1/2}(\sum_{k=2jt_n+1}^{(2j+1)t_n} a_k^2)^{1/2} q)$. 为简单计,我们仅考虑$j=0$的情形且记$B=\{|\xi_0|>5^{1/4}t_n^{1/2}(\sum_{k=1}^{t_n} a_k^2)^{1/2}q\}$. 我们有

$$E\xi_0^4 I_B = \sum_{k=1}^{t_n} a_k^4 E e_k^4 I_B + \sum_{p\neq q} a_p^2 a_q^2 E e_p^2 e_q^2 I_B \tag{13.2.8}$$

$$+\sum_{p\neq q}a_p^3a_qEe_p^3e_qI_B+\sum_{p\neq q\neq r}a_p^2a_qa_rEe_p^2e_qe_rI_B$$
$$+\sum_{p\neq q\neq r\neq s}a_pa_qa_ra_sEe_pe_qe_re_sI_B.$$

显然(13.2.8)右方的前两个和都不超过

$$\left(\sum_{k=1}^{t_n}a_k^2\right)^2\max_{1\leq k\leq t_n}Ee_k^4I_B.$$

记 $M=\max_{1\leq k\leq t_n}Ee_k^4I_B$. 对第三个和,有

$$\left|\sum_{p\neq q}a_p^3a_qEe_p^3e_qI_B\right|\leqslant M\sum_{p=1}^{t_n}|a_p^3|\sum_{q=1}^{t_n}|a_q|\leqslant Mt_n^{1/2}\left(\sum_{k=1}^{t_n}a_k^2\right)^2.$$

类似地第四个和的绝对值有上界 $Mt_n\left(\sum_{k=1}^{t_n}a_k^2\right)^2$. 对于第五个和,它的绝对值有上界 $Mt_n^2\left(\sum_{k=1}^{t_n}a_k^2\right)^2$. 因此我们得到

$$E\xi_0^4I_B\leqslant 5Mt_n^2\left(\sum_{k=1}^{t_n}a_k^2\right)^2. \tag{13.2.9}$$

由这个估计,易知 $E\xi_0^4$ 有上界 $5\nu t_n^2\left(\sum_{k=1}^{t_n}a_k^2\right)^2$. 由此即得 $P(B)\leqslant\nu/q^4$. 从(13.2.2)我们有 $M\leqslant a(q)$. 将它代入(13.2.9)产生

$$\frac{1}{5}t_n^{-2}\left(\sum_{k=1}^{t_n}a_k^2\right)^{-2}E\xi_0^4I_B\leqslant a(q). \tag{13.2.10}$$

对于 $\xi_j$, $j=1,\cdots,h_n$ 我们有几乎相同的结论(仅对 $\xi_{h_n}$,可能相差一个常数). 设 $\{\xi'_j, j=0,1,\cdots,h_n\}$ 是相互独立的随机变量,$\xi'_j$ 与 $\xi_j$ 有相同的分布. 由(13.2.10),

$$\left\{5^{-1/4}t_n^{-1/2}\left(\sum_{k=2jt_n+1}^{(2j+1)t_n}a_k^2\right)^{-1/2}\xi'_j,\ j=0,1,\cdots,h_n\right\}$$

满足引理13.2.1中给出的条件. 取 $\left(\sum_{k=2jt_n+1}^{(2j+1)t_n}a_k^2\right)^{1/2}$ 和 $t_n^{-1/2}n^{1/4}\varepsilon$($\varepsilon>0$ 任意给定)分别作为引理13.2.1中的 $a_j$ 和 $b$,得到

$$t_n^{-2}nP\left\{t_n^{-1/2}\left|\sum_{j=0}^{h_n}\xi'_j\right|\geqslant t_n^{-1/2}n^{1/4}\varepsilon\right\}=O(g(t_n^{-1/2}n^{1/4}\varepsilon)).$$

从 $t_n$ 的选取，我们有

$$(13.2.11)\qquad nP\left\{\left|\sum_{j=0}^{h_n}\xi'_j\right|\geqslant n^{1/4}\varepsilon\right\}=O(t_n^2g(t_n^{-1/2}n^{1/4}))=o(1).$$

进一步，由引理 1.2.9

$$\left|E\exp\left(itn^{-1/4}\sum_{j=0}^{h_n}\xi'_j\right)-E\exp\left(itn^{-1/4}\sum_{j=0}^{h_n}\xi_j\right)\right|$$

$$\leqslant(h_n+1)\varphi(t_n)\leqslant\frac{1}{2}nt_n^{-1}\varphi(t_n).$$

由引理 13.2.1 的条件(ii)，$n^{-1/4}\sum_{j=0}^{h_n}\xi'_j$ 与 $n^{-1/4}\sum_{j=0}^{h_n}\xi_j$ 有相同的极限分布. 因此从(13.2.11)

$$nP\left\{\left|\sum_{j=0}^{h_n}\xi_j\right|\geqslant n^{1/4}\varepsilon\right\}=o(1).$$

对 $\eta_j$ 我们有相同的关系式. 结合这两个结果即得引理的结论.

**定理 13.2.1 的证明.**

显然只需证明弱不变原理就够了，定义 $U_n(\cdot)$ 和 $V_n(\cdot)$ 如下：$U_n(0)=0, V_n(0)=0$，

$$U_n(i/n)=\sum_{j=1}^{n}(e_j^2-\frac{i-r_i}{i}\sigma^2)/\sqrt{n\tau},$$

$$V_n(i/n)=\sum_{j=1}^{\gamma_i}(\sum_{k=1}^{i}a_{jk}^{(i)}e_k)^2/\sqrt{n\tau}, i=1,\cdots,n,$$

$U_n$ 和 $V_n$ 在 $\left[\frac{i-1}{n},\frac{i}{n}\right]$ 内均线性.

我们有

$$Z_n=U_n-V_n.$$

由定理 5.1.1，

$$U_n\Rightarrow W, n\to\infty.$$

因此为了证明定理，只需验证：对任意的 $\varepsilon>0$

$$(13.2.12)\qquad P\left\{\sup_{0\leqslant t\leqslant 1}|V_n(t)|>\varepsilon\right\}\to 0, n\to\infty.$$

因为 $\tau_i\leqslant p$，所以(13.2.12) 等价于对任意的满足 $\sum_{k=1}^{n}a_k^{(n)^2}\leqslant 1$ 的

$\{a_k^{(n)}\}$,

$$P\left\{\max_{1\leq i\leq n}\left|\sum_{k=1}^{i}a_k^{(i)}e_k\right|\geq(n\tau)^{1/4}\varepsilon^{1/2}\right\}\to 0, n\to\infty.$$

由引理 13.2.2 我们有

$$P\left\{\max_{1\leq i\leq n}\left|\sum_{k=1}^{i}a_k^{(i)}e_k\right|\geq(n\tau)^{1/4}\varepsilon^{1/2}\right\}$$

$$\leq n\max_{1\leq i\leq n}P\left\{\left|\sum_{k=1}^{i}a_k^{(i)}e_k\right|\geq(n\tau)^{1/4}\varepsilon^{1/2}\right\}\to 0, n\to\infty.$$

这就完成了定理的证明.

**注 13.2.2** 当$\{e_i\}$是强平稳 $m$ 相依序列时,可取 $t_n$ 为一常数,因此条件(ii)被满足;当$\{e_i\}$是一个有界序列时,我们可取 $a(q)=0$,$t_n$ 为$[n^{2/3}]$,因此条件(ii)也是被满足的.

**注 13.2.3** 林正炎(1984)也对 $\alpha$ 混合的$\{e_i\}$研究了弱不变原理.

13.2.2 强逼近

利用关于 $\varphi$ 混合序列的强逼近结果(参见定理 9.1.1),陆传荣(1986)证明了下列定理.

**定理 13.2.2** 设随机误差序列$\{e_i\}$是强平稳 $\varphi$ 混合的,满足(13.2.1)假设存在 $0<\delta\leq 1$,$E|e_1|^{8+\delta}<\infty$且

(13.2.13) $$\varphi(n)=o(n^{1-4(2+\varepsilon)(10-\delta)/\delta\theta}), n\to\infty,$$

其中 $0<\theta<1$,$\varepsilon>0$. 那么

$$z(t)-W(t)=o(t^{1/4}(\log t)^{9/4+\varepsilon})\quad \text{a.s.}$$

为了证明这个定理,需要下述引理.

**引理 13.2.3** 在定理 13.2.2 的条件下,

$$X=o(n^{1/8})\text{a.s.}$$

其中 $X=\sum_{k=1}^{n}a_ke_k\left(\sum_{k=1}^{n}a_k^2\leq 1\right)$.

**证** 记

$d_n=[n^{\delta\theta/4(10-\delta)}]$, $\quad h_n=[n/2d_n]$,

$$\xi_j=\sum_{k=2jd_n+1}^{(2j+1)d_n}a_ke_k,\quad \eta_j=\sum_{k=(2j+1)d_n+1}^{2(j+1)d_n}a_ke_k,\ j=0,1,\cdots,h_n-1,$$

$$\xi_{h_n}=\sum_{k=2h_nd_n+1}^{n}a_ke_k.$$

类似于(13.2.9),

$$E|\xi_0|^{8+\delta}\leqslant E(\xi_0^{\delta}\sum_{k=1}^{d_n}|a_ke_k|^2)$$

$$\leqslant \sum_{k}|a_k|^{8+\delta}E|e_k|^{8+\delta}+\sum_{k\neq j}\Big(\sum_{i=1}^{8}|a_k|^{i}|a_j|^{8-i+\delta}$$

$$\cdot E|e_k|^{i}|e_j|^{8-i+\delta}\Big)$$

$$+\cdots+\sum_{k_i\neq k_j,i\neq j}|a_{k_1}\cdots a_{k_8}||a_{k_9}|^{\delta}E|e_{k_1}\cdots e_{k_8}||e_{k_9}|^{\delta}$$

$$\leqslant cd_n^{(10-\delta)/2}\Big(\sum_{k=1}^{d_n}a_k^2\Big)^{(8+\delta)/2}.$$

设$\{\xi'_j,j=0,1,\cdots,h_n\}$是相互独立的随机变量,$\xi'_j$与$\xi_j$有相同的分布.记

$$\xi''_j=d_n^{-\frac{10-\delta}{2(8+\delta)}}\Big(\sum_{k=2jd_n+1}^{(2j+1)d_n}a_k^2\Big)^{-1/2}\xi'_j$$

和

$$a'_j=\Big(\sum_{k=2jd_n+1}^{(2j-1)d_n}a_k^2\Big)^{1/2}.$$

那么

$$\sum_{j=0}^{h_n}a_j'^2\leqslant\sum_{k=1}^{n}a_k^2\leqslant 1.$$

应用注 13.2.1 于 $X=\sum_{j=0}^{h_n}a'_j\xi''_j$ 和 $b=\varepsilon d_n^{-(10-\delta)/(2(8+\delta))}n^{1/8}$,我们得

$$(13.2.14)\qquad P\Big\{\Big|\sum_{j=0}^{h_n}\xi'_j\Big|\geqslant\varepsilon n^{1/8}\Big\}$$

$$=P\Big\{d_n^{-\frac{10-\delta}{2(8+\delta)}}\Big|\sum_{j=0}^{h_n}\xi'_j\Big|\geqslant\varepsilon d_n^{-\frac{10-\delta}{2(8+\delta)}}n^{1/8}\Big\}$$

$$= o(d_n^{5-\delta/2} n^{-1-\delta/8}) = o(n^{-1-\delta(1-\theta)/8}).$$

由 $\varphi$ 混合性

$$|P\{\xi_1 + \xi_2 \leqslant x\} - P\{\xi'_1 + \xi'_2 \leqslant x\}|$$

$$\leqslant \int |P\{\xi_1 \leqslant x - u | \xi_2 = u\} - P\{\xi_1 \leqslant x - u\}| dP\{\xi_2 \leqslant x\}$$

$$\leqslant \varphi(d_n).$$

因此

(13.2.15) $$|P\Big\{\sum_{j=0}^{h_n} \xi_j \leqslant x\Big\} - P\Big\{\sum_{j=0}^{h_n} \xi'_j \leqslant x\Big\}| \leqslant O(h_n \varphi(d_n)) = O(n^{-1-\epsilon}).$$

结合(13.2.14)和(13.2.15)得到

$$P\Big\{\Big|\sum_{j=1}^{h_n} \xi_j\Big| \geqslant \epsilon n^{1/8}\Big\} = O(n^{-1-\epsilon}).$$

由 Borel-Cantelli 引理,

$$\Big|\sum_{j=0}^{h_n} \xi_j\Big| = o(n^{1/8}) \text{ a. s.}$$

类似地

$$\Big|\sum_{j=0}^{h_n-1} \eta_j\Big| = o(n^{1/8}) \text{ a. s.}$$

引理证毕.

定理 13.2.2 的证明.

写

$$Z(t) = \sum_{k=1}^{[t]} (e_k^2 - \sigma^2) + r_{[t]}\sigma^2 - \sum_{i=1}^{r[t]} \Big(\sum_{j=1}^{[t]} a_{ij}^{([t])} e_j\Big)^2$$

$$=: X(t) + r_{[t]}\sigma^2 - \sum_{i=1}^{r[t]} \Big(\sum_{j=1}^{[t]} a_{ij}^{([t])} e_j\Big)^2.$$

从引理 13.2.3 我们有

(13.2.16) $$\Big(\sum_{j=1}^{[t]} a_{ij}^{([t])} e_j\Big)^2 = o(t^{1/4}) \text{ a. s.} \quad t \to \infty.$$

由定理 9.1.1,存在 Wiener 过程$\{W(t), t \geqslant 0\}$,使对任意的 $\epsilon > 0$

(13.2.17) $$X(t) - W(t) = O(t^{1/4} (\log t)^{9/4+\epsilon}) \text{ a. s.} \quad t \to \infty.$$

结合(13.2.16)和(13.2.17)即得定理之结论.

## §13.3 密度估计

设$\{X_n, n\geqslant 1\}$是同分布的$R^d$值的随机变量序列,具有相同的密度函数$f(x)$.常见的密度估计有两类,其一称为核估计,它定义作

(13.3.1) $$f_n(x)=(nh_n^d)^{-1}\sum_{i=1}^{n}K\left(\frac{x-X_i}{h_n}\right),$$

其中$K(\cdot)$称为核函数,$h_n$称为窗宽,当$n\to\infty$时,$h_n\downarrow 0$,另一类称为最近邻估计,它定义作

(13.3.2) $$\hat{f}_n(x)=k_n/\{n|S(x,a_n(x))|\},$$

其中$k_n$,$1\leqslant k_n\leqslant n$,是给定的整数,$a_n(x)$是$x$到$(X_1,\cdots,X_n)$中最接近它的$X_i$之间的距离,$S(x,a)$是以$x$为中心,$a$为半径的超球,$|S(x,a)|=L(s(x,a))$,其中$L$是$R^d$中的Lebesgue测度.

本节中总假设$\{X_n\}$是$\varphi$混合的.其它类型的混合序列可类似地处理.

### 13.3.1 核估计

许多作者,例如林正炎(1983)、Masry和Györfi(1987),邵启满(1990),蔡宗武(1991)、Peligrad(1992),樊家琨和薛留根(1993)等都曾研究过混合序列密度函数的核估计.

设$\{X_n, n\geqslant 1\}$是具有共同的未知密度函数$f(x)=f(x_1,\cdots,x_d)$的$R^d$值的$\varphi$混合序列.考虑由(13.3.1)定义的核估计$f_n(x)$.Peligrad(1992)证明了下列结果.

**定理13.3.1** 假设$D$是$R^d$的紧致子集,$f$在$D$的一个$\varepsilon$邻域上连续,假设$K$满足下列条件:

1) $K(\cdot)$是$R^d$上的密度函数,

2) 对任意$x\in R^d$,$K(x)\leqslant K_1<\infty$,

3) 当$x\to\infty$时$\|x\|^{d+1}K(x)\to 0$,

4) $\int \| x \| K(x) dx = K_2 < \infty$,

5) $K(\cdot)$满足 $R^d$ 上的 $\gamma$ 阶 Lipschitz 条件.

那么

(13.3.3) $\quad \sup_{x \in D} |f_n(x) - f(x)| = O(h_n + d_n^{1/2} \log n / (nh_n^d)^{1/2})$ a.s.

其中 $d_n = \exp(2 \sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i))$. 如果附加条件

(13.3.4) $$\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty,$$

则对 $h_n = O((\log^2 n / n)^{1/(d+2)})$,

(13.3.5) $\sup_{x \in D} |f_n(x) - f(x)| = O((\log^2 n/n)^{1/(d+2)})$ a.s.

**注 13.3.1** 如果条件 3)被减弱为

3)′ 当 $x \to \infty$ 时 $\| x \|^d K(x) \to 0$

且条件 4)被删除,而条件(13.3.4)被代之以

$$\lim_{n \to \infty} \varphi(n) < \frac{1}{2}$$

和

$$nh_n^d / (d_n \log^2 n) \to \infty \quad n \to \infty,$$

则有

$$\sup_{x \in D} |f_n(x) - f(x)| \to 0 \quad \text{a.s. } n \to \infty.$$

**注 13.3.2** 在独立情形,核估计与总体密度偏差的上确界的收敛速度是 $O((\log\log n)^{1/2}/n^{1/(2(d-1))})$(Kuelbs1976). (13.3.5)改进这个速度为 $O((\log^2 n/n)^{1/(d+2)})$.

**注 13.3.3** 邵启满(1990)进一步改进这个速度到 $O((\log n/n)^{1/(d+2)})$,但是他要求较强的混合条件:$\varphi(n) = O(n^{-(2+d)})$.

定理 13.3.1 的证明.

显然,由引理 2.2.2,在条件(13.3.4)之下,若在(13.3.3)中取 $h_n = O((\log^2 n/n)^{1/(d+2)})$就得(13.3.5). 因此只需证明(13.3.3)就够了. 写

(13.3.6) $\qquad \sup_{x \in D} |f_n(x) - f(x)|$

$$\leqslant \sup_{x\in D}|f_n(x)-Ef_n(x)|+\sup_{x\in D}|Ef_n(x)-f(x)|$$
$$=:\Delta_1+\Delta_2.$$

首先估计 $\Delta_1$. 因为 $D$ 是紧致的，所以可以选取 $l(n)$ 个中心在 $t_1,\cdots,t_{l(n)}$ 半径为

$$R_n=h_n(n^{-1}h_n^d\log^2 n)^{1/2\gamma}$$

的球 $B_1,\cdots,B_{l(n)}$ 覆盖 $D$；数 $l(n)$ 可以选得小于 $O(h_n^{-d}(n/(h_n^d\log^2 n))^{d/(2\gamma)})$. 不失一般性，可设 $h_n>n^{-2/d}$. 因此

(13.3.7) $$l(n)=O(n^{2-3d/(2\gamma)}).$$

设 $x\in D$，定义

$$S_j(x)=h_n^{-d/2}\sum_{i=1}^{j}\left[K\left(\frac{x-X_i}{h_n}\right)-EK\left(\frac{x-X_i}{h_n}\right)\right].$$

于是

$$\Delta_1=\sup_{x\in D}(nh_n^{d/2})^{-1}S_n(x).$$

由条件 5)，存在 $C>0$，对任意的 $x\in B_k$，我们有

(13.3.8) $$|S_n(x)-S_n(t_k)|\leqslant CnR_n^\gamma h_n^{-d/2-\gamma}\leqslant C(n\log^2 n)^{1/2}.$$

而由引理 2.2.2，存在 $G>0$，使得

(13.3.9) $$ES_n^2(x)\leqslant Gnd_n.$$

此外，从条件 2)

$$h_n^{-d/2}\left|K\left(\frac{x-X_i}{h_n}\right)-EK\left(\frac{x-X_i}{h_n}\right)\right|/(Gnd_n)^{1/2}$$
$$\leqslant 2K_1(Gnd_nh_n^d)^{-1/2}=:C_n.$$

令 $A(>C)$ 是一待定常数，由(13.3.8)我们有

(13.3.10) $$P\{\sup_{x\in D}|S_n(x)|\geqslant 2A(n\log^2 n)^{1/2}\}$$
$$\leqslant\sum_{k=1}^{l(n)}(P\{\sup_{x\in B_k}|S_n(x)-S_n(t_k)|\geqslant A(n\log^2 n)^{1/2}\}$$
$$+P\{|S_n(t_k)|\geqslant A(n\log^2 n)^{1/2}\})$$
$$\leqslant l(n)\max_{1\leqslant k\leqslant l(n)}P\{|S_n(t_k)|\geqslant A(n\log^2 n)^{1/2}\}.$$

估计(13.3.10)的右边的概率. 易知对任意 $\eta$，$0<\eta<1/2$，存在 $p\geqslant 1$ 和 $A>0$ 使对一切 $n\geqslant p$

$$\varphi(p)+\max_{1\le i\le n}P\{|S_n(t_k)-S_i(t_k)|/(Gnd_n)^{1/2}\geqslant A\}\leqslant\eta.$$

结合引理 2.2.7 中的(2.2.18)和(2.2.19)，我们有

(13.3.11) $$P\{\max_{1\le j\le n}|S_j(t_k)|/(Gnd_n)^{1/2}\geqslant x+2A+2PC_n\}\leqslant\frac{\eta}{1-\eta}P\{\max_{1\le j\le n}|S_j(t_k)|/(Gnd_n)^{1/2}\geqslant x\}.$$

记 $B_n=2A+2PC_n$，$M_n=\max_{1\le j\le n}|S_j(t_k)|/(Gnd_n)^{1/2}$. 显然，对任意的 $\alpha_n>0$

$$E\exp(\alpha_n M_n)\leqslant\exp(\alpha_n B_n)+\alpha_n\int_{B_n}^{\infty}\exp(\alpha_n x)P(M_n>x)dx.$$

作变换 $x\to x+B_n$，由(13.3.11)我们得

$$E\exp(\alpha_n M_n)\leqslant\exp(\alpha_n B_n)+\frac{\eta}{1-\eta}E\exp\{\alpha_n(M_n+B_n)\}.$$

令 $\alpha_n=(2B_n)^{-1}\log(\eta^{-1}-1)$，即有

$$E\exp(\alpha_n M_n)\leqslant((\eta^{-1}-1)^{-1/2}-(\eta^{-1}-1)^{-1})^{-1}=:g(\eta).$$

记 $\alpha=\inf_n\alpha_n=(4A)^{-1}\log(\eta^{-1}-1)$，则当 $n\to\infty$ 时对 $k$ 一致地有

(13.3.12) $$\begin{aligned}&P\{\alpha(\log n)^{-1}(Gnd_n)^{-1/2}|S_n(t_k)|\geqslant 3d/(2\gamma)+4\}\\&\leqslant g(\eta)\exp\{-(3d/(2\gamma)+4)\log n\}\\&=O(n^{-(3d/(2\gamma)+4)}).\end{aligned}$$

记 $A=2((3d/2\gamma+4)G^{1/2}/\log(\eta^{-1}-1))^{1/2}=(3d/2\gamma+4)G^{1/2}/\alpha$. 从(13.3.7)、(13.3.10)和(13.3.12)，我们得

(13.3.13) $$\sum_{n=1}^{\infty}P\{\sup_{x\in D}|S_n(x)|\geqslant 2A(nd_n\log^2 n)^{1/2}\}<\infty.$$

因此当 $n\to\infty$ 时

$$\Delta_1=O\left(\left(\frac{d_n\log^2 n}{nh_n^d}\right)^{1/2}\right)\quad\text{a.s.}$$

至于 $\Delta_2$，由熟知的 Bochner-Parzen 定理(参见 Parzen1962)，在定理的条件下有

$$\Delta_2=O(h_n)\qquad n\to\infty.$$

定理证毕.

### 13.3.2 最近邻估计

柴根象(1984)研究了混合序列的密度函数的最近邻估计的强

相合性，利用改进的 Bernstein 不等式(引理 12.4.1)，他的定理在较弱的混合条件下也成立.

**定理 13.3.2** 假设满足条件(13.3.4)且(13.3.2)中的 $k_n$ 满足

(13.3.14) $$k_n \to \infty, k_n/n \to 0 \quad n \to \infty.$$

则由 $k_n/\sqrt{n} \to \infty$ 可推出

(13.3.15) $$\hat{f}_n(x) \xrightarrow{p} f(x) \quad \text{a.s. } x \in R^d(L);$$

且由 $\sum_{n=1}^{\infty} \exp(-ck_n^2/n) < \infty$(任意的 $c>0$)推出

(13.3.16) $$\hat{f}_n(x) \to f(x) \text{a.s.}, \text{a.s. } x \in R^d(L).$$

**证** 记 $V_d$ 是 $R^d$ 中的单位球的体积，$\mu$ 和 $\mu_n$ 分别是 $X_1$ 的分布和 $X_1, \cdots, X_n$ 的经验分布. 对任意给定的 $\varepsilon>0$，记

$$b_n(x) = (f(x)+\varepsilon)V_d n/k_n, b'_n(x) = (f(x)-\varepsilon)V_d n/k_n,$$

$$S_n(x,b) = S(x, b_n^{-1/d}(x)), S_n(x,b') = S(x, b_n'^{-1/d}(x)).$$

则有

(13.3.17) $$\begin{aligned} P\{|\hat{f}_n(x) - f(x)| > \varepsilon\} &\leqslant P\{\hat{f}_n(x) - f(x) > \varepsilon\} + P\{\hat{f}_n(x) - f(x) < -\varepsilon\} \\ &\leqslant P\{\mu_n(S_n(x,b)) - \mu(S_n(x,b)) \geqslant \frac{k_n}{n} - \mu(S_n(x,b))\} \\ &\quad + P\left\{\mu_n(S_n(x,b')) - \mu(S_n(x,b')) \geqslant \frac{k_n}{n} - \mu(S_n, b')\right\} \end{aligned}$$

(如果 $f(x) \leqslant \varepsilon$，第一和第二个不等号的右方的第二项不出现). 由熟知的 Lebesgue 密度定理，当 $n \to \infty$ 时有

$$\mu(S_n(x,b))/|S_n(x,b)| \to f(x) \quad \text{a.s. } x \in R^d(L),$$

$$\mu(S_n(x,b'))/|S_n(x,b')| \to f(x) \quad \text{a.s. } x \in R^d(L).$$

两个例外集分别记作 $D$ 和 $D'$，又记 $E = D^c \cap D'^c$. 则对任意的 $x \in E$ 和充分大的 $n$

$$\mu(S_n(x,b)) \leqslant \frac{k_n}{n}\left(f(x) + \frac{\varepsilon}{2}\right)/(f(x)+\varepsilon),$$

$$\mu(S_n(x,b')) \geqslant \frac{k_n}{n}\left(f(x) - \frac{\varepsilon}{2}\right)/(f(x)-\varepsilon).$$

记 $a(x)=\varepsilon/(2(f(x)+\varepsilon))$ 和 $a'(x)=\varepsilon/(2(f(x)-\varepsilon))$. 对任意的 $x\in E$,

$$
\begin{aligned}
& P\{|\hat{f}_n(x)-f(x)|>\varepsilon\} \\
\leqslant & P\left\{|\mu_n(S_n(x,b))-\mu(S_n(x,b))|\geqslant\frac{k_n}{n}a(x)\right\} \\
& + P\{|\mu_n(S_n(x,b'))-\mu(S_n(x,b'))|\geqslant\frac{k_n}{n}a'(x)\} \\
= & :I_{n1}+I_{n2}.
\end{aligned}
$$

令 $\xi_i=I(X_i\in S_n(x,b))-\mu(S_n(x,b))$, $i=1,\cdots,n$. 那么, 利用引理 12.4.1 就有

$$I_{n1}\leqslant 2\exp\left\{-c\frac{k_n^2}{n}a^2(x)\right\}$$

和

$$I_{n2}\leqslant 2\exp\left\{-c\frac{k_n^2}{n}a'^2(x)\right\}.$$

于是, 当 $k_n/\sqrt{n}\to\infty$ 时得证(13.3.15); 当 $\sum_{n=1}^{\infty}\exp(-ck_n^2/n)<\infty$(对任意的 $c>0$)时得证(13.3.16).

**注 13.3.4** 通过更精细的分析, 如果满足条件(13.3.4)和(13.3.14), 而且 $f(x)>0$ 且满足局部 Lipschitz 条件, 那么可证明下列 a.s. 相合性的速度:

$$\hat{f}_n(x)-f(x)=o(r_n^{-1})\quad \text{a.s.}$$

其中 $r_n=n^\beta$, $0<\beta<1/(2(d+1))$.

下面对 $d=1$ 的情形我们来讨论一致强相合性, 这时 $\hat{f}_n(x)=k_n/(2na_n(x))$, $x\in R$. 首先给出两个引理. 令 $F$ 是 $X_1$ 的分布, $F_n$ 是 $X_1,\cdots,X_n$ 的经验分布, 定义 $\{X_n,n\geqslant 1\}$ 的经验过程:

$$R(s,t)=[t](F_{[t]}(s)-F(s)),\ s\in R,t\geqslant 0.$$

**引理 13.3.1**(Berkes 和 Philipp1977) 假设

(13.3.18) $\qquad \varphi(n)=O(n^{-5-\delta})$, 某个 $\delta\in(0,1/4)$.

那么存在 Kiefer 过程 $K(s,t)$ 使得对某个 $\lambda>0$

$$\sup_{t\le T}\sup_{s\in R}|R(s,t)-K(F(s),t)|=O(T^{1/2}(\log T)^{-\lambda})\text{ a. s.}$$

**引理 13.3.2**（Csörgö和Révész 1981） 对 Kiefer 过程 $K(s,t)$，

$$\limsup_{t\to\infty}\sup_{0\le s\le 1}|K(s,t)|/(t\log\log t)^{1/2}=1/\sqrt{2}\quad\text{a. s.}$$

**定理 13.3.3** 假设满足(13.3.18)且$\{k_n\}$满足

(13.3.19) $$k_n/n\to 0,\ k_n/(n\log\log n)^{1/2}\to\infty.$$

又假设 $f(\cdot)$在 $R$ 上是一致连续的，那么

$$\sup_x|\hat{f}_n(x)-f(x)|\to 0\quad\text{a. s.}$$

**证** 由引理 13.3.1，对任意的 $\varepsilon>0$ 存在常数 $a>0$ 使得

(13.3.20) $$P\{A_n,\ i.o.\}\le\varepsilon,$$

其中 $A_n=\{\sup_x|R(x,n)-K(F(x),n)|\ge an^{1/2}(\log n)^{-\lambda}\}$. 类似地由引理 13.3.2，对任意的 $\varepsilon>0$ 存在常数 $b>0$ 使得

(13.3.21) $$P\{B_{n,i.o.}\}\le\varepsilon,$$

其中 $B_n=\{\sup_{0\le s\le 1}|K(s,n)|\ge b(n\log\log n)^{1/2}\}$. 记

$$d_n(x)=\frac{k_n}{2n(f(x)+\varepsilon)},\ d'_n(x)=\frac{k_n}{2n(f(x)-\varepsilon)}$$

和

$$S_n(x,d_n)=(x-d_n(x),x+d_n(x)),$$
$$S_n(x,d'_n)=(x-d'_n(x),x+d'_n(x)).$$

于是类似于(13.3.17)，

(13.3.22) $$\begin{aligned}&P\{\bigcup_{n\ge m}\{\sup_x|\hat{f}_n(x)-f(x)|>\varepsilon\}\}\\&\le P\{\bigcup_{n\ge m}\bigcup_x\{\mu_n(S_n(x,d_n))-\mu(S_n(x,d_n))\\&\ge\frac{k_n}{n}-\mu(S_n(x,d_n))\}\}\\&\quad+P\{\bigcup_{n\ge m}\bigcup_{x:f(x)>\varepsilon}\{\mu_n(S_n(x,d'_n))-\mu(S_n(x,d'_n))\\&\ge\frac{k_n}{n}-\mu(S_n(x,d'_n))\}\}\\&=:J_{m1}+J_{m2}.\end{aligned}$$

由 $f$ 的一致连续性，$M:=\sup_x f(x)<\infty$，而且进一步对任意的 $x$ 和充分大的 $n$

$$\mu(S_n(x,d_n))\leqslant\frac{k_n}{n}\left(f(x)+\frac{\varepsilon}{2}\right)/(f(x)+\varepsilon),$$

$$\frac{k_n}{n}-\mu(S_n(x,d_n))\geqslant\frac{k_n}{n}\cdot\frac{\varepsilon}{2(f(x)+\varepsilon)}\geqslant\frac{k_n}{n}\cdot\frac{\varepsilon}{2(M+\varepsilon)}=:p_n;$$

且当 $f(x)>\varepsilon$ 时,

$$\mu(S_n(x,d_n'))\geqslant\frac{k_n}{n}\left(f(x)-\frac{\varepsilon}{2}\right)/(f(x)-\varepsilon),$$

$$\mu(S_n(x,d'_n))-\frac{k_n}{n}\geqslant\frac{k_n}{n}\cdot\frac{\varepsilon}{2(f(x)-\varepsilon)}\geqslant\frac{k_n}{n}\cdot\frac{\varepsilon}{2(M-\varepsilon)}=:q_n.$$

因此

$$\begin{aligned}J_{m1}&\leqslant P\{\bigcup_{n\geqslant m}\{\sup_x|\mu_n(S_n(x,d_n))-\mu(S_n(x,d_n))|\geqslant p_n\}\}\\&\leqslant 2P\left\{\bigcup_{n\geqslant m}\left\{\sup_x|F_n(x)-F(x)|\geqslant\frac{p_n}{2}\right\}\right\}\\&\leqslant 2P\left\{\bigcup_{n\geqslant m}\left\{\sup_x\left|\frac{R(x,n)-K(F(x),n)}{n}\right|\geqslant\frac{p_n}{4}\right\}\right\}\\&\quad+2P\left\{\bigcup_{n\geqslant m}\left\{\sup_x\frac{|K(F(x),n)|}{n}\geqslant\frac{p_n}{4}\right\}\right\}\\&=:2J_{m1}^{(1)}+2J_{m1}^{(2)}.\end{aligned}$$

由条件(13.3.19)我们有

$$\sqrt{n}\,p_n(\log n)^\lambda\to\infty,n\to\infty.$$

因此从(13.3.20)可得:当 $m\to\infty$ 时

$$\begin{aligned}J_{m1}^{(1)}&\leqslant P\left\{\bigcup_{n\geqslant m}\left\{\sup_x\left|\frac{R(x,n)-k(F(x),n)}{n^{1/2}(\log n)^{-\lambda}}\right|\geqslant\frac{n^{1/2}p_n(\log n)^\lambda}{4}\right\}\right\}\\&\leqslant P\{\bigcup_{n\geqslant m}A_n\}\to 0.\end{aligned}$$

类似地从(13.3.21)又有:当 $m\to\infty$ 时

$$J_{m1}^{(2)}\leqslant P\{\bigcup_{n\geqslant m}B_n\}\to 0.$$

于是当 $m\to\infty$ 时 $J_{m1}\to 0$. 此外

$$J_{m2}\leqslant P\{\bigcup_{n\geqslant m}\{\sup_x|\mu_n(S_n(x,d'_n))-\mu(S_n(x,d'_n))|\geqslant q_n\}\}.$$

通过对 $J_{m1}$ 相同的处理方式,也有 $J_{m2}\to 0\,(m\to\infty)$. 这样,从(13.3.22),对任意的 $\varepsilon>0$ 都有

$$P\{\bigcup_{n\geq m}\{\sup_x|\hat{f}_n(x)-f(x)|>\epsilon\}\}\to 0,\ m\to\infty.$$

这就完成了定理的证明.

**注 13.3.5** 如果 $f$ 具有有界的二阶导数，那么对于 $k_n=[n^{7/10}]$和任意的 $C_n\to\infty$成立着

$$\sup_x|\hat{f}_n(x)-f(x)|=O(n^{-1/5}(\log\log n)^{1/2}c_n)\text{a. s.}$$

**注 13.3.6** 俞军(1993)给出了另一类密度函数的最近邻估计. 记$(X_{(1)},\cdots,X_{(n)})$为样本$(X_1,\cdots,X_n)$的次序统计量，$m=m_n$ 是一个正整数，定义

$$f_n(x)=\begin{cases}\dfrac{2m_n}{n(X_{(2m_n+j)}-X_{(j+1)})} & x\in[X_{(m_n+j)},X_{(m_n+j+1)}),\\ & j=0,1,\cdots,n-2m_n;\\ 0 & x<X_{(m_n)}\text{或 } x\geq X_{(m-m_n-1)}.\end{cases}$$

在某些附加条件下. 俞军(1993)对 $i,i,d.$ 样本证明了 $f_n(x)$依概率和 a.s. 收敛于 $f(x)$，俞军(1995)对 $\varphi$混合和 $\alpha$ 混合样本给出了 $f_n(x)$强一致收敛于 $f(x)$的速度.

# 第十四章　其他相依随机变量的强逼近

Philipp 和 Stout(1975)已给出了各种弱相依随机变量和的 a. s. 不变原理，如对于缺项三角级数，$\varphi$ 混合和 $\alpha$ 混合序列，一类 Gauss 序列及马氏链的可加泛函等. $\varphi$ 混合及 $\alpha$ 混合序列的部分和的强逼近已在第九章作了研究. 在本章中，我们将研究其他各种相依随机变量的部分和的强逼近，包括加权缺项三角级数，一类 Gauss 序列及马氏过程的可加泛函. 所有这些都本质且全面地改进了 Philipp 和 Stout(1975)中的结果.

## §14.1　加权缺项三角级数

设正实数列 $\{n_k, k\geq 1\}$ 对所有 $k$，某 $q>0$ 和 $0\leq r<\frac{1}{2}$ 满足：

(14.1.1)　$$n_{k+1}/n_k \geq 1+q/k^r.$$

当 $r=0$ 时称它是一缺项序列. 设 $\{a_k, k\geq 1\}$ 是非零实数序列. 令

(14.1.2)　$$A_n^2=\frac{1}{2}\sum_{k=1}^{n}a_k^2,\qquad B_n^2=\frac{1}{2}\sum_{k=1}^{n}a_k^4,$$

假设 $A_n\to\infty$ 且存在常数 $\delta,\beta$，$0\leq\delta<1$，$\beta>0$ 使得

(14.1.3)　$$a_k=O(A_k^{1-\delta}),$$

(14.1.4)　$$A_{2k}=O(A_k),$$

(14.1.5)　$$k^\beta=O(A_k).$$

在本节中，$([0,1),\mathscr{B},P)$ 是一概率空间，其中 $\mathscr{B}$ 是由 $[0,1]$ 的全体 $L$ 可测子集组成的，$P$ 是 $\mathscr{B}$ 上 Lebesgue 测度. 我们考察三角级数

(14.1.6)　$$S(A_n^2,w)=\sum_{k=1}^{n}a_k\cos 2\pi n_k w\quad w\in[0,1).$$

对 $t\geq 0$，令

(14.1.7) $\quad S(t,w)=S(A_n^2,w),\quad$ 当 $A_n^2\leqslant t\leqslant A_{n+1}^2$,

其中 $A_0=0$.

$S(t)$用 Wiener 过程 $W(t)$来强逼近首先是被 Gaposhkin (1966)所讨论. 随后,在 $r=0$ 情形,Philipp 和 Stout(1975)在条件(14.1.3)下对缺项三角级数证明了几乎必然不变原理,得到逼近阶为 $1/2-\lambda$,其中 $\lambda<\delta/32$. 在非加权情形,他们得到逼近阶为 $5/12+\lambda,\lambda>0$ 且指出常数 5/12 可被 1/3 所代替. 孙志刚(1984)证明了这一事实. 邵启满(1987)全面地改进所有这些结果,他研究了加权缺项三角级数的一般情形,指出了孙志刚所获得阶不是最佳的. 进一步,在某些特殊情形,得到对数的逼近阶.

**定理 14.1.1** 假设条件(14.1.3)—(14.1.5)被满足且 $\delta\beta>r$. 那么在其上有一 Wiener 过程$\{W(t),t\geqslant0\}$的较大概率空间上可不改变分布地重新定义过程$\{S(t),t\geqslant0\}$使得

(14.1.8) $$S(t)-W(t)=O\left(t^{\frac{2-\delta}{4}+\frac{r}{4\beta}}\log^2 t\right)\quad \text{a.s.}$$

**定理 14.1.2** 在定理 14.1.1 的条件下且设$|a_k|$是不增的,那么我们有

(14.1.9) $$\begin{aligned}&S(A_n^2)-W(A_n^2)\\&=O\left(A_n^{r/(2\beta)}(B_n^{1/2}VA_n^{r/(2\beta)})\log^2 A_n\right)\quad \text{a.s.}\end{aligned}$$

从上述一般性定理,对无权重项的缺项三角级数直接地可得如下逼近结果.

**推论 14.1.1** 若 $a_k\equiv1$,那么我们有

$$S(t)-W(t)=O\left(t^{\frac{1+2r}{4}}\log^2 t\right)\quad \text{a.s.}$$

在 $r=0$ 情形,(14.1.10)是 Philipp 和 Stout(1975)中定理 3.1 的一个实质性改进,除非不用 Skorohod 嵌入法,1/4 的阶是最佳可能.

**推论 14.1.2** 假设定理 14.1.2 的条件和条件(14.1.5)对某 $\beta>0$ 被满足且 $B_N=O(1)$,$|a_k|\downarrow0$. 那么当 $r=0$ 时

$$S(t)-W(t)=O(\log^2 t)\quad \text{a.s.}$$

定理的证明需要下列引理.

**引理 14.1.1** 设 $\xi_1,\cdots,\xi_n$ 是随机变量. 记

$$S_k=\sum_{j=1}^{k}\xi_j,\quad M_k=\max_{1\leqslant j\leqslant k}|S_j|.$$

假设存在正数列 $\{C_k\}$ 使对所有 $0\leqslant i\leqslant j\leqslant n$ 有

$$E|S_j-S_i|^2\leqslant\sum_{i<k\leqslant j}c_k,$$

那么对每一 $k\leqslant n$

$$EM_k^2\leqslant\Big(\sum_{i=1}^{k}c_i\Big)\Big(\frac{\log 2k}{\log 2}\Big)^2$$

其证明参见 Stout(1974)的定理 2.4.1.

**引理 14.1.2** 对每一常数 $v\geqslant 0$ 我们有

$$\sum_{j\geqslant k}\frac{A_j^v}{n_j}=O\Big(\frac{A_k^v}{n_k}k^r\Big),\tag{14.1.10}$$

$$\sum_{j>k}\frac{A_j^v}{n_j-n_k}=O\Big(\frac{A_k^v}{n_k}k^{2r}\Big),\tag{14.1.11}$$

**证** 由(14.1.4)可知存在常数 $C>0$ 使得对所有 $k,j\geqslant 1$ 有

$$A_{k+j}=O\Big(\Big(\frac{k+j}{j}\Big)^C A_j\Big)$$

且由 $n_{k+1}/n_k\geqslant 1+q/k^r$, 我们有

$$\begin{aligned}\frac{n_{k+j}}{n_k}&\geqslant\prod_{i=1}^{j}\Big(1+\frac{q}{(k+i)^r}\Big)\\&=O\Big(\exp\Big(\frac{q((k+j)^{1-r}-k^{1-r})}{2(1-r)}\Big)\Big),\end{aligned}\tag{14.1.12}$$

其中我们应用了熟知的公式

$$\sum_{k=1}^{n}k^{-r}=\frac{n^{1-r}}{1-r}+u+O(n^{-r}),\Big(0\leqslant r\leqslant\frac{1}{2}\Big),$$

这里 $u$ 是一个常数. 因此

$$\begin{aligned}\sum_{j\geqslant k}\frac{A_j^v}{n_j}&=O\Big(\frac{A_k^v}{n_k k^C}\Big(\sum_{j\geqslant k}j^C\exp\Big(-\frac{qj^{1-r}}{2(1-r)}\Big)\Big)\exp\Big(\frac{qk^{1-r}}{2(1-r)}\Big)\Big)\\&=O\Big(\frac{A_k^v}{n_k}k^r\Big).\end{aligned}$$

这样(14.1.10)被证明. (14.1.11)的证明是类似的.

**引理 14.1.3** 设$\{W(t),t\geqslant 0\}$是 Wiener 过程且$\{t_n\}$是随机变量序列. 假设存在实数列$\{b_n\}$，$b_n=o(n)$，使得

(14.1.13) $$t_n - n = O(b_n) \quad \text{a.s.},$$

那么有

(14.1.14) $$W(t_n) - W(n) = O(b_n^{1/2}\log^{1/2} n) \quad \text{a.s.}$$

**证** 由(14.1.13)可知存在常数$C>0$使得

$$|t_n - n| \leqslant Cb_n \quad \text{a.s.},$$

因此

$$|W(t) - W(n)| \leqslant \sup_{0\leqslant t\leqslant n-b_n} \sup_{0\leqslant s\leqslant 2Cb_n} |W(t+s) - W(t)| \text{a.s.}$$

借助熟知的定理(参见 Hanson 和 Russo 1983 定理 3.2$B$)，我们有

$$\sup_{0\leqslant t\leqslant n-b_k} \sup_{0\leqslant s\leqslant 2Cb_n} |W(t+s) - W(t)|$$

$$= O\left(\left(b_n\left(\log\frac{n+Cb_n}{2Cb_n} + \log\log(n+Cb_n)\right)\right)^{1/2}\right) \text{a.s.},$$

从上述关系式即得(14.1.14).

现在我们定义一个$\sigma$域的增序列$\{\mathscr{F}_k\}$如下：对每一整数$k$记$p=2/\beta+4$，设$r_k$是使下式成立的最大整数$i$

(14.1.15) $$2^i \leqslant A_k^p n_k.$$

定义$\mathscr{F}_k$是形如下的区间所生成的$\sigma$域：

$$U_{v,k} = [v2^{-r_k}, (v+1)2^{-r_k}), \quad 0\leqslant v < 2^{r_k}.$$

令$\xi_k(w)=a_k\cos 2\pi n_k w$，$X_k(w)=E(\xi_k|\mathscr{F}_k)$. 由(14.1.15)即得对每一$k,j\geqslant 1$有

(14.1.16) $$E(\xi_{k+j}|\mathscr{F}_j) = O(a_{k+j}(1\wedge A_j^p n_j/n_{k+j})).$$

**引理 14.1.4** 我们有

(14.1.17) $$\sum_{k=1}^{\infty} |\xi_k - X_k| = O(1).$$

**证** 从(14.1.15)我们有

$$|\xi_k - X_k| = O(A_k^{-p}a_k) = O(A_k^{-p+1}).$$

因此从(14.1.5)即得(14.1.17).

**引理 14.1.5** 我们有

(14.1.18) $$\sum_{1\leqslant j\leqslant N} EX_j^2 - A_N^2 = O(1).$$

**证** 注意到

$$E\xi_j^2 - a_j^2/2 = a_j^2\sin 2\pi n_j/2\pi n_j$$

$$\sum_{j\leqslant N} E\xi_j^2 - A_j^2 = O\Big(\sum_{j\leqslant N} a_j^2/n_j\Big) = O(1).$$

和

$$\sum_{j\leqslant N}\big(EX_j^2 - E\xi_j^2\big) = O\Big(\sum_{j\leqslant N} a_j^2 A_j^{-p}\Big) = O(1),$$

从上述方程得(14.1.18)成立.

现在我们可表 $X_j$ 为

(14.1.19) $$X_j = Y_j + u_j - u_{j+1},$$

其中$\{r_j, \mathscr{F}_j\}$是鞅差序列且

(14.1.20) $$u_j = \sum_{k=0}^{\infty} E(X_{j+k}|\mathscr{F}_{j-1}), \quad j\geqslant 2.$$

**引理 14.1.6** 对所有 $j,n$, $1\leqslant j\leqslant n-1$,我们有

(14.1.21) $$\sum_{j<i<k\leqslant n} |E(X_iX_k|\mathscr{F}_j)| = O\big(n^{2r}A_n^{2-2\delta}\log^2 An\big),$$

其中常数 $C$ 与 $j$ 和 $n$ 无关.

**证** 由 $X_k$ 的定义,我们有

$$E(X_iX_k|\mathscr{F}_j) = E(X_i\xi_k|\mathscr{F}_j).$$

写

$$E(\xi_kX_i|\mathscr{F}_j) = \sum_{v=0}^{2^{r_j}-1} I(U_v, j)b_v.$$

但

$$\frac{2^{-r_j}}{a_ka_i}b_v = \sum_{l=0}^{2^{r_i-r_j}-1} 2^{r_i}\int_{v2^{-r_j}+l2^{-r_i}}^{v2^{-r_j}+(l+1)2^{-r_i}} \cos 2\pi n_k t\,dt\int_{v2^{-r_j}+l2^{-r_i}}^{v2^{-r_j}+(l+1)2^{-r_i}} \cos 2\pi n_i t\,dt$$

$$= 2^{r_i+1}\frac{\sin 2\pi n_i 2^{-r_i-1}\sin 2\pi n_k 2^{-r_i-1}}{(2\pi)^2 n_i n_k}\cdot$$

$$\cdot\sum_{l=0}^{2^{r_i-r_j}-1}\Big(\cos 2\pi(n_k - n_i)\Big(v2^{-r_j} + \Big(l+\frac{1}{2}\Big)2^{-r_i}\Big)$$

$$- \cos 2\pi(n_k + n_i)\left( v2^{-r_j} + \left( l + \frac{1}{2} \right) 2^{-r_i} \right) \Big).$$

利用等式

$$\sum_{v=0}^{n-1} \cos(av + b) = \frac{\sin an/2}{\sin a/2} \cos(b + a(n-1)/2),$$

其中 $a$ 和 $b$ 是使 $\sin(a/2) \neq 0$ 的任何实数，我们得

$$\frac{2^{-r_j} b}{a_k a_i} = \frac{2^{r_i+1} \sin 2\pi n_i 2^{-r_i-1} \sin 2\pi n_k 2^{-r_i-1}}{(2\pi)^2 n_i n_k}$$

$$\left( \frac{\sin 2\pi(n_k - n_i) 2^{-r_j-1}}{\sin 2\pi(n_k - n_i) 2^{-r_i-1}} \cos 2\pi(n_k - n_i)\left( v + \frac{1}{2} \right) 2^{-r_j} \right.$$

$$\left. + \frac{\sin 2\pi(n_k + n_i) 2^{-r_j-1}}{\sin 2\pi(n_k + n_i) 2^{-r_i-1}} \cos 2\pi(n_k + n_i)\left( v + \frac{1}{2} \right) 2^{-r_j} \right)$$

因此对所有 $i, k, v$

$$b_v = O(a_k a_i 2^{r_j} (1 + 2^{-r_j} n_i)/n_k))$$

且对 $i, k: n_k + n_i \leqslant 2^{r_i-1}$ 也有

$$b_v = O(a_k a_i (1 \wedge 2^{r_j}/(n_k - n_i)).$$

对每一 $i: j < i < n$，取 $k_0(i) = \max\{k: (n_k + n_i) \leqslant 2^{r_i - 1}\} \wedge (n-1)$. 由(14.1.2)和(14.1.15)得 $k_0(i) - i = O(i^r \log A_i)$，因此

$$(14.1.22) \qquad \sum_{j < i < k \leqslant n} E(X_i X_k | \mathscr{F}_j)$$

$$= O\left( \sum_{i=j+1}^{n-1} \sum_{k > k_0(i)}^{n} |a_k a_i| \left( \frac{2^{r_i}}{n_k} + \frac{n_i}{n_k} \right) \right.$$

$$\left. + \sum_{i=j+1}^{n-1} \sum_{k=i+1}^{k_0(i)} |a_k a_i| \left( 1 \wedge \frac{2^{r_j}}{n_k - n_i} \right) \right).$$

从引理 14.1.2 和(14.1.15)得(14.1.22)的右边第一部分被界于

$$A_n^{2-2\delta} \sum_{i=j+1}^{n-1} \frac{(2^{r_i} + n_i)}{n_{k_0}(i)} n^r = O\left( A_n^{2-2\delta} \sum_{i=j+1}^{n-1} \frac{(2^{r_j} + n_i)}{n_i A_i^{\rho}} n^r \right)$$

$$= O( A_n^{2-2\delta} n^{2r})$$

若我们取 $i(j) = \max\{i; 2^{r_j} \geqslant n_i i^{-3r}\} \wedge (n-1)$，那么由(14.1.12)和(14.1.15)得 $i(j) - i = O(j^r \log A_j)$. 又从引理 14.1.2，(14.1.22)的右边第二部分被界于

$$A_n^{2-2\delta}\left(\sum_{i=j}^{i(j)}\sum_{i<k\le k_0(i)}1+\sum_{i>i(j)}^{n-1}\sum_{k>i}^{k_0(i)}\frac{2^{r_j}}{n_k-n_i}\right)$$

$$=O\left(\left(n^{2r}\log^2 A_n+\sum_{i>i(j)}^{n-1}\frac{2^{r_j}}{n_{i+1}}i^{2r}\right)A_n^{2-2\delta}\right)$$

$$=O\left(n^{2r}\log^2 A_n+\frac{2^{r_j}(i(j))^{3r}}{n_{i(j)+1}}A_n^{2-2\delta}\right)$$

$$=O(n^{2r}A_n^{2-2\delta}\log^2 A_n).$$

引理 14.1.6 证毕.

**引理 14.1.7** 我们有

$$u_j=O(j^rA_j^{1-\delta}\log A_j).\tag{14.1.23}$$

**证** 利用(14.1.16),(14.1.20)和引理 14.1.2,我们有

$$u_j=\sum_{k=0}^{\infty}E(\xi_{k+j}|\mathscr{F}_{j-1})$$

$$=O\left(\sum_{k=0}^{\infty}A_{k+j}^{1-\delta}(1\wedge A_j^{\rho}n_j/n_{k+j})\right)$$

$$=O(j^rA_j^{1-\delta}\log A_j).$$

**引理 14.1.8** 对每一 $k,n(k<n)$,我们有

$$\sum_{k<i\le n}E(X_iu_{n+1}|\mathscr{F}_i)=O(A_n^{2-2\delta}n^{2r}\log^2 A_n);\tag{14.1.24}$$

$$\sum_{k<i\le n}E(X_iu_k|\mathscr{F}_k)=O(A_n^{2-2\delta}n^r\log A_n).\tag{14.1.25}$$

**证** 由(14.1.20),成立着

$$E(u_{n+1}|\mathscr{F}_i)=\sum_{j=0}^{\infty}E(\xi_{j+n+1}|\mathscr{F}_i)$$

$$=O\left(\sum_{j=0}^{\infty}A_{j+n+1}^{1-\delta}(1\wedge A_j^{\rho}n_i/n_{j+n+1})\right).$$

因此

$$\sum_{i=k+1}^{\infty}E(X_iu_{n+1}|\mathscr{F}_i)$$

$$=O\left(\sum_{i=k+1}^{n}\sum_{j=0}^{\infty}(1\wedge A_i^{\rho}n_i/n_{j+n+1})A_{j+n+1}^{1-\delta}A_n^{1-\delta}\right).$$

取 $k_0=[C_0 n^r \log A_n]$，其中 $C_0$ 待定，我们有

$$\sum_{i=k+1}^{n}\sum_{j=0}^{\infty}(1 \wedge A_i^p n_i/n_{j+n+1})A_{j+n+1}^{1-\delta} \leqslant \sum_{i=n-k_0}^{n}\sum_{j=0}^{k_0}A_{j+n+1}^{1-\delta}$$

$$+\sum_{i=k+1}^{n-k_0}\sum_{j=0}^{\infty}\frac{A_i^p n_i}{n_{j+n+1}}A_{j+n+1}^{1-\delta}+\sum_{i=n-k_0+1}^{n}\sum_{j=k_0+1}^{\infty}\frac{A_i^p n_i}{n_{j+n+1}}A_{j+n+1}^{1-\delta}$$

$$=O\Big(\sum_{i=k+1}^{n-k_0}\frac{A_i^p n_i n^r}{n_{n+1}}A_{n+1}^{1-\delta}+k_0^2 A_n^{1-\delta}$$

$$+\sum_{i=n-k_0+1}^{n}\frac{A_i^p n_i}{n_{k_0+n+1}}(k_0+n+1)^r A_{k_0+n+1}^{1-\delta}\Big)$$

$$=O\Big(n^{3r}A_n^{p+1-\delta}\exp\Big(\frac{(n-k_0)^{1-r}-n^{1-r}}{2(1-r)}q\Big)+k_0^2 A_n^{1-\delta}$$

$$+A_n^p n^r k_0 \exp\Big(\frac{q(n^{1-r}-(n+k_0)^{1-r})}{2(1-r)}\Big)\Big)$$

$$=O(A_n^{1-\delta}n^{2r}\log^2 A_n).$$

其中后一等式当取 $C_0$ 充分大时成立. 这就证明了(14.1.24). (14.1.25)的证明是类似的.

**引理 14.1.9** 我们有

$$\sum_{j=1}^{n}EY_j^2-A_n^2=O(n^{2r}A_n^{2-2\delta}\log^2 A_n). \tag{14.1.26}$$

**证** 由于$\{Y_i\}$是鞅差序列，从而

$$\sum_{j=1}^{n}EY_j^2=E\Big(\sum_{j=1}^{n}Y_j\Big)^2$$

$$=E\Big(\sum_{j=1}^{n}X_j\Big)^2+2Eu_{n+1}\sum_{j=1}^{n}X_j+Eu_{n+1}^2.$$

因此由引理 14.1.5—14.1.8 即得(14.1.26)成立.

**引理 14.1.10** 假设 $\delta\beta>r$，我们有

$$\sum_{j=1}^{n}Y_j^2-A_n^2=O(A_n^{2-\delta}n^r\log^3 A_n)\quad \text{a.s.} \tag{14.1.27}$$

**证** 注意到由(14.1.5)和 $\delta\beta>r$ 有 $n^r=O(A_n^\delta)$. 应用引理 14.1.1，Borel-Cantelli 引理和子序列方法，为证(14.1.27)成立，只

需证明对任意的 $0\leqslant m\leqslant n$ 有

(14.1.28)
$$E\Big(\sum_{m<j\leqslant n}(Y_j^2-EY_j^2)\Big)^2=O\Big(\Big(\sum_{m<j\leqslant n}EY_j^2\Big)A_n^{2-2\delta}n^{2r}\log^2A_n\Big),$$

其中在 $O$ 中所含常数与 $m,n$ 无关. 注意到

$$E\Big(\sum_{m<j\leqslant n}(Y_j^2-EY_j^2)\Big)^2$$
$$=\sum_{m<j\leqslant n}E(Y_j^2-EY_j^2)^2+2\sum_{m<k\leqslant n}E(Y_k^2-EY_k^2)\Big(\sum_{i=k+1}^{n}Y_i^2\Big)$$

而

$$\sum_{m<k\leqslant n}E(Y_k^2-EY_k^2)\Big(\sum_{i=k+1}^{n}Y_i^2\Big)$$
$$=\sum_{m<k\leqslant n}E(Y_k^2-EY_k^2)\Big(\sum_{i=k+1}^{n}Y_i\Big)^2$$
$$=\sum_{m<k\leqslant n}E(Y_k^2-EY_k^2)\Big(\sum_{i=k+1}^{n}X_i^2+2\sum_{k<i<j\leqslant n}X_iX_j$$
$$+2\sum_{k<i\leqslant n}X_i(u_{n+1}-u_{k+1})+(u_{n+1}-u_{k+1})^2\Big)$$
$$=:\sum_{m<k\leqslant n}(I_1(k)+I_2(k)+I_3(k)+I_4(k)).$$

由引理 14.1.6—14.1.8 得

(14.1.29)
$$\max_{2\leqslant i\leqslant 4}I_i(k)=O(n^{2r}A_n^{2-2\delta}EY_k^2\log^2A_n).$$

现在我们来证(14.1.29)式对 $I_1(k)$ 也成立. 由于

$$\Big|\sum_{k<i\leqslant n}X_i^2-\sum_{k<i\leqslant n}\xi_i^2\Big|=O(1),$$

只需证明(14.1.29)式对 $E(Y_k^2-EY_k^2)\Big(\sum_{k<i\leqslant n}\xi_i^2\Big)$ 成立. 注意到 $Y_k$ 是 $\mathscr{F}_k$ 可测的且可写

$$Y_k=\sum_{i=0}^{2^{t_k}-1}d_iI(U_{i,k}),$$

由 $\mathscr{F}_k$ 的定义我们有

$$
\begin{aligned}
&E(Y_k^2 - EY_k^2)\Big(\sum_{j=k+1}^{n}\xi_j^2\Big)\\
&= \sum_{i=0}^{2^r k-1} d_i^2 \sum_{j=k+1}^{n} \int_{i2^{-r}k}^{(i+1)2^{-r}k} a_j^2\cos^2 2\pi n_j t dt\\
&\quad - 2^{-r}k\Big(\sum_{i=0}^{2^r k-1} d_i^2\Big)\sum_{j=k-1}^{n}\int_0^1 a_j^2\cos^2 2\pi n_j t dt\\
&= \sum_{i=0}^{2^r k-1} d_i^2 \sum_{j=k+1}^{n} a_j^2\,\frac{\sin 2\pi n_j 2^{-rk}}{\pi n_j}\cos 4\pi n_j\Big(i+\frac{1}{2}\Big)2^{-rk}\\
&\quad - 2^{-rk}\Big(\sum_{i=0}^{2^r k-1} d_i^2\Big)\sum_{j=k+1}^{n} a_j^2\,\frac{\sin 4\pi n_j}{2\pi n_j}\\
&= O\Big(2^{-rk}\Big(\sum_{i=0}^{2^r k-1} d_i^2\Big)\Big(\sum_{j=k+1}^{n}\frac{a_j^2}{n_j} + \sum_{j=k+1}^{n} a_j^2(1\wedge A_k^2 n_k/n_j)\Big)\Big)\\
&= O(A_n^{2-2\delta} n^r EY_k^2 \log A_n).
\end{aligned}
$$

这就证明了(14.1.29)式对 $I_1(k)$ 成立.对于 $EY_k^4$,我们有

$$EY_k^4 = O(EY_k^2 A_k^{2-2\delta} k^{2r}\log^2 A_k).$$

综上所述证明了(14.1.28)成立,引理证毕.

**引理 14.1.11** 我们有

$$(14.1.30)\quad \sum_{j=1}^{n}(E(Y_k^2|\mathscr{F}_{j-1}) - Y_k^2) = O(n^r A_n^{2-\delta}\log^3 A_n).$$

**证** 令 $R_j = Y_j^2 - E(Y_j^2|\mathscr{F}_{j-1})$. 那么 $\{R_j,\mathscr{F}_j\}$ 是鞅差序列,且有

$$ER_k^2 = O(EY_k^4) = O(EY_k^2 A_k^{2-2\delta} k^{2r}\log^2 A_k).$$

和引理 14.1.10 类似证明得(14.1.30)成立.

现在利用鞅差序列的 Skorohod 嵌入定理,对鞅 $\{\sum_{j=1}^{M} Y_j, M\geqslant 1\}$ 存在一概率空间,在其上有一 Wiener 过程和一列非负随机变量 $\{T_i\}$,使得

$$\{X(\sum_{j\leqslant m} T_j), m\geqslant 1\}\text{和}\{\sum_{j\leqslant m} Y_j, m\geqslant 1\}$$

具有相同的分布.从而在新的概率空间上,不失一般性地可重新定

义$\{Y_j\}$为

$$Y_j = X\left(\sum_{i\leqslant j} T_i\right) - X\left(\sum_{i<j} T_i\right)$$

记

$$\mathscr{L}_m = \sigma\left\{X\left(\sum_{j=1}^{k} T_i\right), k \leqslant m\right\},$$

$$\mathscr{A}_m = \sigma\left\{X(t), 0 \leqslant t \leqslant \sum_{j=1}^{m} T_i\right\}.$$

显然地 $\mathscr{L}_m \subset \mathscr{A}_m, m\geqslant 1$ 且 $T_j$ 是 $\mathscr{A}_j$ 可测的.同时由嵌入定理,对每一 $j\geqslant 1, ET_j = EY_j^2$ 且有

(14.1.31) $\quad E(T_j|\mathscr{A}_{j-1}) = E(Y_j^2|\mathscr{A}_{j-1}) = E(Y_j^2|\mathscr{L}_{j-1})$ a. s.,

进一步,对每一 $v>1$ 有

(14.1.32) $$E|T_j|^v = O(E|Y_j|^{2v})$$

**引理 14.1.12** 在定理 14.1.1 的条件下有

(14.1.33) $$\sum_{j\leqslant n} T_j - A_n^2 = O(A_n^{2-\delta+r/\beta}\log^3 A_n).$$

**证** 写

$$\begin{aligned}
&\sum_{j\leqslant n} T_j - A_n^2 \\
&= \sum_{j\leqslant n}(T_j - E(T_j|\mathscr{F}_{j-1})) + \sum_{j\leqslant n}(E(Y_j^2|\mathscr{F}_{j-1}) - Y_j^2) \\
&\quad + \sum_{j\leqslant n} Y_j^2 - A_n^2 =: I_1 + I_2 + I_3,
\end{aligned}$$

令 $Z_j = T_j - E(T_j|\mathscr{A}_{j-1})$.那么$\{Z_j, \mathscr{A}_j\}$是鞅差序列且 $EZ_j^2 = O(EY_j^4)$.由类似于引理 14.1.11 的证明我们有

(14.1.34) $$I_1 = O(A_n^{2-\delta} n^r \log^3 A_n).$$

从(14.1.5),(14.1.27),(14.1.30)和(14.1.34)即得引理成立.

**定理 14.1.1** 的证明

注意到

$$\sum_{j\leqslant n} Y_j - \sum_{j\leqslant n} X_j = u_{n+1} = O(A_n^{1-\delta} n^r \log A_n).$$

从引理 14.1.3 和 14.1.12 即得定理 1 成立.

**定理 14.1.2** 的证明

在定理的条件下,注意到 $\delta=1$,类似于上述各引理的证明我们有

(14.1.35) $$\sum_{j\leq n}T_j - A_n^2 = O\big(n^r(n^r \vee \overline{B}_n)\log^3 A_n\big) \quad \text{a.s.},$$

其中 $\overline{B}_N = \sum_{k=1}^{N} a_k^2 EY_k^2$. 由于

$$\sum_{j\leq k} EY_j^2 = O\Big(\sum_{j\leq k} a_j^4\Big),$$

借助 Abel 变换并注意到 $a_k^2$ 非增,容易得到

$$\overline{B}_n = O\Big(\sum_{k\leq n} a_k^4\Big).$$

因此

(14.1.36) $$\sum_{j\leq n}T_j - A_n^2 = O\big(n^r(n^r \vee \overline{B}_n)\log^3 A_n\big) \quad \text{a.s.},$$

从(14.1.36)和引理 14.1.3 即得定理 2 成立.

**注 14.1.1** 若对某 $\beta>0$, $k^\beta=O(A_k)$, $|a_k|\downarrow 0$ 且对某 $\theta$, $r\leqslant\theta<1/2$ 有 $|a_k|=O(k^{-\theta})$,那么类似于定理 14.1.1 的证明有

$$S(A_n^2) - X(A_n^2) = O\big(C_n^{1/4}\log^2 A_n\big) \quad \text{a.s.},$$

其中 $C_n = \sum_{k\leq n} |a_k|^{4-2r/\theta}$.

设 $\overline{S}_n(w) = \sum_{k\leq n} \sqrt{2}\cos 2\pi n_k w$, $w\in[0,1)$. 结合推论 14.1.1 与 Wiener 过程的增量结果,我们可得到部分和 $\overline{S}_n$ 的增量的几乎必然极限性质.

## §14.2 一类 Gauss 序列

设$\{X_n, n\geqslant 1\}$是中心化的 Gauss 随机变量序列,在某些条件下,包括:对某 $\varepsilon>0$, $E\Big(\sum_{k=m+1}^{m+n} X_k\Big)^2 = n\sigma^2 + O(n^{1-\varepsilon})$,其中 $\sigma^2>0$ 及 $EX_mX_{m+n}=O(n^{-2})$等,Philipp 和 stout (1975)建立了用 Wiener 过

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$$S(t) = \sum_{k\leqslant t} X_k \qquad t \geqslant 0$$

的阶为 $O(t^{1/2-\lambda})$，其中 $0<\lambda<(1/60)\wedge(4\varepsilon/15)$. 邵启满(1985)应用对称矩阵的性质及关于矩阵特征值的圆盘定理证明着如下的理想结果.

**定理 14.2.1** 假设存在常数 $C_i>0, i=1,2,3$，使对每一 $n$ 满足：

(14.2.1) $E\left(\sum_{k=m+1}^{m+n} X_k\right)^2 \geqslant C_1 n$ 关于 $m$ 一致地成立，

(14.2.2) $$EX_n^2 \leqslant C_2$$

和

(14.2.3) $\gamma(n) = \sup_m |EX_m X_{m+n}| \leqslant C_3 n^{-3/2-\lambda}, n \geqslant 1$,

其中 $\lambda>0$. 那么存在一个概率空间，在其上有一 Wiener 过程 $\{W(t), t\geqslant 0\}$ 且可不改变分布地重新定义过程 $\{S(t), t\geqslant 0\}$ 使得

(14.2.4) $$S(t) - W(b_t) = O(\log^{1/2} t) \quad \text{a.s.}$$

其中

(14.2.5) $b_t = b_{[t]} = a_t + r_t, \quad a_t = ES^2(t), \quad r_t = \sum_{k=1}^{[t]} k^{-1/2+\lambda}$

**推论 14.2.1** 若条件(14.2.3)被加强为对某 $\varepsilon>0$

(14.2.3)′ $$\gamma(n) \leqslant Cn^{-2}(\log n)^{-1-\varepsilon},$$

那么我们有

(14.2.4)′ $$S(t) - W(a_t) = O(\log^{1/2} t) \quad \text{a.s.}$$

**推论 14.2.2** 假设 $\{X_n, n\geqslant 1\}$ 是中心化平稳 Gauss 序列且(14.2.3)′被满足，那么

$$\sigma^2 = EX_1^2 + 2\sum_{k=2}^{\infty} EX_1 X_k$$

绝对收敛. 若 $\sigma^2>0$，不妨设 $\sigma^2=1$，那么有

(14.2.4)″ $$S(t) - W(t) = O(\log^{1/2} t) \quad \text{a.s.}$$

定理的证明需要下述引理.

**引理 14.2.1** 设 $A$ 是 $n$ 阶实对称矩阵，其特征值为 $\lambda_1, \cdots, \lambda_n$.

记 $\lambda = \max\limits_{1\leqslant i\leqslant n} |\lambda_i|$. 那么对任一 $n$ 维行向量 $C$ 有

(14.2.6) $$|CAC'| \leqslant \lambda CC'$$

其中 $C'$ 是 $C$ 的转置向量.

**证** 我们仅需证明矩阵 $\lambda I - A$ 和 $\lambda I + A$ 是非负定的. 由矩阵的熟知性质,存在一实正交矩阵 $U$ 使得 $U'AU$ 为一个对角矩阵 $\Lambda$,且 $\Lambda$ 的对角元恰为 $A$ 的特征值. 所以我们有

$$\lambda I - A = U(\lambda I - \Lambda)U'.$$

显然 $\lambda I - \Lambda$ 的特征值都是非负的,所以实对称矩阵 $\lambda I - A$ 的特征值也均非负,因而是非负定的. 同理可证 $\lambda I + A$ 非负定. 引理证毕.

**引理 14.2.2** 任一 $n$ 阶矩阵 $A = (aij)_{n\times n}$ 的特征值 $\lambda$ 至少满足下列不等式之一:

(14.2.7) $$|\lambda - a_{ii}| \leqslant \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad i = 1, \cdots, n.$$

这一结果属于 Gerschgorin, 叫作圆盘定理(见 Franklin 1968).

记 $\mathscr{F}_n = \sigma\{X_k, 1 \leqslant k \leqslant n\}$. 写

(14.2.8) $$\begin{aligned} Y_n &= \sum_{k=0}^{\infty} (E(X_{n+k} | \mathscr{F}_n) - E(X_{n+k} | \mathscr{F}_{n-1})) \\ &= X_n + u_{n+1} - u_n \end{aligned}$$

其中 $u_1 = 0$ 且

(14.2.9) $$u_n = \sum_{k=0}^{\infty} E(X_{n+k} | \mathscr{F}_{n-1}), n = 1, 2, \cdots.$$

显然 $\{Y_n, \mathscr{F}_n\}$ 是鞅差序列. 下面我们将证明在定理的假定下(14.2.9)中级数是收敛的.

**引理 14.2.3** 若(14.2.1),(14.2.2)和(14.2.3)被满足且对任一 $k \geqslant 1$

(14.2.10) $$EX_k^2 \geqslant \sum_{j=1, j\neq k}^{\infty} |EX_kX_j| + 1,$$

那么

$$\| u_n \|_2 = O(1)$$

**证** 设$\{X_1,\cdots,X_j\}$的协方差阵为$A$，记$C=\{EX_1X_{j+k},\cdots,EX_jX_{j+k}\}$. 由 Philipp 和 Stout (1975)的(5.22)式对任意$j,k\geqslant 1$有

(14.2.11) $$E(E^2(X_{j+k}|\mathscr{F}_j)) = CA^{-1}C'.$$

由条件(14.2.10)和引理 14.2.1 及引理 14.2.2，我们得

(14.2.12) $$CA^{-1}C' \leqslant CC'$$

记

(14.2.13) $$u_{nk} = E(X_{n+k}|\mathscr{F}_{n-1}), k=0,1,\cdots; \quad n=1,2,\cdots.$$

从(14.2.11)，(14.2.12)和(14.2.3)即得

(14.2.14) $$Eu_{nk}^2 \leqslant \sum_{i=1}^{n-1}(EX_{n+k}X_i)^2$$

$$\leqslant c\sum_{i=1}^{n}(n+k-i)^{-3-2\lambda} \leqslant c(k+1)^{-2-2\lambda}$$

这样我们就得证

$$\| u_n \|_2 \leqslant c\sum_{k=0}^{\infty} \| u_{nk} \|_2 = O(1).$$

**定理 14.2.1 的证明.**

1)我们先来证明在条件(4.2.10)下定理成立. 由熟知的 Gauss 序列的性质(参见 Ibragimov 和 Rozanov 1978，$p$. 14)，由(14.2.8)定义的$Y_n$是$X_1,\cdots,X_n$的线性组合，所以$\{Y_n,n\geqslant 1\}$仍是 Gauss 序列，且由(14.2.8)，它是鞅差序列，因而是独立的. 这样可由$\{Y_n,n\geqslant 1\}$构造一个 Wiener 过程$\{W(t),t\geqslant 0\}$，记$b_i=\sum_{k\leqslant i}EY_k^2$，重新定义

$$Y_n = W(b_n) - W(b_{n-1}), \quad n=1,2\cdots$$

且从(14.2.8)我们有

$$\sum_{k\leqslant n}X_k - \sum_{k\leqslant n}Y_k = -u_{n+1}.$$

注意到$u_n,n=1,2,\cdots$，服从正态分布且有一致有界的方差，那么我们有

$$u_n = O(\log^{1/2}n) \quad \text{a.s.}$$

即

$$\sum_{k\leqslant n}X_k-\sum_{k\leqslant n}Y_k=O(\log^{1/2}n)\quad \text{a. s.}$$

亦即

(14.2.15) $$S(t)-W(b_t)=O(\log^{1/2}t)\quad \text{a. s.}$$

而且

$$\begin{aligned}b_n-a_n&=2E\Big(\sum_{k=1}^{n}X_ku_{n+1}\Big)+Eu_{n+1}^2\\&=2E\Big(\sum_{k=1}^{n}\sum_{j=1}^{\infty}X_kX_{n+j}\Big)+Eu_{n+1}^2\\&=O\Big(\sum_{k=1}^{n}\sum_{j=1}^{\infty}\gamma(n+j-k)\Big)=O\Big(\sum_{k=1}^{n}k^{-1/2-\lambda}\Big)\end{aligned}$$

这就证明了在(14.2.10)下定理 14.2.1 成立.

2)现在来证定理对一般情形成立，令 $l=[312C_3^2/C_1^2]$，定义

$$X_m^*=\sum_{(m-1)l<k\leqslant ml}X_k\ m=1,2,\cdots,\quad S_i^*=\sum_{k\leqslant i}X_k^*.$$

那么 $\{X_n^*\}$ 也是 Gauss 序列且满足(14.2.1)，(14.2.2)和(14.2.3). 从(14.2.1)即得

(14.2.16) $$EX_n^{*2}\geqslant C_1l.$$

我们来证 $\{X_n^*\}$ 满足(14.2.10). 事实上

(14.2.17) $$\begin{aligned}\sum_{\substack{j=1\\j\neq n}}^{\infty}|EX_n^*X_j^*|&=\sum_{\substack{j=1\\j\neq n}}^{\infty}\Big|E\Big(\sum_{(n-1)l<k\leqslant nl}X_k\Big)\Big(\sum_{(j-1)l<i\leqslant jl}X_i\Big)\Big|\\&\leqslant C_3\sum_{\substack{j=1\\j\neq n}}^{\infty}\sum_{(n-1)l<k\leqslant nl}\sum_{(j-1)l<i\leqslant jl}|i-k|^{-3/2}\\&\leqslant 2C_3\sum_{k=1}^{l}\sum_{m=0}^{\infty}(m+k)^{-3/2}\\&\leqslant 6C_3\sum_{k=1}^{l}k^{-1/2}\leqslant 6C_3(1+2l^{1/2})\end{aligned}$$

由 $l$ 的定义，容易验证(14.2.17)的右边不超过 $C_1l-1$，这就是说 $\{X_n^*\}$满足(14.2.10).

对任一固定的 $n$，存在 $m$ 使得 $(m-1)l\leqslant n<ml$ 且

(14.2.18) $$|S_n - S_m^*| \leqslant \max_{(m-1)l \leqslant k < ml} |X_k|.$$

注意到$\{X_n, n \geqslant 1\}$是具有一致有界方差的 Gauss 序列，那么我们有

(14.2.19) $$\max_{(m-1)l \leqslant k < ml} |X_k| = O(\log^{1/2} m) \quad \text{a. s.}$$

且

$$\begin{aligned} a_n - a_m^* &= E\Big(\sum_{k=1}^{n} X_k\Big)^2 - E\Big(\sum_{k=1}^{ml} X_k\Big)^2 \\ &= -2E\Big(\sum_{k=1}^{n}\sum_{j=n+1}^{ml} X_k X_j\Big) - E\Big(\sum_{j=n+1}^{ml} X_j\Big)^2 \\ &= O\Big(\sum_{k=1}^{\infty} \gamma(k)\Big) = O(1), \end{aligned}$$

其中 $a_m^* = ES_m^{*2}$. 从已证的 1)及(14.2.18)—(14.2.20)得证定理 14.2.1 成立.

## §14.3 马氏过程的非负可加泛函

设 $\boldsymbol{X} = \{X_t, t \geqslant 0\}$是定义于概率空间$(\Omega, \mathscr{F} P)$上，取值于完全、$\sigma$ 紧、可测距离空间$(E, \rho, \boldsymbol{B})$上的齐次、右连续、强 Feller 马氏过程，满足下述条件：

(i)对任一 $x \in E, t > 0$ 及开集$U \in \boldsymbol{B}$，过程 $\boldsymbol{X}$ 的齐次转移函数

(14.3.1) $$p(t, x, U) > 0,$$

(ii)存在紧集 $K$，使对每一 $a \in \boldsymbol{E}$ 有

(14.3.2) $$P_0\{\text{对某 } t \geqslant 0, X_t(w) \in K\} = 1,$$

其中 $P_a(\cdot) = P(\cdot \mid X_0 = a)$.

从(i)和(ii)即得 $\boldsymbol{X}$ 是常返的强马氏过程.

对每一$U \in \boldsymbol{B}$，令

$$\tau_U(w) = \begin{cases} \inf\{t, t \geqslant 0, X_t(w) \in U\}, & \text{当右边集非空,} \\ \infty & \text{否　　　则.} \end{cases}$$

并定义击中分布

$$h^U(a, S) = P_a\{X_{\tau_U}(w) \in S, \tau_U < \infty\} \quad a \in \boldsymbol{E}, S \in B.$$

记

$$h^U f(x) = \int_{\overline{U}} h^U(x,dy)f(y) \quad f(x) \in B(\overline{U}),$$

其中 $B(\overline{U})$ 是 $\overline{U}$ 上所有有界可测函数的集.

设 $\boldsymbol{H}=\{K,L\}$ 是满足下述条件的集合对全体：

$(H_1)$ $K$ 和 $L$ 都是 $E$ 中闭子集，且至少有一内点，

$(H_2)$ $K$、$L$ 至少有一个是紧集，

$(H_3)$ (i) $K \subset E\backslash L$，$E\backslash L$ 是连通开集，或

(ii) $L \subset E\backslash K$，$E\backslash K$ 是连通开集.

对任意给定的一对 $(K,L) \in \boldsymbol{H}$，$X \in K$，令

$$T^K(x,U) = \int_E h^L(x,dy)H^K(y,U), \tag{14.3.3}$$

其中 $U \in \boldsymbol{B}$，$U \subset K$. 由常返性可知 $T^K(x,U)$ 是一步转移函数. 定义变换

$$T^K f(x) = \int_E T^K(x,dy)f(y) \quad x \in K, f \in B(K). \tag{14.3.4}$$

不失一般性可设 $K$ 是紧集，且 $(H_3)$ 中的 (i) 被满足，那么对 $T^K$ 存在 $K$ 上唯一的不变概率测度 $\mu$. 进一步，假设 $\boldsymbol{X}$ 还满足

(iii) $T^k\{B(k)\} \subset C(K) = \{K$ 上全体连续函数$\}$.

设 $\tau$ 是过程 $\boldsymbol{X}$ 的停时，令 $\boldsymbol{\Omega}_\tau = \{\tau(\omega) < \infty\}$，那么

$$\mathscr{N}_\tau = \{A: A \subset \boldsymbol{\Omega}_\tau, \forall\, t \geqslant 0, A \cap (\tau \leqslant t) \in \mathscr{N}_t\}$$

是 $\boldsymbol{\Omega}_\tau$ 的一个 $\sigma$ 域，其中 $\mathscr{N}_t = \sigma\{X_s, 0 \leqslant s \leqslant t\}$. 记 $\mathscr{N}^*$ 是如Dynkin (1963, 3.5) 所定义 $\Omega$ 的 $\sigma$ 域. 定义 $\mathscr{N}^*$ 到 $\mathscr{N}^*$ 的推移算子 $\theta_t$ 及 $\boldsymbol{N}^*$ 到 $\boldsymbol{\Omega}_\tau \mathscr{N}^*$ 的推移算子 $\theta_\tau$ 满足

$$\theta_\tau A = \bigcup_{t \geqslant 0} \{\theta_t A, \tau(\omega) = t\} \subset \boldsymbol{\Omega}_\tau \qquad A \in \mathscr{N}^*, \tag{14.3.5}$$

其中算子 $\theta_t$ 保持和、交及补的运算且有

$$\theta_\tau\{X_t \in \Gamma\} = \{X_{t+\tau} \in \Gamma\} \quad \Gamma \in B. \tag{14.3.6}$$

对给定的 $(K,L) \in \boldsymbol{H}$，在 $\Omega$ 上定义一列随机函数如下：

$$\tau_1 = \tau_1(K,L,\omega) = \begin{cases} \inf\{t, t \geqslant 0, X_t(\omega) \in K\} & \text{当右边集非空，} \\ \infty & \text{否 则，} \end{cases}$$

$$\sigma_n = \sigma_n(K,L,\omega) = \begin{cases} \inf\{t, t > \tau_n, X_t(\omega) \in L\} & \text{当右边集非空，} \\ \infty & \text{否 则，} \end{cases}$$

$$\tau_{n+1}=\tau_{n+1}(K,L,\omega)=\begin{cases}\inf\{t,t>\sigma_n,X_t(\omega)\in K\} & \text{当右边集非空,}\\ \infty & \text{否　　则,}\end{cases}$$

由Doob(1953)知 $\tau_n,\sigma_n(n\geqslant 1)$都是过程 $X$ 的 a. s. 有限停时. 记 $\xi_n(\omega)=X_{\tau_n}(\omega)$. 它的转移函数 $T^K(x,U)$满足Doeblin条件且是无周期,存在一个唯一的遍历集. 进一步 $T^K(x,U)$还满足

$$|(T^K)^n(x,U)-\mu(U)|<C\delta^n, \tag{14.3.7}$$

其中 $0<\delta<1$,$C$ 是常数. 令

$$P_\mu(B)=\int_E P_a(B)\mu(da)\quad B\in\mathscr{F}. \tag{14.3.8}$$

那么 $P_\mu$ 是$(\Omega,\mathscr{F})$上的概率测度,且从 $P_a(B)=1$ 即得 $P_\mu(B)=1$. 记

$$E_\mu f(\cdot)=\int_\Omega f(\cdot)P_\mu(d\omega)$$

最后,设 $\phi=\{\phi_t^s(\omega)\}$是定义在 $\Omega$ 上,$X$ 的非负强可测齐次可加泛函,即是满足下述条件的实值函数族:

$(\phi_1)$对任给 $s\leqslant t$,$\phi_t^s(\omega)$是非负$\overline{\mathscr{N}_t}$可测的,其中$\overline{\mathscr{N}_t}$是 $\mathscr{N}_t$ 在概率空间$(\Omega,\mathscr{F},P)$中的完备化,

$(\phi_2)$对任一 $\omega\in\Omega$ 和 $s\leqslant t\leqslant u$,$\phi_u^s=\phi_t^s+\phi_u^t$,

$(\phi_3)$对任一 $\omega\in\Omega$ 和 $h\geqslant 0$,$s\leqslant t$,$\theta_h\phi_t^s=\phi_{t+h}^{s+h}$,

$(\phi_4)$对任给的 $0\leqslant u\leqslant v$,作为$(t,w)$的二元函数 $\phi_t^0(\omega)$在$[u,v]\times\Omega$ 上是 $\boldsymbol{B}_{[u,v]}x\overline{\mathscr{N}_v^u}$可测的,其中,$\boldsymbol{B}_{[u,v]}$是区间$[u,v]$上的 $\sigma$ 域,$\overline{\mathscr{N}_v^u}$是 $\mathscr{N}_v^u=\sigma\{X_t,u\leqslant t\leqslant v\}$对于 $P$ 的完备化.

容易验证

$$\theta_{\sigma_1}\tau_n=\tau_{n+1}-\sigma_1 \tag{14.3.9}$$

记

$$y_n=\phi_{\tau_{n-1}}^{\tau_n},\qquad z_n=\tau_{n+1}-\tau_n,$$

那么有

$$\theta_{\sigma_1}y_n=y_{n+1},\quad \theta_{\sigma_1}z_n=z_{n+1} \tag{14.3.10}$$

**引理 14.3.1**　$\{y_n,n\geqslant 1\}$是强平稳 $\varphi$ 混合随机变量序列且 $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$.

特别地，序列$\{z_n, n\geqslant 1\}$和$\{w_n = y_n + C zn, n\geqslant 1\}$也是强平稳$\varphi$混合的.

**证** 为证$\{y_n\}$是强平稳序列，只需证明推移算子$\theta_{\sigma 1}$在$P_\mu$上是保测的，事实上，从(14.3.8)及Dynkin(1963 定理 3.11)，对任一$B\in \mathscr{N}_0 := \sigma\{X_t, t\geqslant 0\}$有

$$P_\mu(\theta_{\sigma 1}B) = \int_{\Omega\sigma 1} Px_{\sigma 1} P_\mu(dx) = P_\mu(B).$$

为证$\{y_n\}$是$\varphi$混合的，首先来证下两事实：

(i)$\sigma\{y_1, \cdots, y_{m-1}\} \subset \sigma\{\xi_1, \cdots, \xi_m\} \subset \mathscr{N}_{\tau m}$,

(ii)$\sigma\{y_{m+k-1}, y_{m+k}, \cdots\} \subset \sigma\{\xi_{m+k}, \xi_{m+k+1}, \cdots\}$,

事实上，由于$\mathscr{N}\tau_{m-1} \subset \mathscr{N}\tau_m$且$\xi_m = X\tau_m$是$\mathscr{N}\tau_m$可测的. 可推出$\xi_k, 1\leqslant k\leqslant m$也是$\mathscr{N}\tau_m$可测的，所以

$$\sigma\{\xi_1, \cdots, \xi_m\} \subset \mathscr{N}_{\tau m}.$$

这样(i)将成立，如果我们能证明

$$\sigma\{y_n \in A, A \in B_T\} \subset \sigma\{\xi_{n+1} \in \Gamma, \Gamma \in \boldsymbol{B}\} \tag{14.3.11}$$

其中$\boldsymbol{B}_K$是$R_+$的$\sigma$域. 为此令

$$\Lambda = \{A: (y_n \in A) \in \sigma\{\xi_{n+1} \in \Gamma, \Gamma \in \boldsymbol{B}\} A \in T\},$$

容易验证$\Lambda$是$\lambda$系且包含$\pi$系

$$\Pi = \{[0,t]: (y_n \in [0,t)) \in \sigma\{\xi_{n+1} \in \Gamma, \Gamma \in \boldsymbol{B}\}\},$$

那么$\sigma(\Pi)\subset\Lambda$，由此即得(14.3.11). (ii)的证明是类似的.

这样对任给$A\in\sigma\{y_k, 1\leqslant k\leqslant m-1\}, B\in\sigma\{y_k, k\geqslant m+n-1\}$，由强马氏性和(14.3.7)我们有

$$\begin{aligned} P_\mu(AB) &= E_\mu(E_\mu(I(A)I(B)|\mathscr{N}\tau_m)) \\ &= E_\mu(I(A)E_\mu(I(B)|\mathscr{N}\tau_m)) + E_\mu(I(A)E_\mu(I(B)|\xi_m)), \end{aligned}$$

和

$$\begin{aligned} &|P_\mu(AB) - P_\mu(A)P_\mu(B)| \\ &= |E_\mu(I(A)E_\mu(I(B)|\xi_m)) - E_\mu I(A)E_\mu I(B)| \\ &= |E_\mu\Big\{I(A)\int_E E_\mu(I(B)|\xi_{m+n} = \eta)[(T^K)^n(\xi_m, d_\eta) \\ &\quad - \mu(d\eta)]\Big\}| \end{aligned} \tag{14.3.12}$$

$$\leqslant E_\mu\left\{I(A)\int_E E_\mu(I(B)|\xi_{m+n}=\eta)V_n(\xi_m,d\eta)\right\}$$
$$\leqslant E_\mu I(A)V_n(\xi_m,E)\leqslant P_\mu(A)C\delta^n,$$
其中 $V_n(\xi_m,E)=|(T^K)^n(\xi_m,A)-\mu(A)|$. 取 $\varphi^{(n)}=c\delta^n(o<\delta<1)$，我们得 $\{y_n\}$ 是 $\varphi$ 混合序列且 $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$.

特别地，令 $\phi_t=t-s$，就有 $y_n=z_n$. 对 $\{w_n\}$，从(i)和(ii)有
$$\sigma\{w_k,1\leqslant k\leqslant m-1\}\subset\sigma\{\xi_k,1\leqslant k\leqslant m\}\subset\mathcal{N}_{\tau m}$$
$$\sigma\{w_k,k\geqslant m+n-1\}\subset\sigma\{\xi_k,k\geqslant m+n\},$$
所以，$\{w_n\}$ 也是强平稳 $\varphi$ 混合序列. 引理 4.3.1 证毕.

**引理 14.3.2** 设 $\phi_t$ 和 $\psi_t$ 是过程 $X$ 的任两非负强可测齐次可加泛函，且有有限的 $\alpha_\varphi=E_\mu y_1$ 和 $\alpha_\psi=E_\mu\psi_{\tau_2}^{\tau_1}\neq 0$. 那么

(a)我们有
$$P_a\{\lim_{t\to\infty}\phi_t^0/\psi_t^0=\alpha_\phi/\alpha_\psi\}=P\{\lim_{t\to\infty}\phi_t^0/\psi_t^0=\alpha_\phi/\alpha_\psi\}=1 \tag{14.3.13}$$
其中
$$P(A)=\int_E P_a(A)P_0(da) \tag{14.3.14}$$
$P_0$ 是初始分布

(b)特别地，若令 $l(t)$ 是正整值随机变量满足
$$\tau_{l(t)}\leqslant t<\tau_{l(t)+1} \tag{14.3.15}$$
记 $\alpha_z=E_\mu z_n$. 那么有
$$P_a\left\{\lim_{t\to\infty}\frac{t}{l(t)}=\alpha_z\right\}=P\left\{\lim_{t\to\infty}\frac{t}{l(t)}=\alpha_z\right\}=1, \tag{14.3.16}$$
$$P_a\left\{\lim_{t\to\infty}\frac{\tau_{l(t)}}{t}=1\right\}=P\left\{\lim_{t\to\infty}\frac{\tau_{l(t)}}{t}=1\right\}=1. \tag{14.3.17}$$

**证** 由 $\{y_n\}$ 的强平稳性和 $\varphi$ 混合性，$\{y_n\}$ 是遍历的且 $\boldsymbol{u}$ 是平凡的. 由(14.3.8)对任一 $A\in\boldsymbol{u}$ 有
$$P_a(A)=P_\mu(A)=0\quad or\quad 1.$$
因此我们有
$$P_\mu\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}yk=\alpha_\phi\right\}=1.$$

且由于$\limsup\frac{1}{n}\sum_{k=1}^{n}y_k$ 和$\liminf\frac{1}{n}\sum_{k=1}^{n}y_k$ 是 $\boldsymbol{u}$ 可测的，有

(14.3.18) $$P_a\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}yk=\alpha_\phi\right\}=1.$$

显然地 $l(t)\to\infty(t\to\infty)$. 从事实

(14.3.19) $$\phi_t^0=\phi_{\tau_1}^0+\sum_{k=1}^{l(t)-1}y_k+\phi_t^{\tau_{l(t)}},$$

及 $\tau_1<\infty$ a.s. 推得对任一 $\varepsilon>0$ 和充分大 $t$ 有

(14.3.20) $$P_a\{\phi_{\tau_1}^0/l(t)\geqslant\varepsilon\}=0,$$

再由 $\phi_t$ 的非负性，对充分大的 $t$ 也有

(14.3.21) $$P_a\{\phi_t^{\tau_{l(t)}}/l(t)\geqslant\varepsilon\}=P_a\left\{\phi\,{}_{\tau_{l(t)-1}}^{\tau_{l(t)}}/l(t)\geqslant\varepsilon\right\}=0,$$

结合(14.3.18)—(14.3.21)，我们得

(14.3.22) $$P_a\{\lim_{t\to\infty}\phi_t^0/l(t)=\alpha_\phi\}=1,$$

类似地我们有

(14.3.23) $$P_a\{\lim_{t\to\infty}\psi_t^0/l(t)=\alpha_\psi\}=1.$$

由(14.3.22)，(14.3.23)和(14.3.14)一起即得(14.3.13)成立.

注意到(14.3.16)是(14.3.22)在 $\phi_t=t-s$ 时的特殊情形. 另一方面，由于

$$\frac{\tau_{l(t)}}{t}=\frac{\tau_1}{t}+\frac{l(t)-1}{t}+\frac{1}{l(t)-1}\sum_{k=1}^{l(t)-1}z_k,$$

从 $\tau_1<\infty$a.s.，(14.3.18)和(14.3.16)得(14.3.17)成立.

现在让我们考察由可加泛函 $\phi_t^0$ 生成的过程

$$S(t)=\phi_t^0-Mt,$$

其中 $M=\alpha_\phi/\alpha_\psi$，$\psi_t=t-s$. 陆传荣(1986)证明着下述定理.

**定理 14.3.1** 设 $\phi_t$ 和 $\psi_t$ 是过程 $X$ 的非负强可测齐次可加泛函，$\omega_n=y_n-Mz_n$. 假设对某 $0<\delta\leqslant2$，

$$E_\mu y_n{}^{2+\delta}<\infty\qquad E_\mu z_n{}^{2+\delta}<\infty,$$

那么 $E_\mu W_n{}^{2+\delta}<\infty$ 且

$$\sigma_\omega^2=E_\mu W_1^2+2\sum_{k=2}^{\infty}EW_1W_k$$

绝对收敛. 不失一般性设 $\sigma_\omega^2/\alpha_\psi=1$，那么存在一较大的概率空间，

在其上有一Wiener过程$\{W(t), t\geqslant 0\}$，不改变分布可在其上重新定义过程$\{S(t), t\geqslant 0\}$使对任给$\varepsilon>0$概率为1地有

$$S(t)-W(t)=\begin{cases} O(t^{\frac{1}{2+\delta}+\varepsilon}) & \text{当 } 0<\delta<2, \\ O(t^{\frac{1}{4}}(\log t)^{\frac{9}{4}+\varepsilon}) & \text{当 } \delta=2. \end{cases}$$

**证** 写

$$S(t)=\phi_t^0-Mt=\phi_{\tau_1}^0+\sum_{k=1}^{l(t)-1}\omega_k+\phi_t^{\tau l(t)}-M(t-\tau_{l(t)}+\tau_1),$$

记$Z(t)=\sum_{k=1}^{l(t)-1}\omega_k$，我们有

(14.3.24) $\quad S(t)-Z(t)=\phi_{\tau_1}^0+\phi_t^{\tau l(t)}-M(t-\tau_{l(t)}+\tau_1).$

我们来证(14.3.24)的右边a.s.地等于$O((t\log\log t)^{1/(2+\delta)})$. 由引理14.3.1知$\{y_n, n\geqslant 1\}$是强平稳$\varphi$混合序列，$\varphi(n)=Ce^{-\theta n}$，$\theta>0$. 所以$\{y_n^{(2+\delta)/2}, n\geqslant 1\}$也是强平稳$\varphi$混合序列且

$$E(y_n^{(2+\delta)/2})^2=Ey_n^{2+\delta}<\infty.$$

所以由重对数律我们有

$$\sum_{k=1}^{n}(y_k^{(2+\delta)/2}-Ey_k^{(2+\delta)/2})=O((n\log\log n)^{1/2})\ \text{a.s.}$$

从可加泛函$\phi_t^s$的非负性和引理14.3.2我们得

$$\begin{aligned}\phi_t^{\tau l(t)}\leqslant y_{l(t)}&=O((l(t)\log\log l(t))^{1/(2+\delta)})\\&=O((t\log\log t)^{1/(2+\delta)})\ \text{a.s.}\end{aligned}$$

同样讨论我们也有

(14.3.25) $\quad t-\tau_{l(t)}\leqslant z_{l(t)}=O((t\log\log t)^{1/(2+\delta)})$ a.s.

显然地$\phi_{\tau_1}^0=O(1)$，$M\tau_1=O(1)$. 这样我们有

$$S(t)-Z(t)=O((t\log\log t)^{1/(2+\delta)})\ \text{a.s.}$$

对强平稳$\varphi$混合序列$\{W_n\}$，从定理9.1.1和9.1.2. 概率为1地有

(14.3.26) $$\sum_{k=1}^{n}\omega_k-W(n\sigma_w^2)=\begin{cases} O(t^{\frac{1}{2+\delta}+\varepsilon}) & \text{当 } 0<\delta<2, \\ O(t^{\frac{1}{4}}(\log t)^{\frac{9}{4}+\varepsilon}) & \text{当 } \delta=2. \end{cases}$$

现在我们写

(14.3.27) $\quad Z(t)-W(t)$

$$= \sum_{k=1}^{l(t)-1} \omega_k - W((l(t)-1)\sigma_\omega^2) + W((l(t)-1)\sigma_\omega^2) - W(t)$$

$$=: I_1 + I_2.$$

从引理 14.3.2 和(14.3.26),概率为 1 地有

$$(14.3.28)\quad I_1 = \begin{cases} O\left(l(t)^{\frac{1}{2+\delta}+\epsilon}\right) = O\left(t^{\frac{1}{2+\delta}+\epsilon}\right) & \text{当 } 0<\delta<2, \\ O\left(l(t)^{\frac{1}{4}}(\log l(t))^{\frac{9}{4}+\epsilon}\right) = O\left(t^{\frac{1}{4}}(\log\log t)^{\frac{9}{4}+\epsilon}\right) & \text{当 } \delta=2. \end{cases}$$

另一方面,由关于强平稳 $\varphi$ 混合序列 $\{z_n\}$ 的重对数律,我们有

$$\sum_{k=1}^{n} z_k - n\alpha_\varphi = O((n \log\log n)^{1/2}) \text{ a. s.}$$

因此

$$\begin{aligned} \tau_{l(t)} - l(t)\alpha_\varphi &= O((l(t)\log\log l(t))^{1/2}) \\ &= O((t \log\log t)^{1/2}) \text{ a. s.} \end{aligned}$$

所以

$$t - l(t)\alpha_\varphi = t - l(t)\sigma_\omega^2 = O((t \log\log t)^{1/2}) \text{ a. s.}$$

由Csörgö和 Révész(1981)的定理 1.2.2 我们有

$$I_2 = W((l(t)-1)\alpha_2) - W(t) = O((t \log\log t)^{1/4}) \text{ a. s.}$$

把它与(14.3.25)—(14.3.27)相结合,概率为 1 地有

$$S(t) - W(t) = \begin{cases} O\left(t^{\frac{1}{2+\delta}+\epsilon}\right) & \text{当 } 0<\delta<2, \\ O\left(t^{\frac{1}{4}}\ (\log t)^{\frac{9}{4}+\epsilon}\right) & \text{当 } \delta=2. \end{cases}$$

定理 14.3.1 证毕.

**注 14.3.1** 从定理 14.3.1 我们可给出马氏过程非负可加泛函的弱不变原理和重对数律.

# 附　　录

**定义 A1**　对某 $A\geqslant 0$，在$[A,\infty)$上正的可测函数 $R(x)$说是在无穷远点是具有指数 $\alpha$ 正则变化的，若对任一 $a>0$

$$(A1)\qquad \lim_{x\to\infty} R(ax)/R(x) = a^{\alpha}.$$

把具有指数 $\alpha$ 的正则变化函数 $R(x)$重写成

$$(A2)\qquad R(x) = x^{\alpha}L(x).$$

那么由(A1)，我们有

$$\lim_{x\to\infty} L(ax)/L(x) = 1.$$

**定义 A2**　一个具有指数 $\alpha=0$ 正则变化函数 $L(x)$称为缓变函数.

我们将列举缓变函数若干主要性质. 它们的证明参见 Senata (1976)和 Ibragimov 和 Linnik(1971).

缓变函数的 Karamata 表示定理如下：

**定理 A1**　设 $L(x)$是定义在$[A,\infty)$上的缓变函数，$A\geqslant 0$. 那么存在正的 $B\geqslant A$ 使对任一 $x\geqslant B$

$$L(x) = \exp\left\{\eta(x) + \int_B^x \frac{\varepsilon(t)}{t}dt\right\},$$

其中 $\eta(x)$是$[B,\infty)$上有界可测函数，当 $x\to\infty$时 $\eta(x)\to c(|c|<\infty)$，$\varepsilon(x)$是$[B,\infty)$上的连续函数，当 $x\to\infty$时 $\varepsilon(x)\to 0$.

利用这一表示定理，我们可导出很多有用的性质. 在下面常设 $L(x)$，$L_1(x)$，$L_2(x)$是缓变函数.

**性质 A1**　对任一 $a\geqslant 0$

$$\lim_{x\to\infty} L(x+a)/L(x) = 1.$$

**性质 A2**　对任一 $\varepsilon>0$

$$\lim_{x\to\infty} x^{\varepsilon}L(x) = \infty,\qquad \lim_{x\to\infty} x^{-\varepsilon}L(x) = 0.$$

**性质 A3** 当 $k\to\infty$ 时

$$\sup_{2^k\leqslant t\leqslant 2^{k+1}} L(t)/L(2^k) \to 1.$$

**性质 A4** 设当 $x\to\infty$ 时 $a=a(x)\to 0$. 那么对任一 $\varepsilon>0$

$$\lim_{x\to\infty} a^{\varepsilon}\frac{L(ax)}{L(x)} = \lim_{x\to\infty} a^{\varepsilon}\frac{L(x)}{L(ax)} = 0.$$

**性质 A5** 当 $x\to\infty$ 时

$$(\log L(x))/\log x \to 0.$$

**性质 A6** 对任一实数 $\alpha$, $L^{\alpha}(x)$, $L_1(x)L_2(x)$ 和 $L_1(x)+L_2(x)$ 都是缓变函数. 其次, 若当 $x\to\infty$ 时 $L_2(x)\to\infty$, 那么 $L_1(L_2(x))$ 也是缓变函数.

**性质 A7** 由下式定义 $\overline{L}(x)$ 和 $\underline{L}(x)$:

$$x^{\gamma}\overline{L}(x) = \sup_{B\leqslant t\leqslant x} t^{\gamma}L(t),$$

$$x^{\gamma}\underline{L}(x) = \inf_{x\leqslant t\leqslant \infty} t^{\gamma}L(t),$$

其中 $\gamma>0$ 是任给的常数. 那么 $\overline{L}\sim L$ 且 $\underline{L}\sim L$.

**注 A1** 作为性质 7 的一个推论, $x^{\gamma}L(x)$ 渐近地等于一个指数 $\gamma$ 的非降正则变化函数.

**性质 A8** 对 $R_1(x)=x^{\gamma}L_1(x)$, $\gamma>0$, 存在一正则变化函数 $R_2(x)=x^{1/\gamma}L_2(x)$ 使得当 $x\to\infty$ 时

$$R_1(R_2(x)) \sim x, \qquad R_2(R_1(x)) \sim x.$$

此处 $R_2(x)$ 渐近唯一地被定义, 即若当 $R_2$ 被 $R_3$ 代替时上述关系成立, 且当 $x\to\infty$ 时 $R_3(x)\to\infty$, 那么 $R_3(x)\sim x^{1/\gamma}L_2(x)$.

**性质 A9** 设 $L(x)$ 是 $[A,\infty)$ 上正的缓变函数. 若对某 $\gamma>0$, $R(x)=x^{\gamma}L(x)$ 在 $[A,\infty)$ 上是不减的. 对 $x\geqslant R(A)$ 设

$$R^{*}(x) = \inf\{y: y\in[A,\infty), R(y)\geqslant x\}.$$

那么 $R^{*}(x)=x^{1/\gamma}L^{*}(x)$, 当 $L^{*}(x)$ 是缓变函数时, $R^{*}(x)$ 是具有性质 $A8$ 意义下 $R(x)$ 的逆函数.

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# 索　引

Mathematics and Its Applications

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# Limit Theory for Mixing Dependent Random Variables

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Mathematics and Its Applications

# Limit Theory for Mixing Dependent Random Variables

LIN ZHENGYAN

and

LU CHUANRONG

*Department of Mathematics,*
*Hangzhou University,*
*Hangzhou, People's Republic of China*

For many practical problems, observations are not independent. In this book, limit behaviour of an important kind of dependent random variables, the so-called mixing random variables, is studied. Many profound results are given, which cover recent developments in this subject, such as basic properties of mixing variables, powerful probability and moment inequalities, weak convergence and strong convergence (approximation), limit behaviour of some statistics with a mixing sample, and many useful tools are provided.

*Audience*

This volume will be of interest to researchers and graduate students in the field of probability and statistics, whose work involves dependent data (variables).

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# Limit Theory for Mixing Dependent Random Variables

by

Lin Zhengyan

and

Lu Chuanrong
*Department of Mathematics,*
*Hangzhou University,*
*Hangzhou, The People's Republic of China*

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'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'

JulesVerne

The series is divergent; therefore we may be able to do something with it.

O.Heaviside

One service methematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded non-secse'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and nonlinearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.
Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguable true. And all statements obtainable this way form part of the raison d'être of this series.
This series, *Mathematics and Its Applications,* started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote
"Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantume fields, crystal defects anf mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrial engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental methematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."
By and large, all this this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such book available.
If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to

be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the nonlinear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I sm hinting at; if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, $p$-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading-fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five subseries: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include book treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an improtant role in several different mathematical and/or scientific specialization areas;
- New applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

The present volume, one of the first in the 'Chinese subseries' of MIA, also appropriately enough, one dealing with fundamental issues: interrelations between logic and computer science. The advent of computers has sparked off revived interest in a host of fundamental issues in science and mathematics such as computability, recursiveness, computational complexity and automated theorem proving to which latter topic ths author has made seminal contributions for which he was awarded the ATP prize in 1982.

It is a pleasure to welcome this volume in this series.

The shortest path between two truths in the real domain passes through the complex domain

J. Hadamard

La physique ne nous donne pas seulement ;'occasion de rèsoudr des problèmes ... elle nous fait presentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Bussum, August 1989

Michiel Hazewinkel

# Preface

The classical limit theorems of probability theory for independent random variables had been developed successfully in the thirties and forties. The basic results were summed up in Gnedenko and Kolmogorov's monograph "*Limit Distributions for Sums of Indenpendent Random Variables*" (1954) and Petrov's monograph "*Sums of Independent Random Variables*" (1975). The modern limit theorems of probability theory, such as weak convergence of probability measures and strong approximations etc, have been studied by many authors since the fifties. The limit theory for weakly dependent random variables was also discussed deeply. In fact, the limit distributions of sums for non-independent random variables were studied early by some probabilitists and statisticians, such as Bernstein (1927), Hopf (1937), Hoeffding and Robbins (1948), etc. The dependence of random variables as a concept is developed not only in some branches of probability theory and mathematical statistics, such as Markov chains, random field theory and time series analysis, etc, but also appears in many practical problems. Although the assumption of independence is reasonable sometimes, it is difficult to check the independence of a sample. Moreover in many practical problems, the samples are not independent observations.

The definition of strong mixing ($\alpha$-mixing) was first introduced by Rosenblatt (1956). Ibragimov (1959), Rozanov and Volconski (1959) also introduced this concept independently at the same time as they introduced the definition of $\varphi$-mixing. The definition of $\rho$-mixing was introduced by Kolmogorov and Rozanov (1960). All these concepts describe the asymptotic independence of random variables when the difference of their indices goes to infinity. The 1971 monograph by Ibragimov and Linnik, "*Independent and Stationary Sequence of Random Variables*" summed up main results of convergence in distribution for a mixing sequence up to the sixties. Since the theory of weak convergence of probability measures appears, particularly, following the monograph by Billingsley, "*Convergence of Probability Measures*" (1968), the weak convergence for a sequence of mixing random variables attracts the attentions of many authors and some

ideal results have been obtained. The theory of strong approximations for a sequence of dependent random variables is discussed systematically in Philipp and Stout's monograph "*Almost Sure Invariance Principle for Partial Sums of Weakly Dependent Random Variables*" (1975). The results of this monograph have been improved comprehensively by us.

The modern limit theory for a sequence of mixing random variables has been studied deeply by many authors. This book will introduce them comprehensively, including Z. Y. Lin, C. R. Lu and Q. M. Shao's work for the weak convergence and strong approximations.

The book consists of four parts. The first part contains two chapters. We shall introduce the definitions of various mixing sequences and give a series of inequalities for mixing random variables, some of them are due to Shao. These inequalities are indispensable tools for the proofs of various limit theorems.

In the second part, which is separated into five chapters, the weak convergence, Berry-Esseen inequality and the rate of weak convergence are discussed. Some ideal results, such as weak convergence of $\rho$-mixing sequences, will be introduced.

In the third part, the almost sure convergence and strong approximations for the mixing random variables are studied. There are four chapters in this part. Some best results will be presented, such as strong approximations of partial sums for mixing sequences, which are done by Shao and Lu; the limiting behaviour of the increments of partial sums for a mixing sequence is obtained by Lin, et al.

In the fourth part, the weak convergence and strong approximations for some statistics with mixing dependent samples and some other kinds of dependent random variables are studied. Most results are profound.

Our best thanks are due to Dr. Q. M. Shao, whose results enrich greatly the book. We also want to thank all colleagues who help us to complete the book. We express our most gratitude to National Science Foundation of China and Zhejiang Province for their financial supports as well.

Lin, Z. Y.

Lu, C. R.

Hangzhou University, May 1996

# Contents

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# Part I Introduction

In this part, we introduce some common and important definitions of weakly dependent random variables, establish some bounds of covariances for the various mixing sequences, and also discuss the relations between each other for different definitions. These will be given in Chapter 1.

In Chapter 2, we give the estimations of some kind of moments of partial sums of a mixing sequence, which play important roles in the limit theorems and will be used often.

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# Chapter 1 Definitions and Basic Inequalities

In this book, we always assume that $\{X_n, n \geq 1\}$ is a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. There are many ways to describe weak dependence or asymptotic independence of $\{X_n\}$. In Section 1.1, we give some common and important definitions of this kind. In Section 1.2, some basic inequalities on covariances of $\{X_n\}$ are established, which are useful for studying limit properties of $\{X_n\}$. In these sections, we also discuss the relations between each other for different definitions.

## 1.1 Definitions

Let $\mathcal{A}$ and $\mathcal{B}$ be sub-$\sigma$-fields of $\mathcal{F}, L_p(\mathcal{A})$ a set of all $\mathcal{A}$-measurable random variables with *p-th* moments. Define

$$\begin{aligned}
\alpha(\mathcal{A},\mathcal{B}) &= \sup_{A\in\mathcal{A},B\in\mathcal{B}} |P(AB)-P(A)P(B)|,\\
\rho(\mathcal{A},\mathcal{B}) &= \sup_{X\in L_2(\mathcal{A}),Y\in L_2(\mathcal{B})} \frac{|EXY-EXEY|}{\sqrt{\mathrm{Var}X\ \mathrm{Var}Y}},\\
\varphi(\mathcal{A},\mathcal{B}) &= \sup_{A\in\mathcal{A},B\in\mathcal{B},P(A)>0} |P(B|A)-P(B)|,\\
\psi(\mathcal{A},\mathcal{B}) &= \sup_{A\in\mathcal{A},B\in\mathcal{B},P(A)P(B)>0} \frac{|P(AB)-P(A)P(B)|}{P(A)P(B)},\\
\beta(\mathcal{A},\mathcal{B}) &= E(\mathrm{tvar}_{B\in\mathcal{B}}|P(B|\mathcal{A})-P(B)|),\\
\lambda(\mathcal{A},\mathcal{B}) &= \sup_{X\in L_{1/\alpha}(\mathcal{A}),Y\in L_{1/\beta}(\mathcal{B})} \frac{|EXY-EXEY|}{\|X\|_{1/\alpha}\|Y\|_{1/\beta}},
\end{aligned}$$

where tvar means total variation and $\|X\|_p = (E|X|^p)^{1/p}$. Let $\mathcal{F}_a^b = \sigma(X_i,\, a \leq i \leq b)$, $\mathbb{Z}$ a set of all integers, $\mathbb{Z}^+$ a set of all non-negative integers, $\mathbb{N}$ a set of all positive integers. Some commom and important

definitions of mixing sequences are as follows:

**Definition 1.1.1.** A sequence $\{X_n, n \geq 1\}$ is said to be *α-mixing* or *strong mixing* if

$$\alpha(n) = \sup_{k \in \mathbb{N}} \alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

**Definition 1.1.2.** A sequence $\{X_n, n \geq 1\}$ is said to be *ρ-mixing* if

$$\rho(n) = \sup_{k \in \mathbb{N}} \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

**Definition 1.1.3.** A sequence $\{X_n, n \geq 1\}$ is said to be *φ-mixing* or *uniformly strong mixing* if

$$\varphi(n) = \sup_{k \in \mathbb{N}} \varphi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

**Definition 1.1.4.** A sequence $\{X_n, n \geq 1\}$ is said to be *ψ-mixing* or **-mixing* if

$$\psi(n) = \sup_{k \in \mathbb{N}} \psi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

**Definition 1.1.5.** A sequence $\{X_n, n \geq 1\}$ is said to be *absolutely regular* if

$$\beta(n) = \sup_{k \in \mathbb{N}} \beta(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

**Definition 1.1.6.** Let $0 \leq \alpha, \beta \leq 1, \alpha + \beta = 1$. A sequence $\{X_n, n \geq 1\}$ is said to be $(\alpha, \beta)$ *-mixing* if

$$\lambda(n) = \sup_{k \in \mathbb{N}} \lambda(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

**Remark 1.1.1.** The versions of the above definitions for a sequence with time-parameter set $R^+$ or $R$ or $\mathbb{Z}$ are trivial.

**Remark 1.1.2.** The concept of α-mixing was introduced by Rosenblatt (1956). The concept of ρ-mixing was introduced by Kolmogorov and Rozanov (1960). Dobrushin (1956) first introduced the definition of φ-mixing for a Markov process. This definition for a stationary process was presented by Ibragimov(1959) and Rozanov and Volconski (1959) respectively (one can also trace back to Hirschfeld 1935 and Gebelein 1941).

Absolute regularity was introduced by Kolmogorov (1959),(cf. Rosanov and Volconski 1959). Blum, Hanson and Koopmans (1963) presented the concept of $\psi$-mixing. $(\alpha, \beta)$-mixing was introduced by Bradley (1985a) and Shao(1989a) independently.

**Remark 1.1.3.** Doob(1953) showed that a Döeblin irreducible Markov chain is $\varphi$-mixing with $\varphi(n) \leq ab^n$ for some $a > 0$ and $0 \leq b < 1$; Rosenblatt (1971) showed that a purely non-deterministic Markov chain is $\alpha$-mixing; Davydov (1973) gave a class of Markov chains which are $\beta$-mixing.

**Remark 1.1.4.** For simplicity, we always assume that the mixing coefficients $\alpha(n), \rho(n), \cdots, \lambda(n)$ all are non-increasing.

It is clear from the definitions that

$$\rho(n) = \lambda_{1/2,1/2}(n), \quad \lambda_{1,0}(n) = \varphi(n) \leq \psi(n),$$

and further

$$\alpha(n) \leq \rho(n)$$

by taking $X = 1_A$ and $Y = 1_B$ in the definition of $\rho$-mixing.

Kolmogorov and Rozanov (1960) investigated the relation between $\alpha$-mixing and $\rho$-mixing for a Gaussian sequence.

**Theorem 1.1.1.** *For a Gaussian sequence $\{X_n, n \geq 1\}$, we have*

$$\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \leq \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \leq 2\pi\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty).$$

**Proof.** The former inequality is obvious.

For any $\varepsilon > 0$, there exist two normal random variables $X \in L_2(\mathcal{F}_1^k), Y \in L_2(\mathcal{F}_{k+n}^\infty)$ such that $EX = EY = 0, \text{Var}X = \text{Var}Y = 1$ and

$$r := EXY \geq \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) - \varepsilon.$$

Noting that $A := \{X > 0\} \in \mathcal{F}_1^k, \quad B := \{Y > 0\} \in \mathcal{F}_{k+n}^\infty$, we have

$$P(AB) = \frac{1}{4} + \frac{1}{2\pi} \arcsin r, \quad P(A)P(B) = \frac{1}{4} \tag{1.1.1}$$

by elementary calculations (see Cramér 1946, p.290). If $\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) > \frac{1}{4}$, it is clear that

$$2\pi\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) > \frac{\pi}{2} \geq \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty);$$

if $\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \leq \frac{1}{4}$, by (1.1.1) we obtain

$$\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \geq P(AB) - P(A)P(B) = \frac{1}{2\pi} \arcsin r,$$

which implies

$$\rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) - \varepsilon \leq r \leq \sin 2\pi\alpha \leq 2\pi\alpha.$$

The theorem is proved by arbitrariness of $\varepsilon$.

Kolmogorov and Rozanov (1960) also studied the relation between the spectral function of a (weakly) stationary sequence and $\rho$-mixing property. At first, we give some notations and concepts about a stationary sequence $\{X_n, n \in \mathbb{N}\}$. Let the covariance function of $\{X_n\}$

$$R(n) = EX_m X_{m+n}.$$

By the Herglotz theorem, there exists the spectral resolution for $R(n)$ as follows:

$$R(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, dF(\lambda),$$

where $F(\lambda)$ is called the spectral function of the stationary sequence. When the spectral function is absolutely continuous, its derivative $f(\lambda) = F'(\lambda)$ is called the spectral density of the stationary sequence.

**Theorem 1.1.2.** *If the spectral function of a stationary sequence is not absolutely continuous, then $\rho(n) \equiv 1$, i.e. the sequence is not $\rho$-mixing. Conversely, if the spectral function is absolutely continuous, then*

$$\rho(n) = \inf_h \operatorname{ess\,sup}_\lambda \left| f(\lambda) - e^{i\lambda n} h(e^{-i\lambda}) \right| / f(\lambda),$$

*where the inf is extended over all functions which is analytically continuable in unit circle; and further, if there exists an analytic function $h_0(z)$ in unit circle with the boundary value $h_0(e^{-i\lambda})$ such that $|f(\lambda)/h_0(e^{-i\lambda})| \geq \varepsilon > 0$ and $(f(\lambda)/h_0(e^{-i\lambda}))^{(k)}$ is bounded uniformly, then*

$$\rho(n) \leq cn^{-k}$$

*for some $c > 0$. In particular, when $f(\lambda)$ is a rational function of $e^{i\lambda}$,*

$$\rho(n) = e^{-cn}$$

*for some $c > 0$.*

The Proof of Theorem 1.1.2 is omitted(Kolmogorov, Rozanov 1960).

## 1.2 Basic inequalities

Let $X$ be $\mathcal{F}_{-\infty}^{k}$ measurable and $Y$ be $\mathcal{F}_{k+n}^{\infty}$ measurable.

In this section, we establish some bounds of the covariance $\mathrm{Cov}(X,Y) = EXY - EXEY$ for the various mixing sequences.

At first, we consider the $\alpha$-mixing case.

**Lemma 1.2.1.** *Let $\{X_n, n \in \mathbb{Z}\}$ be an $\alpha$-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $|X| \le C_1$ and $|Y| \le C_2$. Then*

$$|EXY - EXEY| \le 4C_1C_2\alpha(n). \tag{1.2.1}$$

**Proof.** By the property of conditional expectation, we have

$$\begin{aligned}|EXY - EXEY| &= \left|E\{X(E(Y|\mathcal{F}_{-\infty}^{k}) - EY)\}\right| \\ &\le C_1E|E(Y|\mathcal{F}_{-\infty}^{k}) - EY| \\ &= C_1|E\xi\{E(Y|\mathcal{F}_{-\infty}^{k}) - EY\}|,\end{aligned}$$

where $\xi = \mathrm{sgn}(E(Y|\mathcal{F}_{-\infty}^{k}) - EY) \in \mathcal{F}_{-\infty}^{k}$, i.e.

$$|EXY - EXEY| \le C_1|E\xi Y - E\xi EY|.$$

With the same argument procedure it follows that

$$|E\xi Y - E\xi EY| \le C_2|E\xi\eta - E\xi E\eta|,$$

where $\eta = \mathrm{sgn}\ (E(\xi|\mathcal{F}_{k+n}^{\infty}) - E\xi)$. Therefore

$$|EXY - EXEY| \le C_1C_2|E\xi\eta - E\xi E\eta|. \tag{1.2.2}$$

Put $A = \{\xi = 1\}, B = \{\eta = 1\}$. It is clear that $A \in \mathcal{F}_{-\infty}^{k}, B \in \mathcal{F}_{k+n}^{\infty}$. Using the definition of $\alpha$-mixing, we obtain

$$\begin{aligned}&|E\xi\eta - E\xi E\eta| \\ &\quad = |P(AB) + P(A^cB^c) - P(A^cB) - P(AB^c) \\ &\qquad - (P(A) - P(A^c))(P(B) - P(B^c))| \\ &\quad \le 4\alpha(n).\end{aligned}$$

Inserting it into (1.2.2) yields (1.2.1).

**Lemma 1.2.2.** *Let $\{X_n, n \in \mathbb{Z}\}$ be an $\alpha$-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $E|X|^p < \infty$ for some $p > 1$ and $|Y| \le C$. Then*

$$|EXY - EXEY| \le 6C\|X\|_p(\alpha(n))^{1/q}, \qquad (1.2.3)$$

*where $1/p + 1/q = 1$.*

**Proof.** Let $X_N = XI(|X| \le N)$, $X_N' = X - X_N$. Write

$$|EXY - EXEY| \le |EX_NY - EX_NEY| + |EX_N'Y - EX_N'EY|.$$

By Lemma 1.2.1, $|EX_NY - EX_NEY| \le 4CN\alpha(n)$. For the second term of the right hand side of the above inequality, we have

$$|EX_N'Y - EX_N'EY| \le 2CE|X_N'| \le 2CN^{-p+1}E|X|^p.$$

Taking $N = \|X\|_p(\alpha(n))^{-1/p}$ yields (1.2.3).

For a random variable $X$ and a continuous non-decreasing function $f(x)$ on $R^+$ with $f(0) = 0$, which doesn't identically equal to zero, define

$$\|X\|_f = \inf\{t > 0, Ef(|X|/t) \le 1\}.$$

From this definition, it is easy to know that

$$\|X\|_f = 0 \Longleftrightarrow X = 0 \quad \text{a.s.} \qquad (1.2.4)$$

and if $0 < \|X\|_f < \infty$, then $Ef(|X|/\|X\|_f) \le 1$. Moreover, if $|X_1| \le |X_2|$ a.s., then $\|X_1\|_f \le \|X_2\|_f$.

**Lemma 1.2.3.** *Let $\{X_n,\ n \in \mathbb{Z}\}$ be an $\alpha$-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$, $Y \in \mathcal{F}_{k+n}^{\infty}$, $f(x)$ and $g(x)$ be two continuous functions on $R^+$ with $f(0) = g(0) = 0$, $f(x)/x^{\frac{r+s}{r}} \nearrow \infty$ and $g(x)/x^{\frac{r+s}{s}} \nearrow \infty$ for some $r > 0, s > 0, \|X\|_f < \infty, \|Y\|_g < \infty$. Then*

$$|EXY - EXEY| \le 10 \operatorname{inv} f\Big(\frac{1}{\alpha(n)}\Big) \operatorname{inv} g\Big(\frac{1}{\alpha(n)}\Big)\alpha(n)\|X\|_f\|Y\|_g. \quad (1.2.5)$$

**Proof.** It is easy to see that $E|X|^{1+s/r} < \infty$ and $E|Y|^{1+r/s} < \infty$ by the conditions of the lemma . If either $\|X\|_f = 0$ or $\|Y\|_g = 0$, (1.2.4) implies that (1.2.5) holds. If $\alpha(n) = 0$, (1.2.5) is trivial by independence of $X$ and $Y$. Now we assume that $\|X\|_f > 0, \|Y\|_g > 0$ and $\alpha(n) > 0$. There are $M > 0$ and $N > 0$ such that

$$\alpha(n) = 1/f(M/\|X\|_f) = 1/g(N/\|Y\|_g).$$

Let

$$X_M = XI(|X| \le M), \quad X'_M = X - X_M,$$
$$Y_N = YI(|Y| \le N), \quad Y'_N = Y - Y_N.$$

We have

$$\begin{aligned}
&|EXY - EXEY| \\
&\quad \le |EX_MY_N - EX_MEY_N| + |EX'_MY_N - EX'_MEY_N| \\
&\qquad + |EX_MY'_N - EX_MEY'_N| + |EX'_MY'_N - EX'_MEY'_N| \\
&\quad =: I_1 + I_2 + I_3 + I_4. \qquad (1.2.6)
\end{aligned}$$

By Lemma 1.2.1, $I_1 \le 4MN\alpha(n)$. Noting that $f(x)/x \nearrow \infty$ and $g(x)/x \nearrow \infty$, we have

$$\begin{aligned}
E|X'_M| &= E(|X'_M|/\|X'_M\|_f)\|X'_M\|_f \\
&\le Ef(|X'_M|/\|X'_M\|_f)M/f(M/\|X'_M\|_f) \\
&\le M/f(M/\|X\|_f).
\end{aligned}$$

Therefore

$$I_2 \le 2MN/f(M/\|X\|_f) = 2\,\mathrm{inv}\, f\Big(\frac{1}{\alpha(n)}\Big)\,\mathrm{inv}\, g\Big(\frac{1}{\alpha(n)}\Big)\alpha(n)\|X\|_f\|Y\|_g.$$

Similarly, we have the same estimation for $I_3$.

Furthermore, noting that $f(x)/x^{\frac{r+s}{r}} \nearrow \infty$ and $g(x)/x^{\frac{r+s}{s}} \nearrow \infty$, we have

$$\begin{aligned}
EX'_MY'_N &\le \Big(E(|X'_M|/\|X'_M\|_f)^{\frac{r+s}{r}}\Big)^{\frac{r}{r+s}} \\
&\quad \cdot \Big(E(|Y'_N|/\|Y'_N\|_g)^{\frac{r+s}{s}}\Big)^{\frac{s}{r+s}}\|X'_M\|_f\|Y'_N\|_g \\
&\le \Big(Ef(|X'_M|/\|X'_M\|_f)\Big)^{\frac{r}{r+s}}\Big(Eg(|Y'_N|/\|Y'_N\|_g)\Big)^{\frac{s}{r+s}} \\
&\quad \cdot MN/\Big(f(M/\|X'_M\|_f)\Big)^{\frac{r}{r+s}}\Big(g(N/\|Y'_N\|_g)\Big)^{\frac{s}{r+s}} \\
&\le MN/\Big(f(M/\|X\|_f)\Big)^{\frac{r}{r+s}}\Big(g(N/\|Y\|_g)\Big)^{\frac{s}{r+s}}.
\end{aligned}$$

Hence

$$\begin{aligned}
I_4 &\le 2MN/\Big(f(M/\|X\|_f)\Big)^{\frac{r}{r+s}}\Big(g(N/\|Y\|_g)\Big)^{\frac{s}{r+s}} \\
&\le 2\,\mathrm{inv}\, f\Big(\frac{1}{\alpha(n)}\Big)\,\mathrm{inv}\, g\Big(\frac{1}{\alpha(n)}\Big)\alpha(n)\|X\|_f\|Y\|_g.
\end{aligned}$$

Now, inserting these estimations into (1.2.6) yields (1.2.5).

As some consequences of this lemma, we have

**Lemma 1.2.4.** *Let $\{X_n, n \in \mathbb{Z}\}$ be an $\alpha$-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $E|X|^p < \infty$ and $E|Y|^q < \infty$, $\frac{1}{p} + \frac{1}{q} < 1$. Then*

$$|EXY - EXEY| \leq 10\|X\|_p\|Y\|_q(\alpha(n))^{1-\frac{1}{p}-\frac{1}{q}}. \tag{1.2.7}$$

**Lemma 1.2.5.** *Let $\{X_n, n \in \mathbb{Z}\}$ be an $\alpha$-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $E|X|^{2+\delta} \leq C_1, E|Y|^{2+\delta} \leq C_2$. Then*

$$|EXY - EXEY| \leq 10(C_1C_2)^{\frac{1}{2+\delta}}(\alpha(n))^{\frac{\delta}{2+\delta}}. \tag{1.2.8}$$

For an $(\alpha, \beta)$-mixing sequence and a $\rho$-mixing sequence, we have the following lemmas.

**Lemma 1.2.6.** *Let $\{X_n, n \in \mathbb{Z}\}$ be an $(\alpha, \beta)$-mixing sequence, $X \in L_p(\mathcal{F}_{-\infty}^{k})$ and $Y \in L_q(\mathcal{F}_{k+n}^{\infty})$ with $p, q \geq 1$ and $1/p + 1/q = 1$. Then*

$$|EXY - EXEY| \leq 4\lambda(n)^{\frac{1}{\alpha p}\wedge\frac{1}{\beta q}}\|X\|_p\|Y\|_q. \tag{1.2.9}$$

**Proof.** Without loss of generality, assume that $\alpha p \geq 1$, which implies that $\beta q \leq 1$. Put

$$Y_1 = YI(|Y| \leq C), \quad Y_2 = Y - Y_1,$$

where $C$ is a positive constant specified later on. Write

$$|EXY - EXEY| \leq |EXY_1 - EXEY_1| + |EXY_2 - EXEY_2|. \tag{1.2.10}$$

By the definition of $(\alpha, \beta)$-mixing and the Hölder inequality

$$\begin{aligned}|EXY_1 - EXEY_1| &\leq \lambda(n)\|X\|_{1/\alpha}\|Y_1\|_{1/\beta}\\ &\leq \lambda(n)C^{1-\beta q}\|X\|_p\|Y\|_q^{\beta q},\end{aligned}$$

$$\begin{aligned}|EXY_2| &\leq (E|Y_2|^q)^{1-\frac{1}{\alpha p}}(E|X|^{\alpha p}|Y_2|^{\beta q})^{\frac{1}{\alpha p}}\\ &\leq (E|Y_2|^q)^{1-\frac{1}{\alpha p}}\left(E|X|^{\alpha p}E|Y_2|^{\beta q} + \lambda(n)(E|X|^p)^\alpha(|Y_2|^q)^\beta\right)^{\frac{1}{\alpha p}}\\ &\leq (E|Y|^q)^{1-\frac{1}{\alpha p}}\left(E|X|^{\alpha p}E|Y|^qC^{-\alpha q} + \lambda(n)(E|X|^p)^\alpha(E|Y|^q)^\beta\right)^{\frac{1}{\alpha p}}\\ &\leq \|X\|_p\|Y\|_q^qC^{-\frac{q}{p}} + \lambda^{\frac{1}{\alpha p}}(n)\|X\|_p\|Y\|_q\end{aligned}$$

and

$$|EXEY_2| \le \|X\|_p \|Y\|_q^q C^{-q/p}.$$

Inserting these estimations into (1.2.10) and taking $C = \|Y\|_q (\lambda(n))^{-1/\alpha q}$ we obtain (1.2.9).

Let $p = q = 2$ in (1.2.9). It is easy to see that

$$\rho(n) \le 4\lambda(n)^{\frac{1}{2\alpha} \wedge \frac{1}{2\beta}}. \tag{1.2.11}$$

As a consequence of Lemma 1.2.6, noting that $\rho(n) = \lambda_{1/2,1/2}(n)$, we have

**Lemma 1.2.7.** *Let $\{X_n,\ n \in \mathbb{Z}\}$ be a $\rho$-mixing sequence, $X \in L_p(\mathcal{F}_{-\infty}^k)$ and $Y \in L_q(\mathcal{F}_{k+n}^\infty)$ with $p, q \ge 1$ and $1/p + 1/q = 1$. Then*

$$|EXY - EXEY| \le 4\rho(n)^{\frac{2}{p} \wedge \frac{2}{q}} \|X\|_p \|Y\|_q.$$

For the $\varphi$-mixing case, we have the following three results.

**Lemma 1.2.8.** *Let $\{X_n,\ n \in \mathbb{Z}\}$ be a $\varphi$-mixing sequence, $X \in L_p(\mathcal{F}_{-\infty}^k)$ and $Y \in L_q(\mathcal{F}_{k+n}^\infty)$ with $p, q \ge 1$ and $1/p + 1/q = 1$. Then*

$$|EXY - EXEY| \le 2(\varphi(n))^{\frac{1}{p}} \|X\|_p \|Y\|_q. \tag{1.2.12}$$

**Proof.** At first, we assume that $X$and $Y$are simple functions, i.e.

$$X = \sum_i a_i I_{A_i}, \quad Y = \sum_j b_j I_{B_j},$$

where both $\sum_i$ and $\sum_j$ are finite sums and $A_i \cap A_k = \emptyset \quad (i \ne k)$, $B_j \cap B_l = \emptyset \quad (j \ne l)$, $A_i \in \mathcal{F}_{-\infty}^k$, $B_j \in \mathcal{F}_{k+n}^\infty$. So

$$EXY - EXEY = \sum_{i,j} a_i b_j P(A_i B_j) - \sum_{i,j} a_i b_j P(A_i) P(B_j).$$

By the Hölder inequality we have

$$
\begin{aligned}
&|EXY-EXEY|\\
&\quad=\Big|\sum_i a_i(P(A_i))^{1/p}\sum_j(P(B_j|A_i)-P(B_j))b_j(P(A_i))^{1/q}\Big|\\
&\quad\le\Big(\sum_i|a_i|^pP(A_i)\Big)^{1/p}\Big(\sum_i P(A_i)|\sum_j b_j(P(B_j|A_i)-P(B_j))|^q\Big)^{1/q}\\
&\quad\le\|X\|_p\Big|\sum_i P(A_i)\Big(\sum_j|b_j|^q(P(B_j|A_i)\\
&\qquad+P(B_j))\Big)\Big(\sum_j|P(B_j|A_i)-P(B_j)|\Big)^{\frac{q}{p}}\Big|^{\frac{1}{q}}\\
&\quad\le 2^{1/q}\|X\|_p\|Y\|_q\max_i\Big(\sum_j|P(B_j|A_i)-P(B_j)|\Big)^{1/p}. \qquad (1.2.13)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_j|P(B_j|A_i)-P(B_j)|&=(P(\cup_j^+B_j|A_i)-P(\cup_j^+B_j))\\
&\quad-(P(\cup_j^-B_j|A_i)-P(\cup_j^-B_j))\\
&\le 2\varphi(n), \qquad (1.2.14)
\end{aligned}
$$

where the union $\cup_j^+(\cup_j^-)$ is carried out over $j$ such that $P(B_j|A_i)-P(B_j)>0$ $(P(B_j|A_i)-P(B_j)<0)$. Inserting (1.2.14) into (1.2.13) yields (1.2.12) for the simple function case.

In order to complete the proof of the lemma, let

$$
X_N=\begin{cases}0 & \text{if } |X|>N.\\ k/N & \text{if } k/N<X\le(k+1)/N,\ |X|\le N;\end{cases}
$$

$$
Y_N=\begin{cases}0 & \text{if } |Y|>N.\\ k/N & \text{if } k/N<Y\le(k+1)/N,\ |Y|\le N.\end{cases}
$$

We have showed that (1.2.12) is true for $X_N$ and $Y_N$. Moreover, note

$$
E|X-X_N|^p\to 0,\ E|Y-Y_N|^q\to 0, \quad \text{as } N\to\infty.
$$

Letting $N\to\infty$, we obtain (1.2.12) for the general case.

Let $p=q=2$ in (1.2.12). It is easy to see that

$$
\rho(n)\le 2\varphi^{1/2}(n). \qquad (1.2.15)
$$

From the proof of Lemma 1.2.8, we can see that

**Lemma 1.2.9.** *Let $\{X_n, n \in \mathbb{Z}\}$ be a $\varphi$-mixing sequence, $X \in \mathcal{F}_{-\infty}^k$ and $Y \in \mathcal{F}_{k+n}^\infty$ with $|X| \le C_1$ and $|Y| \le C_2$. Then*

$$|EXY - EXEY| \le 2C_1C_2\varphi(n). \tag{1.2.16}$$

Let $p = 1$ and $q = \infty$ in (1.2.12). From Lemma 1.2.8, we also have

**Lemma 1.2.10.** *Let $\{X_n, n \in \mathbb{Z}\}$ be a $\varphi$-mixing sequence, $X \in \mathcal{F}_{-\infty}^k$ and $Y \in \mathcal{F}_{k+n}^\infty$ with $E|X| < \infty$ and $|Y| \le C$. Then*

$$|EXY - EXEY| \le 2C\varphi(n)E|X|. \tag{1.2.17}$$

Finally, we consider the $\psi$-mixing case.

**Lemma 1.2.11.** *Let $\{X_n, n \in \mathbb{Z}\}$ be a $\psi$-mixing sequence, $X \in \mathcal{F}_{-\infty}^k$ and $Y \in \mathcal{F}_{k+n}^\infty$ with $E|X| < \infty$ and $E|Y| < \infty$. Then $E|XY| < \infty$ and*

$$|EXY - EXEY| \le \psi(n)E|X|E|Y|. \tag{1.2.18}$$

**Proof.** At first, we assume that $X$ and $Y$ are non-negative simple functions. We have

$$\begin{aligned}|EXY - EXEY| &= |\sum_{i,j} a_ib_j(P(A_iB_j) - P(A_i)P(B_j))| \\ &\le \sum_{i,j} a_ib_j\psi(n)P(A_i)P(B_j) \\ &= \psi(n)EXEY.\end{aligned}$$

From this, (1.2.18) holds for non-negative random variables $X$ and $Y$.

For the general case, write $X = X^+ - X^-$, $Y = Y^+ - Y^-$. We have

$$\begin{aligned}&|EXY - EXEY| \\ &\quad\le |EX^+Y^+ - EX^+EY^+| + |EX^+Y^- - EX^+EY^-| \\ &\quad\quad + |EX^-Y^+ - EX^-EY^+| + |EX^-Y^- - EX^-EY^-| \\ &\quad\le \psi(n)(EX^+ + EX^-)(EY^+ + EY^-) \\ &\quad\le \psi(n)E|X|E|Y|.\end{aligned}$$

Finally, we summarize the relations between one and another of variaous mixing properties. It is easy to verify that

$$2\alpha(n) \le \beta(n) \le \varphi(n). \tag{1.2.19}$$

With a necessary and sufficient condition for Markov processes to be $\psi$-mixing, one can show that a $\varphi$-mixing (Markov) sequence is not $\psi$-mixing (Blum, Hanson and Koopmans 1963). Ibragimov and Solev (1969) given an example of a stationary $\alpha$-mixing Gaussian process which is not $\beta$-mixing; such a process is $\rho$-mixing but not $\beta$-mixing. Davydov (1973) constructed a stationary $\alpha$-mixing Markov process with less than geometric rate of decay of the mixing coefficients, which is not $\rho$-mixing. It is possible that a geometrically ergodic Markov process which is not Doeblin recurrent is $\beta$-mixing and not $\varphi$-mixing (Andrews 1984). Combining these results and recalling Remark 1.1.4, (1.2.11) and (1.2.15) we have

$$\psi-\text{mixing}\left\{\begin{matrix}\Longrightarrow\\ \not\Longleftarrow\end{matrix}\right\}\varphi-\text{mixing}\left\{\begin{matrix}\Longrightarrow\\ \not\Longleftarrow\end{matrix}\right.\left\{\begin{array}{l}\beta-\text{ mixing}\left\{\begin{matrix}\Longrightarrow\\ \not\Longleftarrow\end{matrix}\right\}\alpha-\text{ mixing}\\ \;\not\Uparrow\quad\;\not\Downarrow\\ \rho-\text{ mixing}\left\{\begin{matrix}\Longrightarrow\\ \not\Longleftarrow\end{matrix}\right\}\alpha-\text{ mixing}\end{array}\right.$$

$$\Uparrow$$

$$\lambda-\text{mixing}$$

# Chapter 2 Moment Estimations of Partial Sums

The estimations of some kind of moments of partial sums of a mixing sequence play important roles in showing limit theorems. In Section 2.1, we give some forms of the variances of partial sums of mixing sequences of various kinds. Section 2.2 is devoted to deduce some inequalities for the moments of partial sums. In passing we also give some probability inequalities in this section.

## 2.1 Variances of partial sums

Let $\{X_n, n \in \mathbb{Z}\}$ be a (weakly) stationary sequence with $EX_1 = 0$, $EX_1^2 < \infty$. Put $S_n = \sum_{i=1}^n X_i$. We investigate its variance $\mathrm{Var}S_n$. Let $R(n)$ be the correlation function and $F(\lambda)$ be the spectral function of the sequence $\{X_n\}$.

At first, we give the representations of $\mathrm{Var}S_n$ by $R(n)$ or $F(\lambda)$.

**Theorem 2.1.1.**

$$\mathrm{Var}S_n = \sum_{|j|<n} (n - |j|)R(j), \tag{2.1.1}$$

$$\mathrm{Var}S_n = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{n\lambda}{2}}{\sin^2 \frac{\lambda}{2}} dF(\lambda). \tag{2.1.2}$$

*If the spectral function is absolutely continuous, i.e., there is a spectral density $f(\lambda)$, and further, if $f(\lambda)$ is continuous at $\lambda = 0$, then*

$$\mathrm{Var}S_n = 2\pi f(0)n + o(n) \quad \text{as } n \to \infty. \tag{2.1.3}$$

The proof of the theorem can be found in the book of Ibragimov and Linnik (1971) and is not presented here.

When a stationary sequence satisfies a certain mixing condition, $\mathrm{Var}S_n$ possesses more evident form.

**Theorem 2.1.2.** *Suppose that a stationary sequence $\{X_n\}$ is $\varphi$-mixing and $\mathrm{Var}S_n \to \infty$ as $n \to \infty$. Then*

$$\mathrm{Var}S_n = nh(n), \tag{2.1.4}$$

*where $h(n)$ is a slowly varying function of $n$ and its domain of definition can be extended to $R$ such that $h(x)$ is also slowly varying on $R$.*

**Proof.** We first prove that $h(n)$ is slowly varying. Put $\sigma_n^2 = \mathrm{Var}S_n$. Equivalently, we show that for every positive integer $k$

$$\lim_{n\to\infty} \sigma_{kn}^2/\sigma_n^2 = k. \tag{2.1.5}$$

Let

$$\begin{aligned}
\xi_j &= \sum_{s=1}^{n} X_{(j-1)n+(j-1)r+s}, \quad j = 1, 2, \cdots, k, \\
\eta_j &= \sum_{s=1}^{r} X_{jn+(j-1)r+s}, \quad j = 1, 2, \cdots, k-1, \\
\eta_k &= -\sum_{s=1}^{(k-1)r} X_{nk+s},
\end{aligned}$$

where $r = [\log \sigma_n^2]$. By Theorem 1.1.2, $\{X_n\}$ possesses a spectral density $f(\lambda)$. Using (2.1.2) we obtain

$$\begin{aligned}
\sigma_n^2 &= \int_{-\pi}^{\pi} \frac{\left(\sin^2 \frac{n\lambda}{2}\right)}{\left(\sin^2 \frac{\lambda}{2}\right)} f(\lambda)\, d\lambda \\
&\le n^2 \int_{-\pi}^{\pi} f(\lambda)\, d\lambda.
\end{aligned} \tag{2.1.6}$$

Hence $r = O(\log n)$. And further

$$\sigma_{kn}^2 = \mathrm{Var}S_{kn} = \sum_{j=1}^{k} E\xi_j^2 + 2\sum_{i\neq j} E\xi_i\xi_j + \sum_{i,j} E\xi_i\eta_j + \sum_{i,j} E\eta_i\eta_j. \tag{2.1.7}$$

By stationarity of the sequence, $E\xi_j^2 = \sigma_n^2 = \mathrm{Var}S_n$. From Lemma 1.2.8, we have

$$|E\xi_i\xi_j| \le 2\varphi(|i-j|r)^{1/2}\|\xi_i\|_2\|\xi_j\|_2 \le 2\varphi(r)^{1/2}\sigma_n^2 \tag{2.1.8}$$

for $i \neq j$. Using the Schwarz inequality and (1.2.6), we have

$$|E\xi_i\eta_j| \leq \|\xi_i\|_2\|\eta_j\|_2 = \sigma_n\sigma_r = O(\sigma_n \log \sigma_n), \tag{2.1.9}$$

$$|E\eta_i\eta_j| \leq \sigma_r^2 = O((\log \sigma_n)^2). \tag{2.1.10}$$

Inserting (2.1.8), (2.1.9) and (2.1.10) into (2.1.7) and noting $\varphi(r) = o(1)$ as $n \to \infty$, we obtain

$$\sigma_{kn}^2 = k\sigma_n^2 + o(\sigma_n^2),$$

which implies (2.1.5).

Next, we prove that the domain of $h(n)$ can be extended to $R$ such that $h(x)$ is also slowly varying on $R$. Recalling (2.1.6), we define

$$\psi(x) = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{x\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda)\, d\lambda,$$

$$h(x) = \psi(x)/x.$$

In order to show that $h(x)$ is slowly varying, it is enough to verify that for any $a > 0$

$$\lim_{x\to\infty} \psi(ax)/\psi(x) = a. \tag{2.1.11}$$

It is not difficult to know from the definition of $\psi$ that

$$\psi(x) = \psi([x])(1 + o(1))$$

as $x \to \infty$. When $a$ in (2.1.11) is an integer, we have

$$\frac{\psi(ax)}{\psi(x)} = \frac{[ax]h([ax])}{[x]h([x])}(1 + o(1)) = a(1 + o(1)).$$

Therefore, for $a = p/q$, where both $p$ and $q$ are integers, we obtain

$$\lim_{x\to\infty} \frac{\psi(ax)}{\psi(x)} = \lim_{x\to\infty} \frac{\psi(p\frac{x}{q})}{\psi(\frac{x}{q})} \frac{\psi(\frac{x}{q})}{\psi(q\frac{x}{q})} = \frac{p}{q} = a.$$

For any positive real number $a$, put

$$\psi_1(a) = \varliminf_{x\to\infty} \frac{\psi(ax)}{\psi(x)}, \quad \psi_2(a) = \varlimsup_{x\to\infty} \frac{\psi(ax)}{\psi(x)}.$$

For any rational number $a$, $\psi_1(a) = \psi_2(a)$ by the above proof. Hence, it suffices to show that both $\psi_1(x)$ and $\psi_2(x)$ are continuous. Because

$$\begin{aligned}
&\Big|\frac{\psi((a+\varepsilon)x) - \psi(ax)}{\psi(x)}\Big| \\
&\leq \frac{1}{\psi(x)}\Big|\int_{-\pi}^{\pi} \frac{\sin^2 \frac{\varepsilon x\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda)\, d\lambda + \int_{-\pi}^{\pi} \frac{\sin \varepsilon x\lambda \sin ax\lambda}{2\sin^2 \frac{\lambda}{2}} f(\lambda)\, d\lambda\Big| \\
&\leq \frac{\psi(\varepsilon x)}{\psi(x)} + \Big(\frac{\psi(\varepsilon x)}{\psi(x)}\Big)^{\frac{1}{2}},
\end{aligned}$$

it is enough to show that $\psi_1(a)$ and $\psi_2(a)$ are continuous at $a = 0$. Using Property A4 about a slowly varying function(see Appendix A), for $\varepsilon > 0$ small enough, we have

$$\begin{aligned}\frac{\psi(\varepsilon x)}{\psi(x)} &= \frac{[\varepsilon x]h\Big(\frac{[\varepsilon x]}{[x]}[x]\Big)}{[x]h([x])}(1+o(1)) \\ &\le \varepsilon^{1/2}(1+o(1))\end{aligned}$$

as $x \to \infty$. Hence both $\psi_1(a)$ and $\psi_2(a)$ are continuous at $a = 0$. Theorem 2.1.2 are proved.

**Remark 2.1.1.** In the proof of Theorem 2.1.2, $\varphi$-mixing property is only used to give the inequality

$$E\Big|S_n \sum_{j=n+p}^{m+n+p} X_j\Big| \le 2\varphi(p)^{1/2}\|S_n\|_2\|S_m\|_2.$$

Then, for $\alpha$-mixing case, we have also

**Theorem 2.1.3.** *Let $\{X_n\}$ be a strictly stationary $\alpha$-mixing sequence satisfying that $EX_1 = 0$, $EX_1^2 < \infty$, $\sigma_n^2 = ES_n^2 \to \infty$ and $\{S_n^2/\sigma_n^2,\, n \ge 1\}$ is integrable uniformly. Then the conclusions of Theorem 2.1.2 hold true.*

**Proof.** By the proof of Theorem 2.1.2 and Remark 2.1.1, it suffices to show the following facts.

1. $\sigma_n^2 \to \infty$;
2. for any $\varepsilon > 0$, there exist $p = p(\varepsilon)$, $N = N(\varepsilon)$ such that

$$\Big|ES_n \sum_{j=n+p}^{m+n+p} X_j\Big| \le \varepsilon\sigma_n\sigma_m \qquad \text{if } n, m \ge N(\varepsilon).$$

The first fact is an assumption of the theorem. Consider the latter. From uniform integrability of $\{S_n^2/\sigma_n^2\}$, for any $\varepsilon > 0$, there exists a $K > 0$ such that for $p$ large enough,

$$\int_{S_n^2/\sigma_n^2 \ge K} S_n^2/\sigma_n^2\, dP < \frac{\varepsilon}{4}, \quad K\alpha(p) < \varepsilon/16.$$

Then, by Lemma 1.2.1, Schwarz's inequality and strict stationarity, we

obtain

$$\begin{aligned}
&\left|ES_n \sum_{j=n+p}^{m+n+p} X_j\right| \Big/ \sigma_n\sigma_m \\
&\le \int_{|\frac{S_n}{\sigma_n}|<\sqrt{K},\, |\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}|<\sqrt{K}} \left|\frac{S_n}{\sigma_n}\cdot\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right| dP \\
&\quad + \int_{|\frac{S_n}{\sigma_n}|\ge\sqrt{K}} \sqrt{K}\left|\frac{S_n}{\sigma_n}\right| dP \\
&\quad + \int_{|\frac{S_{m+n-p}-S_{n+p-1}}{\sigma_m}|\ge\sqrt{K}} \sqrt{K}\left|\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right| dP \\
&\quad + \int_{|\frac{S_n}{\sigma_n}|\ge\sqrt{K},\, |\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}|\ge\sqrt{K}} \left|\frac{S_n}{\sigma_n}\cdot\frac{S_{m+n+p}-S_{n+p-1}}{\sigma_m}\right| dP \\
&\le 4K\alpha(p) + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 \le \varepsilon.
\end{aligned}$$

For a $\rho$-mixing sequence, Peligrad (1982) showed the following general result. Denote $S_k(n) = S_{k+n} - S_k = \sum_{j=k+1}^{k+n} X_j$.

**Theorem 2.1.4.** *Let $\{X_n,\ n \ge 1\}$ be a $\rho$-mixing sequence of random variables with $EX_n = 0$. Assume that*

*(i)* $\sup_n EX_n^2 = \sigma_0^2 < \infty$;

*(ii)* $ES_n^2 \to \infty$ *as* $n \to \infty$;

*(iii)* $\lim_{n\to\infty} \dfrac{ES_k^2(n)}{ES_n^2} = 1$ *uniformly in* $k$.

*Then*

$$ES_n^2 = nh(n),$$

*where $h(n)$ is a slowly varying function and its domain of definition can be extended to $R$ such that $h(x)$ is also slowly varying on $R$. If, in addition, assume that*

*(iv)* $\sum_{n=1}^{\infty} \rho(2^n) < \infty$,

*then* $ES_n^2/n \to \sigma^2 > 0$.

In order to prove Theorem 2.1.4, we need the following lemma.

**Lemma 2.1.1.** *Let $\{X_n,\ n \ge 1\}$ be a $\rho$-mixing sequence with $EX_n = 0$. If condition (i) in Theorem 2.1.4 is satisfied, then, for natural numbers $p, q, m$ with $p + q = m$,*

$$\begin{aligned}
&(1-\rho(n))(ES_{km}^2(p) + ES_{km+p}^2(q)) - C_1 \\
&\qquad \le ES_{km}^2(m) \\
&\qquad \le (1+\rho(n))(ES_{km}^2(p) + ES_{km+p}^2(q)) + C_1, \quad (2.1.12)
\end{aligned}$$

*where $k$ and $n$ are positive integers and*

$$C_1 = C_1(m,p,n) \le 20\sigma_0^2 n^2 + 12\sigma_0 n(\|S_{km}(p)\|_2 + \|S_{km+p}(q)\|_2),$$

*and further*

$$(1-\rho(n))^{1/2}\|S_{km}(p)\|_2 \le \|S_{km}(m)\|_2 + C_2 \tag{2.1.13}$$

*where $C_2 \le 2\sigma_0 n$.*

**Proof.** By the definition of $\rho$-mixing, we have

$$\begin{aligned}\Big|E(S_{km}(p) + S_{km+p+i}(q))^2 - (ES^2_{km}(p) + ES^2_{km+p+i}(q))\Big| \\ \le \rho(i)(ES^2_{km}(p) + ES^2_{km+p+i}(q)).\end{aligned} \tag{2.1.14}$$

Noting that $S_{km+p+n}(q) = S_{km+p}(q) - S_{km+p}(n) + S_{(k+1)m}(n)$, we obtain

$$\|S_{km+p+n}(q)\|_2 = \|S_{km+p}(q)\|_2 + \theta_1, \tag{2.1.15}$$

where $|\theta_1| \le 2\sigma_0 n$. Hence, from (2.1.14) and (2.1.15), it follows that

$$\begin{aligned}(1-\rho(n))(ES^2_{km}(p) + ES^2_{km+p}(q) + \theta_2) \\ \le E(S_{km}(p) + S_{km+p+n}(q))^2 \\ \le (1+\rho(n))(ES^2_{km}(p) + ES^2_{km+p}(q) + \theta_2),\end{aligned} \tag{2.1.16}$$

where $|\theta_2| \le 4\sigma_0^2 n^2 + 4\sigma_0 n\|S_{km+p}(q)\|_2$. Write

$$S_{km}(m) = S_{km}(p) + S_{km+p}(n) + S_{km+p+n}(q) - S_{(k+1)m}(n).$$

Then

$$\|S_{km}(m)\|_2 = \|S_{km}(p) + S_{km+p+n}(q)\|_2 + \theta_3, \tag{2.1.17}$$

where $|\theta_3| \le 2\sigma_0 n$. Hence $ES^2_{km}(m) = E(S_{km}(p) + S_{km+p+n}(q))^2 + \theta_4$, where

$$|\theta_4| \le 12\sigma_0^2 n^2 + 4\sigma_0 n(\|S_{km}(p)\|_2 + \|S_{km+p}(q)\|_2).$$

Inserting it into (2.1.16) we obtain (2.1.12), where

$$\begin{aligned}C_1 &= \max(|(1-\rho(n))\theta_2 + \theta_4|, |(1+\rho(n))\theta_2 + \theta_4|) \\ &\le 20\sigma_0^2 n^2 + 12\sigma_0 n(\|S_{km}(p)\|_2 + \|S_{km+p}(q)\|_2).\end{aligned}$$

We turn to (2.1.13). (2.1.14) implies

$$(1-\rho(n))ES^2_{km}(p) \le E(S_{km}(p) + S_{km+p+n}(q))^2.$$

Then (2.1.13) is showed from (2.1.17).

**Proof of Theorem 2.1.4.** At first, we prove that for any $h \in \mathbb{N}$

$$\lim_{n\to\infty} ES_{hn}^2/ES_n^2 = h. \tag{2.1.18}$$

By (2.1.12) with $k = 0$, $m = hn$, $p = (h-1)n$, $q = n$, $n = [(ES_n^2)^{1/3}]$, we have

$$\begin{aligned}(1-\rho(n))&(ES_{(h-1)n}^2 + ES_{(h-1)n}^2(n)) - C_0 \\ &\le ES_{hn}^2 \\ &\le (1+\rho(n))(ES_{(h-1)n}^2 + ES_{(h-1)n}^2(n)) + C_0,\end{aligned}$$

where $C_0 = 20\sigma_0^2 n^2 + 12\sigma_0 n(\|S_{(h-1)n}\|_2 + \|S_{(h-1)n}(n)\|_2)$. Using conditions (ii) and (iii), (2.1.18) follows by induction on $h$. Therefore,

$$h(n) := ES_n^2/n$$

is a slowly varying function. Extend its domain by letting

$$h(t) = ES_{[t]}^2/t.$$

We show that

$$\lim_{n\to\infty} h((1-\varepsilon_n)n)/h(n) = 1, \tag{2.1.19}$$

where $\varepsilon_n \downarrow 0$ suah that $n\varepsilon_n$ are integers. By Property A4 (in Appendix),

$$\lim_{n\to\infty} \frac{h(n\varepsilon_n)}{h(n)}\varepsilon_n^\delta = 0. \tag{2.1.20}$$

Let $h_n = \max(h : hn\varepsilon_n < (1-\varepsilon_n)n)$. Note that

$$S_{n-n\varepsilon_n} = S_n - S_{h_n n\varepsilon_n}(n\varepsilon_n) + S_{h_n n\varepsilon_n}(p) - S_{h_n n\varepsilon_n + n\varepsilon_n}(p),$$

where $p = n - (h_n+1)n\varepsilon_n \le n\varepsilon_n$. By (2.1.13) with $m = n\varepsilon_n$, $k = h_n$ and $h_n + 1$, we have

$$\begin{aligned}&|\|S_{n-n\varepsilon_n}\|_2 - \|S_n\|_2| \\ &\quad\le \|S_{h_n n\varepsilon_n}(n\varepsilon_n)\|_2 + \frac{1}{(1-\rho(i))^{1/2}}(\|S_{h_n n\varepsilon_n}(n\varepsilon_n)\|_2 \\ &\qquad + \|S_{(h_n+1)n\varepsilon_n}(n\varepsilon_n)\|_2 + 4\sigma_0 i),\end{aligned}$$

where $i$ is such that $\rho(i) < 1$. Dividing both sides of the above inquality by $\|S_n\|_2$ and using (iii) and (2.1.20), we obtain (2.1.19). For integer $k > 0$, there exists $n_k$ such that for every $n \ge n_k$,

$$\left|\log \frac{h(nk)}{h(n)}\right| < \frac{1}{k} \tag{2.1.21}$$

since $h(n)$ is slowly varying. Without loss of generality, assume that $n_k$ is strictly increasing on $k$. Let $t > 1$. For integer $n > 0$, define $k = q_n$ such that $n_k \le nt < n_{k+1}$. Then, (2.1.21) implies that

$$\lim_{n\to\infty} \log(h([nt]q_n)/h(nt)) = 0. \tag{2.1.22}$$

Put $p_n = [q_n t]$. Then $p_n = [kt] \ge k$. Hence, (2.1.21) also implies that

$$\lim_{n\to\infty} \log(h(np_n)/h(n)) = 0. \tag{2.1.23}$$

Moreover, from (2.1.19) we have

$$\lim_{n\to\infty} h([nt]q_n)/h(np_n) = 1.$$

Combining it with (2.1.22) and (2.1.23) yields

$$\lim_{n\to\infty} h(nt)/h(n) = 1.$$

Therefore, by Property A1 of Appendix

$$\lim_{x\to\infty} \frac{h(xt)}{h(x)} = \lim_{x\to\infty} \frac{h([x]t)}{h([x])} = 1$$

as required.

Now we consider the second part of the theorem. By (2.1.12) with $m = 2N$, $p = q = N$, $n = [N^{1/3}]$, we have

$$\begin{aligned}(1-&\rho([N^{1/3}]))(ES^2_{2kN}(N) + ES^2_{(2k+1)N}(N))(1-\alpha_N)\\ &\le ES^2_{2kN}(2N)\\ &\le (1+\rho([N^{1/3}]))(ES^2_{2kN}(N) + ES^2_{(2k+1)N}(N))(1+\alpha_N)\end{aligned} \tag{2.1.24}$$

where

$$\alpha_N = \sup_k \frac{20\sigma_0^2 N^{2/3} + 12\sigma_0^2 N^{1/3}(\|S_{2kN}(N)\|_2 + \|S_{(2k+1)N}(N)\|_2)}{(1-\rho([N^{1/3}]))(ES^2_{2kN}(N) + ES^2_{(2k+1)N}(N))}$$

and $N_0$ is so large that $\rho([N^{1/3}]) < 1$ for $N \ge N_0$. Conditions (i) and (iii) imply that

$$\alpha_N = O\Big(\frac{N^{2/3} + N^{1/3}\|S_N\|_2}{\|S_N\|_2^2}\Big).$$

By Property A2 of Appendix, for any $0 < \varepsilon < 1/6$

$$\lim_{N\to\infty} N^{\varepsilon} ES_N^2/N = \infty.$$

Hence $(ES_N^2)^{-1} = O(N^{-1+\varepsilon})$, and further,

$$\alpha_N = O(N^{-\frac{1}{6}+\varepsilon}). \tag{2.1.25}$$

Then (2.1.24) implies that for integers $r > p \geq N_0$ with $\rho([2^{N_0/3}]) < 1$,

$$\begin{aligned}\prod_{i=p}^{r-1}&(1-\rho([2^{i/3}]))(1-\alpha_{2^i}) \sum_{i=0}^{2^{r-p}-1} ES_{i2^p}^2(2^p)\\ &\leq ES_{2^r}^2\\ &\leq \prod_{i=p}^{r-1}(1+\rho([2^{i/3}]))(1+\alpha_{2^i}) \sum_{i=0}^{2^{r-p}-1} ES_{i2^p}^2(2^p).\end{aligned} \tag{2.1.26}$$

By conditon (iv), $\sum_i \rho([2^{i/3}]) < \infty$. Moreover, (2.1.25) implies that $\sum_i \alpha(2^i) < \infty$. Therefore, from (2.1.26) we obtain

$$\lim_{r>p\to\infty} ES_{2^r}^2 \Big/ \sum_{i=0}^{2^{r-p}-1} ES_{i2^p}^2(2^p) = 1.$$

Consequently, it follows by condition (iii) that

$$\lim_{r>p\to\infty} h(2^r)/h(2^p) = 1,$$

and further, $h(2^r)$ converges to a positive constant. Applying Property A3 of Appendix to $h(t)$ and $1/h(t)$, we obtain that $h(n)$ converges to the same limit as $h(2^r)$. Theorem 2.1.4 is proved.

For a strictly stationary $\rho$-mixing sequence, we also have the following result.

**Theorem 2.1.5.**(Ibragimov 1975) *Let $\{X_n,\, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence with $EX_1 = 0$, $EX_1^2 < \infty$ and $\sum_{n=1}^{\infty} \rho(2^n) < \infty$. Then $\{X_n\}$ possesses a continuous spectral density $f(\lambda)$ and*

$$\operatorname{Var} S_n = 2\pi f(0)n + o(n) \quad \textit{as} \quad n \to \infty$$

*if $f(0) \neq 0$.*

For the proof of Theorem 2.1.5 we refer to Lemma 17 of Ibragimov and Rozanov (1978). They first show that $\{X_n\}$ possesses a bounded spectral density if $\sum_{n=1}^{\infty} \rho(2^n) < \infty$, and then show that

$$E_n(f) \leq 128 \max_{\lambda} f(\lambda) \sum_{k=0}^{\infty} \rho(2^{k-1}n),$$

where $E_n(f)$ denotes the error of best approximation of $f(\lambda)$ by a trigonometric polynomial of degree less than or equal to $n$ on $[-\pi, \pi]$. With the help of this result, we may know that $f(\lambda)$ is a continuous function on $[-\pi, \pi]$. The rest of the proof can be completed by using Theorem 2.1.1.

## 2.2 Further inequalities

In order to show limit theorems for a mixing sequence, we often need some further inequalities besides the basic inequalities in Section 1.2.

The following extended *Ottaviani inequality* for an $\alpha$-mixing sequence was given by Lin (1982).

Recall the notation $\mathcal{F}_a^b = \sigma(X_i, a \le i \le b)$ for a sequence $\{X_n\}$ of random variables.

**Lemma 2.2.1.** *Let $\{X_n, n \ge 1\}$ be an $\alpha$-mixing sequence. For any given integers $p, q$ and $k$, let $\xi_j$ be $\mathcal{F}_{(j-1)(p+q)+1}^{jp+(j-1)q}$ measurable, $j = 1, 2, \cdots, k$. If*

$$P\{|\xi_{l+1} + \cdots + \xi_k| \le C\} \ge \frac{1}{2}, \quad l = 1, \cdots, k-1,$$

*then*

$$P\Big\{\max_{1 \le l \le k} |\xi_1 + \cdots + \xi_l| > 2C\Big\} \le 2P\{|\xi_1 + \cdots + \xi_k| > C\} + 2k\alpha(q).$$

**Proof.** Let events

$$\begin{aligned}
&A = \Big\{\max_{1 \le l \le k} |\xi_1 + \cdots + \xi_l| > 2C\Big\}, \quad B = \{|\xi_1 + \cdots + \xi_k| > C\}, \\
&A_1 = \{|\xi_1| > 2C\}, \\
&A_l = \Big\{\max_{1 \le r \le l-1} |\xi_1 + \cdots + \xi_r| \le 2C, \ |\xi_1 + \cdots + \xi_l| > 2C\Big\}, \ l = 2, \cdots, k, \\
&B_l = \{|\xi_{l+1} + \cdots + \xi_k| \le C\}, \ l = 1, \cdots, k-1, \ B_k = \Omega.
\end{aligned}$$

Then

$$A_i A_j = \emptyset \ (i \ne j), \quad A = \bigcup_{l=1}^{k} A_l, \quad \bigcup_{l=1}^{k} A_l B_l \subseteq B.$$

By the conditions of the lemma,

$$P(A_l B_l) \ge P(A_l) P(B_l) - \alpha(q) \ge \frac{1}{2} P(A_l) - \alpha(q),$$

and hence

$$
\begin{aligned}
P(B) &\geq \sum_{l=1}^{k} P(A_l B_l) \geq \frac{1}{2} \sum_{l=1}^{k} P(A_l) - k\alpha(q) \\
&= \frac{1}{2} P(A) - k\alpha(q)
\end{aligned}
$$

as required.

The following lemmas all are about the bounds of the moments of partial sums. For a $\rho$-mixing sequence, earlier work was due to Peligrad (1982, 1987). Shao (1988b, 1989a,b) improved and generalized her results.

**Lemma 2.2.2.** *Let $\{X_n, n \geq 1\}$ be a $\rho$-mixing sequence with $EX_n = 0$, $EX_n^2 < \infty$ for each $n \geq 1$. Then for any $\varepsilon > 0$, there exists a $C = C(\varepsilon) > 0$ such that*

$$
ES_k^2(n) \leq Cn \exp\Big\{(1+\varepsilon) \sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \max_{k<i\leq k+n} EX_i^2
$$

*for each $k \geq 1$ and $n \geq 1$, where $S_k(n) = \sum_{i=k+1}^{k+n} X_i$.*

**Proof.** Without loss of generality, assume $0 < \varepsilon < \frac{1}{4}$. Let $C_n$ be a non-decreasing sequence of numbers such that

$$
ES_k^2(n) \leq C_n n \max_{k<i\leq k+n} EX_i^2. \tag{2.2.1}
$$

For $n \leq 2^{1/\varepsilon}$ we need only to take $C_n \geq 2^{1/\varepsilon}$ by the Minkowski inequality. Let $C_1 = 2^{1/\varepsilon}$. We suppose that $C_m$, $m = 1, \cdots, n-1$, are already defined as demanded in the lemma. Put $n_1 = [n/2]$, $n_2 = n - n_1$, $n_3 = [n^{1/(1+\varepsilon)}] + 1$. It is clear that

$$
ES_k^2(n) = ES_k^2(n_1) + ES_{k+n_1}^2(n_2) + 2ES_k(n_1)S_{k+n_1}(n_2), \tag{2.2.2}
$$

and further

$$
\begin{aligned}
&|ES_k(n_1)S_{k+n_1}(n_2)| \\
&\quad \leq |ES_k(n_1)S_{k+n_1}(n_3)| + |ES_k(n_1)S_{k+n_1+n_3}(n_2 - n_3)| \\
&\quad \leq \|S_k(n_1)\|_2 \|S_{k+n_1}(n_3)\|_2 \\
&\qquad + \rho(n_3)\|S_k(n_1)\|_2 \|S_{k+n_1+n_3}(n_2 - n_3)\|_2 \\
&\quad \leq 2\|S_k(n_1)\|_2 \|S_{k+n_1}(n_3)\|_2 + \rho(n_3)\|S_k(n_1)\|_2 \|S_{k+n_1}(n_2)\|_2.
\end{aligned}
$$

Inserting the above inequality into (2.2.2) and noting (2.2.1) we obtain

$$\begin{aligned} ES_k^2(n) &\le (ES_k^2(n_1) + ES_{k+n_1}^2(n_2))(1+\rho(n_3)) \\ &\quad + 4\|S_k(n_1)\|_2\|S_{k+n_1}(n_3)\|_2 \\ &\le C_{n_2}\cdot n(1+\rho(n_3))\max_{k<i\le k+n} EX_i^2 + 4C_{n_1} n_1^{\frac12} n_3^{\frac12}\max_{k<i\le k+n} EX_i^2 \\ &\le C_{n_2}(1+\rho(n_2^{\frac{1}{1+\varepsilon}}) + 4n_2^{-\frac{\varepsilon}{2(1+\varepsilon)}})n\max_{k<i\le k+n} EX_i^2, \end{aligned}$$

where

$$\rho(x) = (\rho(i+1)-\rho(i))(x-i)+\rho(i) \quad \text{if } i\le x< i+1.$$

Hence for $n\ge 2$, we define

$$C_n = C_{n_2}\Big(1+\rho(n_2^{\frac{1}{1+\varepsilon}}) + 4n_2^{-\frac{\varepsilon}{2(1+\varepsilon)}}\Big).$$

Obviously $C_n$ is nondecreasing, and

$$\begin{aligned} C_{2^n} &= C_{2^{n-1}}(1+\rho(2^{\frac{n-1}{1+\varepsilon}}) + 4\cdot 2^{-\frac{\varepsilon(n-1)}{2(1+\varepsilon)}}) \\ &= C_1\prod_{i=0}^{n-1}\Big(1+\rho(2^{\frac{i}{1+\varepsilon}}) + 4\cdot 2^{-\frac{\varepsilon i}{2(1+\varepsilon)}}\Big) \\ &\le C_1\exp\Big\{\sum_{i=o}^{n-1}\Big(\rho(2^{\frac{i}{1+\varepsilon}}) + 4\cdot 2^{-\frac{\varepsilon i}{2(1+\varepsilon)}}\Big)\Big\} \\ &\le C_1\exp\Big\{3+\int_2^{n-1}\rho(2^{\frac{x}{1+\varepsilon}})\,dx + c_\varepsilon\Big\} \\ &\le C_1\exp\Big\{3+(1+\varepsilon)\int_{2/(1+\varepsilon)}^{n-1}\rho(2^x)\,dx + c_\varepsilon\Big\} \\ &\le C_1\exp\Big\{3+(1+\varepsilon)\sum_{i=1}^{n-1}\rho(2^i) + c_\varepsilon\Big\}, \end{aligned} \tag{2.2.3}$$

where $c_\varepsilon = 4/(1-2^{-\varepsilon/(2(1+\varepsilon))})$. Put $d_\varepsilon = 2^{1/\varepsilon}\exp(3+c_\varepsilon)$. We get

$$C_{2^n}\le d_\varepsilon\exp\Big\{(1+\varepsilon)\sum_{i=1}^{n-1}\rho(2^i)\Big\}.$$

For any $n$, there exists an $m$ such that $2^m\le n<2^{m+1}$. Using the monotonicity of $C_n$, it follows that

$$\begin{aligned} C_n \le C_{2^{m+1}} &\le d_\varepsilon\exp\Big\{(1+\varepsilon)\sum_{i=1}^{m}\rho(2^i)\Big\} \\ &\le d_\varepsilon\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}. \end{aligned}$$

The lemma is proved.

**Lemma 2.2.3.** *Let* $\{X_n, n \geq 1\}$ *be a* $\rho$*-mixing sequence with* $EX_n = 0, EX_n^2 < \infty$ *for each* $n \geq 1$. *Suppose that*

$$ES_k^2(n) / \min_{k<i\leq k+n} EX_i^2 \to \infty \quad \text{as } n \to \infty \tag{2.2.4}$$

*uniformly in* $k$ *and*

$$\max_{k<i\leq k+n} EX_i^2 \leq a \min_{k<i\leq k+n} EX_i^2 \quad \text{for some } a \geq 1. \tag{2.2.5}$$

*Then, for any* $\varepsilon > 0$, *there exist* $C' = C'(\varepsilon, \rho(\cdot), a) > 0$ *and an integer* $N$ *such that for each* $k \geq 0$ *and* $n \geq N$

$$ES_k^2(n) \geq C' n \exp\Big\{-(1+\varepsilon) \sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \min_{k<i\leq k+n} EX_i^2.$$

**Proof.** Without loss of generality assume that $0 < \varepsilon < 1/400$. Consequently,

$$1 - 5\varepsilon^2 > (3/2)^{-\varepsilon/6}.$$

Hence, noting $\rho(n) \to 0$ as $n \to \infty$, we have

$$1 - 5\varepsilon^2 - \rho(m_0) > (3/2)^{-\varepsilon/6}$$

for some large $m_0$. It is not hard to verify that $\exp\{2\sum_{i=0}^{[\log n]} \rho(2^i)\}$ is a slowly varying function. Then, by Lemma 2.2.2 and condition (2.2.4), there exists an $n_0$ such that for $n \geq n_0$

$$ES_k^2(n) \leq n^{1+\varepsilon^4} \max_{k<i\leq k+n} EX_i^2, \tag{2.2.6}$$

$$ES_k^2(n) \geq \frac{4am_0^2}{\varepsilon^4} \min_{k<i\leq k+n} EX_i^2. \tag{2.2.7}$$

When $n \geq 2n_0$, put $n_1 = [n/2]$, $n_2 = n - n_1$. Then

$$\begin{aligned}
ES_k^2(n) &= ES_k^2(n_1) + ES_{k+n_1}^2(n_2) \\
&\quad + 2ES_k(n_1)S_{k+n_1}(m_0) + 2ES_k(n_1)S_{k+n_1+m_0}(n_2 - m_0) \\
&\geq ES_k^2(n_1) + ES_{k+n_1}^2(n_2) - 2\|S_k(n_1)\|_2\|S_{k+n_1}(m_0)\|_2 \\
&\quad - 2\rho(m_0)\|S_k(n_1)\|_2\|S_{k+n_1+m_0}(n_2 - m_0)\|_2 \\
&\geq (1 - \rho(m_0))(ES_k^2(n_1) + ES_{k+n_1}^2(n_2)) \\
&\quad - 4\|S_k(n_1)\|_2\|S_{k+n_1}(m_0)\|_2 \\
&\geq (1 - \rho(m_0))(ES_k^2(n_1) + ES_{k+n_1}^2(n_2)) - 4\varepsilon^2 ES_k^2(n_1) \\
&\quad - \frac{4}{\varepsilon^2} m_0^2 \max_{k<i\leq k+n} EX_i^2 \\
&\geq (1 - 4\varepsilon^2 - \rho(m_0))(ES_k^2(n_1) + ES_{k+n_1}^2(n_2)) \\
&\quad - \frac{4}{\varepsilon^2} m_0^2 a \min_{k<i\leq k+n} EX_i^2 \\
&\geq (1 - 5\varepsilon^2 - \rho(m_0))(ES_k^2(n_1) + ES_{k+n_1}^2(n_2)) \\
&\geq (3/2)^{-\varepsilon/6}(ES_k^2(n_1) + ES_{k+n_1}^2(n_2)).
\end{aligned} \tag{2.2.8}$$

We first show that for each $n \geq n_0$

$$ES_k^2(n) \geq C_2 n^{1-\varepsilon/6} \min_{k<i\leq k+n} EX_i^2, \tag{2.2.9}$$

where $C_2 = 2am_0^2 n_0^{-1}\varepsilon^{-4}$. By (2.2.7), (2.2.9) holds for $n_0 \leq n < 2n_0$. When $n \geq 2n_0$, we assume that (2.2.9) is true for each positive integer less than $n$. Then it is also true for $n$. In fact, using (2.2.8) we obtain

$$\begin{aligned}
ES_k^2(n) &\geq (3/2)^{-\varepsilon/6} C_2 (n_1^{1-\varepsilon/6} + n_2^{1-\varepsilon/6}) \min_{k<i\leq k+n} EX_i^2 \\
&\geq \left(\frac{3}{2}\right)^{-\varepsilon/6} C_2 \left(\left(\frac{1}{3}\right)^{1-\varepsilon/6} + \left(\frac{2}{3}\right)^{1-\varepsilon/6}\right) \\
&\quad \cdot n^{1-\varepsilon/6} \min_{k<i\leq k+n} EX_i^2 \\
&\geq C_2 n^{1-\varepsilon/6} \min_{k<i\leq k+n} EX_i^2
\end{aligned}$$

as required.

Next, we turn to the assertion of the lemma. For $n \geq n_0^{1+\varepsilon}$, put $n_1 = [\frac{n}{2}]$, $n_2 = n - n_1$, $n_3 = [n^{1/(1+\varepsilon)}] + 1$. From (2.2.6) and (2.2.9), it

follows that

$$\begin{aligned}
&ES_k^2(n)\\
&\quad = ES_k^2(n_1) + ES_{k+n_1}^2(n_2) + 2ES_k(n_1)ES_{k+n_1}(n_3)\\
&\qquad + 2ES_k(n_1)S_{k+n_1+n_3}(n_2 - n_3)\\
&\quad \geq ES_k^2(n_1) + ES_{k+n_1}^2(n_2) - 4\|S_k(n_1)\|_2\|S_{k+n_1}(n_3)\|_2\\
&\qquad - 2\rho(n_3)\|S_k(n_1)\|_2\|S_{k+n_1}(n_2)\|_2\\
&\quad \geq (1-\rho(n_3))(ES_k^2(n_1) + ES_{k+n_1}^2(n_2))\\
&\qquad - 4(n_1 n_3)^{(1+\varepsilon^4)/2} \max_{k<i\leq k+n} EX_i^2\\
&\quad \geq (1-\rho(n_3))(ES_k^2(n_1) + ES_{k+n_1}^2(n_2))\\
&\qquad - 4n^{1-(\varepsilon-3\varepsilon^4)/2(1+\varepsilon)} \max_{k<i\leq k+n} EX_i^2\\
&\quad \geq (1-\rho(n_3))(ES_k^2(n_1) + ES_{k+n_1}^2(n_2)) - 4an^{1-\varepsilon/5} \min_{k<i\leq k+n} EX_i^2\\
&\quad \geq (1-\rho(n_3))(ES_k^2(n_1) + ES_{k+n_1}^2(n_2)) - 8aC_2^{-1}n^{-\varepsilon/30}ES_k^2(n_1)\\
&\quad \geq (1-\rho(n_3) - n^{-\varepsilon/40})(ES_k^2(n_1) + ES_{k+n_1}^2(n_2)). \qquad (2.2.10)
\end{aligned}$$

The last inequality holds for $n \geq (8aC_2^{-1})^{120/\varepsilon}$. Put

$$n_o' = \max(n_0^{1+\varepsilon}, (8aC_2^{-1})^{120/\varepsilon}).$$

Let $C_n$ be non-increasing so that

$$ES_k^2(n) \geq C_n n \min_{k<i\leq k+n} EX_i^2$$

for $n \geq n_0$. Then, by (2.2.10)

$$\begin{aligned}
ES_k^2(n) &\geq (1-\rho(n_3) - n_2^{-\varepsilon/40})C_{n_2} n \min_{k<i\leq k+n} EX_i^2\\
&\geq (1-\rho(n_2^{\frac{1}{1+\varepsilon}}) - n_2^{-\varepsilon/40})C_{n_2} n \min_{k<i\leq k+n} EX_i^2
\end{aligned}$$

for $n \geq n_0'$. Hence we can choose

$$C_n = C_{n_2}\Big(1 - \rho(n_2^{\frac{1}{1+\varepsilon}}) - n_2^{-\varepsilon/40}\Big) \qquad (2.2.11)$$

for $n \geq n_0'$. It is easy to see that there exists an $n_0''$ such that for $n \geq n_0''$

$$1 - \rho(n_2^{\frac{1}{1+\varepsilon}}) - n_2^{-\varepsilon/40} \geq \exp\Big\{-(1+\varepsilon)\Big(\rho(n_2^{\frac{1}{1+\varepsilon}}) + n_2^{-\varepsilon/40}\Big)\Big\}. \qquad (2.2.12)$$

Put $n_0^* = n_0' \vee n_0''$. In view of (2.2.7), we take $C_{n_0^*} = 4am_0^2/(\varepsilon^4 n_0^*)$. Obviously $\{C_n, n \ge 2n_0^*\}$ defined by (2.2.11) is non-increasing. From (2.2.11) and (2.2.12), we obtain that for $2^m > n_0^*$

$$\begin{aligned} C_{2^m} &= C_{2^{m-1}}(1 - \rho(2^{\frac{m-1}{1+\varepsilon}}) - 2^{-(m-1)\varepsilon/40}) \\ &\ge C_{2^{m-1}} \exp\{-(1+\varepsilon)(\rho(2^{\frac{m-1}{1+\varepsilon}}) + 2^{-(m-1)\varepsilon/40})\}, \end{aligned}$$

which implies that

$$\begin{aligned} C_{2^m} &\ge C_{n_0^*} \prod_{i=0}^{m-1} \exp\{-(1+\varepsilon)(\rho(2^{i/(1+\varepsilon)}) + 2^{-i\varepsilon/40})\} \\ &= C_{n_0^*} \exp\Big\{-(1+\varepsilon)\sum_{i=0}^{m-1}(\rho(2^{i/(1+\varepsilon)}) + 2^{-i\varepsilon/40})\Big\}. \end{aligned}$$

Similarly to (2.2.3), there exists a $d_\varepsilon > 0$ such that

$$\begin{aligned} &\exp\Big\{-(1+\varepsilon)\sum_{i=0}^{m-1}(\rho(2^{i/(1+\varepsilon)}) + 2^{-i\varepsilon/40})\Big\} \\ &\qquad\qquad \ge d_\varepsilon \exp\Big\{-(1+\varepsilon)^2 \sum_{i=0}^{m} \rho(2^i)\Big\}. \end{aligned}$$

Therefore

$$C_{2^m} \ge d_\varepsilon C_{n_0^*} \exp\Big\{-(1+\varepsilon)^2 \sum_{i=0}^{m} \rho(2^i)\Big\}.$$

For arbitrary $n > n_0^*$, there exists an $m$ such that $2^m \le n < 2^{m+1}$. By monotonicity of $C_n$, we find that

$$\begin{aligned} C_n &\ge C_{2^{m+1}} \ge d_\varepsilon C_{n_0^*} \exp\Big\{-(1+\varepsilon)^2 \sum_{i=0}^{m+1} \rho(2^i)\Big\} \\ &\ge \frac{1}{8} d_\varepsilon C_{n_0^*} \exp\Big\{-(1+\varepsilon)^2 \sum_{i=0}^{[\log n]} \rho(2^i)\Big\}, \end{aligned}$$

and hence we arrive at the assertion of the lemma.

Sometimes we need the bounds of moments of higher than two orders.

**Lemma 2.2.4.** *Let* $\{X_n, n \ge 1\}$ *be a* $\rho$*-mixing sequence with* $EX_n = 0$, $\sup_n E|X_n|^{2+\delta} < \infty$ *for some* $0 < \delta < 1$ *and*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty. \tag{2.2.13}$$

*Then there exists a* $C = C(\delta, \rho(\cdot)) > 0$ *such that for each* $n \geq 1$

$$\sup_{k\geq 1} E|S_k(n)|^{2+\delta} \leq C\Big\{n^{1+\delta/2}(\sup_{k\geq 1} EX_k^2)^{1+\delta/2} + n\exp\Big\{(C\log n)^{\delta/(2+\delta)}\Big\}\sup_{k\geq 1} E|X_k|^{2+\delta}\Big\}.$$

**Proof.** It is not difficult to verify that

$$\begin{aligned}(1+x)^{2+\delta} &\leq 1 + (2+\delta)^2(x + x^{1+\delta}) + x^{2+\delta} \\ &\leq 1 + 9(x + x^{1+\delta}) + x^{2+\delta} \end{aligned} \tag{2.2.14}$$

for $x \geq 0$. Put

$$a_m = \sup_{k\geq 1} \|S_k(m)\|_{2+\delta}, \quad \sigma_m = \sup_{k\geq 1} \|S_k(m)\|_2.$$

Obviously,

$$\|S_k(2m)\|_{2+\delta} \leq \|S_k(m) + S_{k+m+[m^{1/5}]}(m)\|_{2+\delta} + 2m^{1/5}a_1.$$

By (2.2.14), putting $m_1 = m + [m^{1/5}]$, we have

$$\begin{aligned}E|S_k(m) + S_{k+m_1}(m)|^{2+\delta} \leq\ & 2a_m^{2+\delta} + 9E|S_k(m)|^{1+\delta}|S_{k+m_1}(m)| \\ & + 9E|S_k(m)||S_{k+m_1}(m)|^{1+\delta}.\end{aligned}$$

Moreover, by the Schwarz inequality and Lemma 1.2.7, we have

$$\begin{aligned}&E|S_k(m)|^{1+\delta}|S_{k+m_1}(m)| \\ &\leq \|S_k(m)\|_{2+\delta}^{\delta}\|S_k(m)S_{k+m_1}(m)\|_{(2+\delta)/2} \\ &\leq a_m^{\delta}\{\sigma_m^{2+\delta} + 4\rho([m^{1/5}])a_m^{2+\delta}\}^{2/(2+\delta)} \\ &\leq a_m^{\delta}\sigma_m^2 + 4\rho^{2/(2+\delta)}([m^{1/5}])a_m^{2+\delta}.\end{aligned}$$

Similarly

$$E|S_k(m)||S_{k+m_1}(m)|^{1+\delta} \leq a_m^{\delta}\sigma_m^2 + 4\rho^{2/(2+\delta)}([m^{1/5}])a_m^{2+\delta}.$$

Combining these inequalities yields that

$$\begin{aligned}&E|S_k(m) + S_{k+m_1}(m)|^{2+\delta} \\ &\leq 2a_m^{2+\delta} + 18(a_m^{\delta}\sigma_m^2 + 4\rho^{2/(2+\delta)}([m^{1/5}])a_m^{2+\delta}) \\ &\leq \Big\{[2(1 + 36\rho^{2/(2+\delta)}([m^{1/5}]))]^{1/(2+\delta)}a_m + 18\sigma_m\Big\}^{2+\delta},\end{aligned}$$

which implies that

$$a_{2m} \le \left\{2(1+36\rho^{2/(2+\delta)}([m^{1/5}]))\right\}^{1/(2+\delta)} a_m + 18\sigma_m + 2m^{1/5}a_1. \quad (2.2.15)$$

Noting monotonecity of $\rho(n)$ and condition (2.2.13), we have

$$\rho(n) \le c/\log n,$$

here, and in the sequel, $c$ stands for a positive constant, which may take different values at different places. Hence, applying Lemma 2.2.2, we obtain

$$\begin{aligned}
a_{2^r} &\le \{2(1+36\rho^{2/(2+\delta)}([2^{(r-1)/5}]))\}^{1/(2+\delta)} a_{2^{r-1}} \\
&\quad + 18\sigma_{2^{r-1}} + 2\cdot 2^{(r-1)/5}a_1 \\
&\le 2^{(r-1)/(2+\delta)} \prod_{i=0}^{r-1} (1+36\rho^{2/(2+\delta)}([2^{i/5}]))^{1/(2+\delta)} a_1 \\
&\quad + c\sigma_1 \sum_{i=0}^{r-1} 2^{i/2} \prod_{j=i+1}^{r-1} \left\{2(1+9\rho^{2/(2+\delta)}([2^{j/5}]))\right\}^{1/(2+\delta)} \\
&\quad + 2a_1 \sum_{i=0}^{r-1} 2^{i/5} \prod_{j=i+1}^{r-1} \left\{2(1+9\rho^{2/(2+\delta)}([2^{j/5}]))\right\}^{1/(2+\delta)} \\
&\le C2^{r/2}\sigma_1 + 2^{r/(2+\delta)} \exp(Cr)^{\delta/(2+\delta)} a_1. \qquad (2.2.16)
\end{aligned}$$

This implies the conclusion of the lemma.

Similarly, by finer estimation, Shao (1989a) showed the following results, whose proof will not be presented here.

**Lemma 2.2.5.** *Let $\{X_n,\, n \ge 1\}$ be a $\rho$-mixing sequence with*

$$EX_n = 0, \quad \sup_n E|X_n|^{2+\delta} < \infty \qquad \text{for some } \delta \ge 0.$$

*Then, for any $\varepsilon > 0$, there exists a $C = C(\delta, \rho(\cdot), \varepsilon) > 0$, such that for each $n \ge 2$*

$$\begin{aligned}
E|S_k(n)|^{2+\delta} \le C\Big\{ &\Big(n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \max_{k<i\le k+n} EX_i^2\Big)^{1+\delta/2} \\
&+ n\exp\Big\{C\sum_{i=0}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big\} \max_{k<i\le k+n} E|X_i|^{2+\delta}\Big\}.
\end{aligned}$$

**Lemma 2.2.6.** *Let $\{X_n, n \geq 1\}$ be a $\rho$-mixing sequence with*

$$EX_n = 0, \quad E|X_n|^q < \infty,\ q \geq 2, \quad ES_k^2(n) \leq nh(n) \max_{k<i\leq k+n} EX_i^2.$$

*Suppose that there exists a function $h(n)$ such that for every $k \geq 0$, $n \geq 1$ and there exist a positive integer $n_0$ and a constant $0 < \theta < 2^{1-2/(q\wedge 3)}$ such that*

$$\max(h([n/2]), h(n-[n/2])) \leq \theta h(n)$$

*for $n \geq n_0$. Furthermore, when $q > 3$ assume that there exists a $C > 0$ such that*

$$h(n) \geq \frac{1}{C}\exp\Big\{-C\sum_{i=0}^{[\log n]} \rho^{2/q}(2^i)\Big\}.$$

*Then there exists a constant $K = K(q, n_0, \theta, C, \rho(\cdot))$, such that for every $k \geq 0$, $n \geq 1$*

$$\begin{aligned} E|S_k(n)|^q \leq K\Big\{&(nh(n)\max_{k<i\leq k+n} EX_i^2)^{q/2} \\ &+ n\exp\Big\{K\sum_{i=0}^{[\log n]}\rho^{2/q}(2^i)\Big\}\max_{k\leq i\leq k+n} E|X_i|^q\Big\}. \end{aligned}$$

Next, we turn our attention to a $\varphi$-mixing sequence. Peligrad (1985) showed the following *inequality of tail probability* (see Shao 1988a).

**Lemma 2.2.7.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence, $0 < \eta < 1$. Suppose that there exists an integer $p$, $1 \leq p \leq n$, a number $A > 0$ such that*

$$\varphi(p) + \max_{p\leq i\leq n} P\{|S_n - S_i| \geq A\} \leq \eta. \tag{2.2.17}$$

*Then, for any $a \geq 0$, $b \geq 0$, we have*

$$\begin{aligned} &P\Big\{\max_{1\leq i\leq n}|S_i| \geq a + A + b\Big\} \\ &\quad\leq \frac{1}{1-\eta}P\{|S_n| \geq a\} + \frac{1}{1-\eta}P\Big\{\max_{1\leq i\leq n}|X_i| \geq \frac{b}{p-1}\Big\}. \end{aligned} \tag{2.2.18}$$

$$\begin{aligned} &P\{|S_n| \geq a + A + b\} \\ &\quad\leq \eta P\Big\{\max_{1\leq i\leq n}|S_i| \geq a\Big\} + P\Big\{\max_{1\leq i\leq n}|X_i| \geq \frac{b}{p}\Big\}. \end{aligned} \tag{2.2.19}$$

**Proof.** Put $E_i = \{\max_{1\le j<i}|S_j| < a+A+b \le |S_i|\}$. Then

$$P\Big\{\max_{1\le i\le n}|S_i| \ge a+A+b\Big\} \le P(|S_n|\ge a) + \sum_{i=1}^{n-1} P\Big(E_i \cap \{|S_n - S_i| \ge A+b\}\Big)$$

$$\begin{aligned}\sum_{i=1}^{n-1} P\Big(E_i\cap\{|S_n-S_i|\ge A+b\}\Big) &\le \sum_{i=1}^{n-p-1} P\Big(E_i\cap\{|S_{i+p-1}-S_i|\ge b\}\Big)\\ &\quad+\sum_{i=1}^{n-p-1} P\Big(E_i\cap\{|S_n-S_{i+p-1}|\ge A\}\Big)\\ &\quad+\sum_{i=n-p}^{n-1} P\Big(E_i\cap\{|S_n-S_i|\ge A+b\}\Big)\\ &\le \sum_{i=1}^{n-1} P\Big(E_i\cap\Big\{\max_{1\le j\le n}|X_j|\ge\frac{b}{p-1}\Big\}\Big)\\ &\quad+\sum_{i=1}^{n-p-1} P(E_i)\Big(P\{|S_n-S_{i+p-1}|\ge A\}+\varphi(p)\Big)\\ &\le P\Big\{\max_{1\le j\le n}|X_j|\ge\frac{b}{p-1}\Big\}+\eta P\Big\{\max_{1\le i\le n}|S_i|\ge a+A+b\Big\},\end{aligned}$$

where condition (2.2.17) is used in the last inequality. Consequently, (2.2.18) is proved.

As for (2.2.19), putting $E_i' = \{\max_{1\le j<i}|S_j| < a \le |S_i|\}$ and noting

$$|S_n - S_{j+p-1}| \ge ||S_n| - |S_{j-1}| - p\max_{1\le i\le n}|X_i|| \text{for} 1 \le j \le n-p,$$

we have

$$\begin{aligned}P\{|S_n|\ge a+A+b\} &\le P\Big\{|S_n|\ge a+A+b, \max_{1\le i\le n-p}|S_i|\ge a, \max_{1\le i\le n}|X_i|\le\frac{b}{p}\Big\}\\ &\quad+P\Big\{\max_{1\le i\le n}|X_i|>\frac{b}{p}\Big\}\\ &\le \sum_{i=1}^{n-p} P(E_i'\cap\{|S_n-S_{i+p-1}|>A\})+P\Big\{\max_{1\le i\le n}|X_i|\ge\frac{b}{p}\Big\}\\ &\le \eta P\Big\{\max_{1\le i\le n}|S_i|\ge a\Big\}+P\Big\{\max_{1\le i\le n}|X_i|\ge\frac{b}{p}\Big\}\end{aligned}$$

as required.

Lemma 2.2.8 is due to Shao and Lu (1986).

**Lemma 2.2.8.** *Let $\{X_n,\, n \geq 1\}$ be a $\varphi$-mixing sequence with $EX_n = 0$ and $\sup_n E|X_n|^{2+\delta} < \infty$ for some $\delta > 0$. Suppose that*

$$\sup_k ES_k^2(n) \leq Mn \sup_k EX_k^2 \quad \text{for some} M > 0. \tag{2.2.20}$$

*Then there exists a $C = C(\delta,\, M,\, \varphi(\cdot)) > 0$ such that for each $n \geq 1$*

$$\sup_k E|S_k(n)|^{2+\delta} \leq Cn^{1+\delta/2} \sup_k E|X_k|^{2+\delta}.$$

**Proof.** It is easy to see that for $r \geq 1$ and $x \geq 0$

$$(1+x)^r \leq \sum_{k=0}^{[r]} \binom{r}{k} x^k + \delta_r x^r, \tag{2.2.21}$$

where $\delta_r = 1$ if $r$ is not an integer, otherwise $\delta_r = 0$. We now prove the lemma by induction on $r := 2+\delta$. Assume that the lemma holds for $l \leq [r], r$ being non-integer. Denoting $a_m = \sup_k \|S_k(m)\|_r$, from (2.2.21) we obtain

$$\begin{aligned}
&E|S_k(m) + S_{k+m+k_0}(m)|^r \\
&\quad \leq E|S_k(m)|^r + E|S_{k+m+k_0}(m)|^r \\
&\qquad + \textstyle\sum_{j=1}^{[r]} \binom{r}{j} E|S_k(m)|^j |S_{k+m+k_0}(m)|^{r-j} \\
&\quad \leq \Big(2 + 2\textstyle\sum_{j=1}^{[r]} \binom{r}{j} \varphi^{1/r}(k_0)\Big) a_m^r \\
&\qquad + \textstyle\sum_{j=1}^{[r]} \binom{r}{j} E|S_k(m)|^j E|S_{k+m+k_0}(m)|^{r-j} \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{2.2.22}$$

By the induction hypothesis, we have

$$\begin{aligned}
I_2 &\leq \textstyle\sum_{j=1}^{[r]} \binom{r}{j} (E|S_k(m)|^{[r]})^{j/[r]} (E|S_{k+m+k_0}(m)|^{[r]})^{(r-j)/[r]} \\
&\leq (m^{[r]/2} \sup_k E|X_k|^{[r]})^{r/[r]} \leq c\, m^{r/2} a_1^r.
\end{aligned}$$

Substituting the above inequality into (2.2.22), we obtain

$$a_{2m} \leq \Big(2 + 2\sum_{j=1}^{[r]} \binom{r}{j} \varphi^{1/r}(k_0)\Big)^{1/r} a_m + c\, m^{1/2} a_1.$$

Now choosing a sufficiently large $k_0$ and proceeding as in the proof of Lemma 2.2.4, we conclude that the lemma holds in this case and similarly we have the lemma for $[r]+1$. This proves the lemma.

Using Lemma 2.2.7, Shao (1988a) proved the following Lemma.

**Lemma 2.2.9.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence satisfying (2.2.17) and $q > 0$ satisfying $\eta 4^q < 1 - \eta$. Then*

$$E \max_{1\leq i\leq n} |S_i|^q \leq (1-\eta-\eta 4^q)^{-1}\Big\{(8A)^q + 2(4p)^q E \max_{1\leq i\leq n} |X_i|^q\Big\},$$

*where $\eta$, $p$, $A$ are defined in Lemma 2.2.7.*

**Proof.** By Lemma 2.2.7, we have for $x \geq 8A$

$$\begin{aligned}
&P\Big\{\max_{1\leq i\leq n}|S_i| \geq x\Big\}\\
&\quad\leq \frac{1}{1-\eta}\Big(P\Big\{|S_n| \geq \frac{5}{8}x\Big\} + P\Big\{\max_{1\leq i\leq n}|X_i| \geq \frac{x}{4p}\Big\}\Big)\\
&\quad\leq \frac{1}{1-\eta}\Big(\eta P\Big\{\max_{1\leq i\leq n}|S_i| \geq \frac{x}{4}\Big\} + 2P\Big\{\max_{1\leq i\leq n}|X_i| \geq \frac{x}{4p}\Big\}\Big).
\end{aligned}$$

Hence for any $B > 8A$

$$\begin{aligned}
&\int_0^B qy^{q-1}P\Big\{\max_{1\leq i\leq n}|S_i| \geq y\Big\}\,dy\\
&\quad\leq \int_0^{8A} qy^{q-1}P\Big\{\max_{1\leq i\leq n}|S_i| \geq y\Big\}\,dy\\
&\qquad+\frac{\eta}{1-\eta}\int_{8A}^B qy^{q-1}P\Big\{\max_{1\leq i\leq n}|S_i| \geq \frac{y}{4}\Big\}\,dy\\
&\qquad+\frac{2}{1-\eta}\int_{8A}^B qy^{q-1}P\Big\{\max_{1\leq i\leq n}|X_i| \geq \frac{y}{4p}\Big\}\,dy\\
&\quad\leq (8A)^q + \frac{\eta}{1-\eta}4^q\int_0^B qy^{q-1}P\Big\{\max_{1\leq i\leq n}|S_i| \geq y\Big\}\,dy\\
&\qquad+\frac{2(4p)^q}{1-\eta}\int_0^\infty qy^{q-1}P\Big\{\max_{1\leq i\leq n}|X_i| \geq y\Big\}\,dy.
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_0^B qy^{q-1}P\Big\{\max_{1\leq i\leq n}|S_i| \geq y\Big\}\,dy\\
&\quad\leq \Big(1-\frac{\eta 4^q}{1-\eta}\Big)^{-1}\Big((8A)^q + \frac{2(4p)^q}{1-\eta}\int_0^\infty qy^{q-1}P\Big\{\max_{1\leq i\leq n}|X_i| \geq y\Big\}\,dy\Big)\\
&\quad\leq (1-\eta-\eta 4^q)^{-1}\Big((8A)^q + 2(4p)^q E\max_{1\leq i\leq n}|X_i|^q\Big).
\end{aligned}$$

Letting $B \to \infty$ yields the assertion of the lemma.

A similar result is

**Lemma 2.2.10.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence. Suppose that there exists an array $\{c_{kn}\}$ of positive numbers such that*

$$\max_{1 \leq i \leq n} ES_k^2(i) \leq c_{kn}. \tag{2.2.23}$$

*Then, for any $q \geq 2$, there exists a $C = C(q, \varphi(\cdot))$ such that*

$$E \max_{1 \leq i \leq n} |S_k(i)|^q \leq C\Big(c_{kn}^{q/2} + E \max_{k < i \leq k+n} |X_i|^q\Big). \tag{2.2.24}$$

**Proof.** Take $\eta = 4^{-2q}$, $A^2 = 2c_{kn}/\eta$.There exists a $p_0$ such that $\varphi(p_0) \leq \eta/2$ since $\varphi(p) \to 0$ as $p \to \infty$. Using (2.2.23) we can verify that (2.2.17) is satisfied. Hence, we get (2.2.24) from Lemma 2.2.9.

# Part II Weak Convergence

In this part, we investigate weak convergence of probability measures (or distributions ) of normalized sums of the form

$$\frac{1}{B_n}\sum_{j=1}^{n} X_j - A_n. \tag{II1}$$

We have found a series of this kind of results for an independent sequence (cf. e.g. Petrov 1971 and Billingsley 1968). A natural question is what about a weakly dependent sequence. Only assumption of weak dependence is not enough for weak convergence. For instance, let $\{\xi_n,\, n \geq 1\}$ be a sequence of i.i.d. random variables with a common characteristic function $f(t)$, and let

$$X_n = \xi_{n+1} - \xi_n.$$

Then $\{X_n,\, n \geq 1\}$ is a strictly stationary sequence and satisfies any mixing conditions mentioned in §1.1. The sum

$$\sum_{k=1}^{n} X_k = \xi_{n+1} - \xi_1 \tag{II2}$$

has the characteristic function $|f(t)|^2$ for all $n$. It is reasonable to introduce some restrictions to make the variance of the sum $\sum_{k=1}^{n} X_k$ increase when $n$ is increasing. Hence we assume always that $B_n$ in (II1) tends to infinity as $n \to \infty$.

First of all, we state the so-called *Bernstein's blocking technique* utilized frequently in showing limit theorems for mixing random variables. Let positive integers $p = p(n)$, $q = q(n)$ and $k = k(n)$ with $1 \leq p \leq n$, $q =$

$o(p)$, $k = [n/(p+q)]$,[1] and let

$$\xi_j = \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} X_i, \qquad \eta_j = \sum_{i=jp+(j-1)q+1}^{j(p+q)} X_i, \; j = 1, \cdots, k,$$

$$\eta_{k+1} = \sum_{i=k(p+q)+1}^{n} X_i. \tag{II3}$$

Then

$$S_n = \sum_{j=1}^{k} \xi_j + \sum_{j=1}^{k+1} \eta_j. \tag{II4}$$

By weak dependence, $\xi_1, \xi_2, \cdots, \xi_k$ are asymptotically independent as $q = q(n)$ is large enough. On the other hand, the sum $\sum_{j=1}^{k+1} \eta_j$ is negligible, compared with $S_n$ by noting $q = o(p)$. Consequently, the Bernstein method allows us to consider the sums of mixing random variables as independent sums.

Using this method, by the procedure similar to that for an independent sequence, for $\alpha$-mixing sequence, we may prove the following theorem about the class of possible limit distributions of sums.

**Theorem II1.** *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\alpha$-mixing sequence, $\{A_n\}$ and $\{B_n\}$ two sequences of real numbers with $B_n \to \infty$ as $n \to \infty$. Suppose that the distribution function $F_n(x)$ of the sum*

$$\frac{1}{B_n} \sum_{k=1}^{n} X_k - A_n$$

*converges weakly to a distribution function $F(x)$. Then $F(x)$ is stable with some exponent $a$. Moreover,*

$$B_n = n^{1/a} h(n),$$

*where $h(n)$ is a slowly varying function with positive integer argument.*

[1] In this book, the sign $[\cdot]$ sometimes denotes the greatest integer part or at other times denotes brackets. It will be clear from the context.

# Chapter 3 Weak Convergence for $\alpha$-mixing Sequences

## 3.1 Necessary and sufficient conditions for the CLT

For an $\alpha$-mixing sequence $\{X_n, n \geq 1\}$, Ibragimov (1959, 1962) first gave necessary and sufficient conditions for the *central limit theorem* (*CLT*). In this chapter we always assume that $\{X_n, n \geq 1\}$ is a strictly stationary $\alpha$-mixing sequence unless special indication. Put $S_n = \sum_{j=1}^n X_j$, $\sigma_n^2 = \mathrm{Var} S_n$.

**Theorem 3.1.1.** *Suppose that $EX_1 = 0$ and $EX_1^2 < \infty$. Then in order that $\{X_n\}$ obeys the CLT and $\lim_{n\to\infty} \sigma_n^2 = \infty$, it is necessary that*

*(1) $\sigma_n^2 = nh(n)$, where $h(x)$ is a slowly varying function of the continuous variable $x > 0$,*

*(2) for any pair of sequences $p = p(n)$, $q = q(n)$ satisfying that as $n \to \infty$.*

*(a) $p \to \infty$, $q \to \infty$, $q = o(p)$, $p = o(n)$,*

*(b) $n^{1-\beta} q^{1+\beta} p^{-2} \to 0$ for all $\beta > 0$,*

*(c) $np^{-1}\alpha(q) \to 0$, and*

$$\lim_{n\to\infty} \frac{n}{p\sigma_n^2} \int_{|x|>\varepsilon\sigma_n} x^2 \, dF_p(x) = 0 \tag{3.1.1}$$

*for any $\varepsilon > 0$, where $F_p(x) = P(S_p < x)$. Conversely, if (1) holds and if (3.1.1) is satisfied for some choice of the functions $p$ and $q$ satsifying the given conditions, then the CLT is satisfied.*

We do not prove this theorem here. For its proof we refer to Ibragimov and Linnik's book (1971).

A simpler necessary and sufficient condition for the CLT was given by Denker (1985).

**Theorem 3.1.2.** *Suppose that* $EX_1 = 0$, $EX_1^2 < \infty$ *and* $\sigma_n^2 \to \infty$ *as* $n \to \infty$. *Then in order that* $\{X_n\}$ *obeys the CLT, it is necessary and sufficient that* $\{S_n^2/\sigma_n^2,\ n \geq 1\}$ *is integrable uniformly.*

**Proof.** Necessity. Suppose that

$$S_n/\sigma_n \xrightarrow{d} N(0,1),$$

where $N(0,1)$ stands for a standard normal variable. Therefore, for any $\varepsilon > 0$ there exists a $K > 0$ such that

$$\lim_{n\to\infty} \int_{|S_n/\sigma_n|>K} S_n^2/\sigma_n^2\, dP = \int_{|N|>K} N^2\, dP < \varepsilon,$$

which implies uniform integrability of $\{S_n^2/\sigma_n^2,\ n \geq 1\}$.

Sufficiency. Suppose that $\{S_n^2/\sigma_n^2,\ n \geq 1\}$ is integrable uniformly. By Theorem 2.1.3, we have

$$\sigma_n^2 = nh(n), \tag{3.1.2}$$

where $h(x)$ is a slowly varying function on $[1, \infty)$. Assume that $p$ and $q$ are the functions satisfying conditions (a) and (b) in Theorem 3.1.1. Furthermore, we choose $p$ and $q$ that satisfy condition (c) in Theorem 3.1.1 and

$$n^2 p^{-2} \sigma_q^2 \sigma_p^{-2} = n^2 p^{-3} q h(q)/h(p) \to 0 \quad \text{as } n \to \infty. \tag{3.1.3}$$

For any given $\delta > 0$, there exists a $K > 0$ such that

$$\int_{|S_n|>K\sigma_n} S_n^2\, dP < \delta\sigma_n^2 \tag{3.1.4}$$

for each $n \geq 1$. By Property A4 of a slowly varying function,

$$\frac{\sigma_p^2}{\sigma_n^2} = \frac{ph(p)}{nh(n)} = o(1) \quad \text{as}\, n \to \infty,$$

which implies that $K\sigma_p < \varepsilon\sigma_n$ for all large $n$. Hence

$$\begin{aligned} \frac{n}{p} \int_{|S_p|>\varepsilon\sigma_n} S_p^2\, dP &\leq \frac{n}{p} \int_{|S_p|>K\sigma_p} S_p^2\, dP \\ &< \frac{n}{p} \delta\sigma_p^2 < 2\delta\sigma_n^2. \end{aligned}$$

The last inequality is due to (II4), condition (c) and (3.1.3). Consequently, condition (3.1.1) is satisfied. From Theorem 3.1.1, sufficiency is proved.

It is well known that boundedness of $\{E|S_n/\sigma_n|^{2+\delta}, n \geq 1\}$ implies uniform integrability of $\{S_n^2/\sigma_n^2, n \geq 1\}$. The former requires existence of higher order moments. This explains that the moment conditions imposed on $\{S_n\}$ are important for the CLT. A related result is given by Dehling, Denker and Philipp (1986).

**Theorem 3.1.3.** *Suppose that* $EX_1 = 0$, $EX_1^2 = 1$, $\sigma_n^2 = nh(n)$, *where* $h(n)$ *is a slowly varying function. Then in order that the distributions of* $\{S_n/\sigma_n, n \geq 1\}$ *tend to the standard normal distribution* $\Phi(x)$, *it is necessary and sufficient that*

$$\limsup_{n\to\infty} \sigma_n/E|S_n| \leq \sqrt{\pi/2}. \tag{3.1.5}$$

In order to prove this theorem, we need the following lemma.

At first, we introduce some notations. Let integer $p$ and real number $g$ satisfy

$$2 \leq g \leq \alpha^{-1/4}(\sigma_p^{1/4}) \wedge \sigma_p^{1/4},$$

where $\alpha(x) = \alpha([x])$, and let

$$v^2 = \sigma_p^{-2} \int_{g^{1/2}<|S_p|/\sigma_p \leq g} S_p^2\, dP, \tag{3.1.6}$$

$$u^2 = \int_{|S_p|\leq g\sigma_p} S_p^2\, dP, \tag{3.1.7}$$

$$r = [g^2 d] \tag{3.1.8}$$

where $d$ satisfies $2g^{-1/2} \vee v^2 \leq d < 1$,

$$n = r(p + [\sigma_p^{1/4}]) \text{ and} \tau^2 = ru^2. \tag{3.1.9}$$

**Lemma 3.1.1.** *Suppose that* $EX_1 = 0$ *and* $EX_1^2 = 1$. *We have*

$$\begin{aligned} &|E\exp(itS_n/\tau) - \exp(-t^2/2)| \\ &\quad \leq 2d + |t|\sigma_p d^{1/2}/u + |t|^3\sigma_p/(ug^{1/4}) + |t|^3\sigma_p^3 v/u^3 \\ &\quad\quad + 4\alpha^{1/2}(\sigma_p^{1/4}) + t^4/g + t^2\sigma_p^2/(u^2 g). \end{aligned}$$

**Proof.** Note that $u \leq \sigma_p$. Hence, for $|t| > r^{1/2}$, we have

$$|t|^3\sigma_p/(ug^{1/4}) \geq r^{3/2}/g^{1/4} \geq ((g^2 d - 1)g^{-1/6})^{3/2} \geq g^2.$$

Then the conclusion of Lemma 3.1.1 holds obviously.

Now we assume that $|t| \le r^{1/2}$. Let $q = [\sigma_p^{1/4}]$ and

$$\xi_j = \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} X_i,$$

$$\eta_j = \sum_{i=jp+(j-1)q+1}^{j(p+q)} X_i, \qquad j = 1, \cdots, r.$$

Recalling the definition of $n$, we can write

$$S_n = \sum_{j=1}^{r} \xi_j + \sum_{j=1}^{r} \eta_j =: S_n' + S_n''.$$

By the Minkowski inequality

$$ES_n''^2 \le r^2 \sigma_p^{1/2} \le g^4 \sigma_p^{1/2} \le \sigma_p^{3/2}.$$

Hence

$$\begin{aligned}
&|E\exp(itS_n/\tau) - E\exp(itS_n'/\tau)| \\
&\quad \le |E\exp(itS_n''/\tau) - 1| \\
&\quad \le t^2 ES_n''^2/\tau^2 \le t^2\sigma_p^{3/2}/(u^2 r) \\
&\quad \le t^2\sigma_p^2/(u^2 g).
\end{aligned} \tag{3.1.10}$$

Moreover, by Lemma 1.2.1, we have

$$\begin{aligned}
&|E\exp(itS_n'/\tau) - (E\exp(itS_p/\tau))^r| \\
&\quad \le 4r\alpha(\sigma_p^{1/4}) \le 4\alpha^{1/2}(\sigma_p^{1/4}).
\end{aligned} \tag{3.1.11}$$

We now estimate $|E\exp(itS_p/\tau) - (1 - t^2/(2r))|$. By the Chebyshev inequality we have

$$\Big|\int_{|S_p|>g\sigma_p} \exp(itS_p/\tau)\, dP\Big| \le g^{-2} \le d/r. \tag{3.1.12}$$

By Taylor's theorem

$$\begin{aligned}
&\Big|\int_{|S_p|\le g\sigma_p} \exp(itS_p/\tau)\, dP - (1 - t^2/(2r))\Big| \\
&\le \Big|P(|S_p| \le g\sigma_p) + \frac{it}{\tau}\int_{|S_p|\le g\sigma_p} S_p\, dP \\
&\quad - \frac{t^2}{2\tau^2}\int_{|S_p|\le g\sigma_p} S_p^2\, dP - \Big(1 - \frac{t^2}{2r}\Big)\Big| \\
&\quad + \frac{|t|^3}{\tau^3}\int_{|S_p|\le g\sigma_p} |S_p|^3\, dP.
\end{aligned} \tag{3.1.13}$$

Similarly to (3.1.12),

$$|1 - P(|S_p| \le g\sigma_p)| \le g^{-2} \le d/r. \tag{3.1.14}$$

Noting $ES_p = 0$, we obtain

$$\begin{aligned}\tau^{-1}\Big|\int_{|S_p|\le g\sigma_p} S_p\,dP\Big| &= \tau^{-1}\Big|\int_{|S_p|>g\sigma_p} S_p\,dP\Big| \\ &\le \sigma_p/\tau g \le \sigma_p d^{1/2}/(ru). \end{aligned}\tag{3.1.15}$$

It is clear that

$$\frac{t^2}{2\tau^2}\int_{|S_p|\le g\sigma_p} S_p^2\,dP = \frac{t^2}{2r}. \tag{3.1.16}$$

The cubic term in (3.1.13) is estimated as follows.

$$\begin{aligned}&\tau^{-3}\int_{g^{1/2}\sigma_p<|S_p|\le g\sigma_p} |S_p|^3\,dP \\ &\le \tau^{-3} g\sigma_p^3 v^2 \\ &\le \sigma_p^3 v/(ru^3)\end{aligned}\tag{3.1.17}$$

and

$$\begin{aligned}&\tau^{-3}\int_{|S_p|\le g^{1/2}\sigma_p} |S_p|^3\,dP \\ &\le \tau^{-3} g^{1/2}\sigma_p u^2 \\ &\le \sigma_p g^{1/2}/(ur^{3/2}) \\ &\le \sigma_p/(rug^{1/4}).\end{aligned}\tag{3.1.18}$$

Hence substituting (3.1.14)-(3.1.18) into (3.1.13) we obtain by (3.1.12)

$$|E\exp(itS_p/\tau) - (1 - t^2/(2r))| \le \eta/r,$$

where

$$\eta = 2d + |t|\sigma_p d^{1/2}/u + |t|^3\sigma_p/(ug^{1/4}) + |t|^3\sigma_p^3 v/u^3.$$

Note that $|a^r - b^r| \le r|a-b|$ for $|a|\le 1$, $|b|\le 1$. We obtain for $|t| < r^{1/2}$

$$|E\exp(itS_n{}'/\tau) - (1 - t^2/(2r))^r| \le \eta + 4\alpha^{1/2}(\sigma_p^{1/4}) \tag{3.1.19}$$

by (3.1.11). Moreover

$$|\exp(-t^2/2) - (1 - t^2/(2r))^r| \le \frac{1}{4}t^4 r^{-1} \quad \text{for } |t| < r^{1/2},$$

since $|e^x - (1+x)| \le x^2$ for $|x| \le \frac{1}{2}$. Consequently, the lemma follows from (3.1.19), (3.1.10).

**Proof of Theorem 3.1.3.** Necessity. If the distribution of $S_n/\sigma_n$ tends to $\Phi(x)$, then for any $a > 0$

$$\liminf_{n\to\infty} E|S_n|/\sigma_n \ge \int_{-a}^{a} \frac{|x|}{\sqrt{2\pi}} e^{-x^2/2}\, dx,$$

which implies (3.1.5).

Sufficiency. Let $\rho_n = \sqrt{\pi/2}E|S_n|$. If we can show that

$$S_n/\rho_n \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty, \tag{3.1.20}$$

then for any $\alpha > 0$

$$\begin{aligned}\int_{|N|\le\alpha} N^2 dP &= \lim_{n\to\infty} \rho_n^{-2} \int_{|S_n|/\rho_n \le \alpha} S_n^2\, dP \\ &\le \limsup_{n\to\infty} \sigma_n^2/\rho_n^2 \le 1\end{aligned}$$

by (3.1.5). Letting $\alpha \to \infty$ yields that $\sigma_n/\rho_n \to 1$ as $n \to \infty$. Hence (3.1.20) implies $S_n/\sigma_n \xrightarrow{d} N(0,1)$.

Now we are going to prove (3.1.20). At first, for each $n \in \mathbb{N}$ we show that there is an infinite sequence $Q \subset \mathbb{N}$ and real numbers $\tau_n$, $n \in Q$ such that

$$S_n/\tau_n \xrightarrow{d} N(0,1) \quad n \to \infty,\, n \in Q. \tag{3.1.21}$$

For this purpose we prove that there exist a sequence $\{g(p),\, p \ge 1\}$ and a monotone sequence $\{c(p),\, p \ge 1\}$ with the following properties:

$$g(p) \to \infty,\, c(p) \to 0 \quad \text{as } p \to \infty, \tag{3.1.22}$$

$$g(p) \le \alpha^{-1/4}(\sigma_p^{1/4}) \wedge \sigma_p^{1/4}, \tag{3.1.23}$$

$$v^2(p) := \sigma_p^{-2} \int_{g(p)^{1/2} < |S_p|/\sigma_p \le g(p)} S_p^2\, dP \to 0 \quad \text{as } p \to \infty \tag{3.1.24}$$

and

$$2g(p)^{-1/2} \vee v^2(p) \le c(p) < 1. \tag{3.1.25}$$

We first choose a sequence $\{z(p),\, p \ge 1\}$ with

$$\lim_{p\to\infty} z(p) = \infty, \quad z(p) \le \alpha^{-1/4}(\sigma_p^{1/4}) \wedge \sigma_p^{1/4}. \tag{3.1.26}$$

Next, we choose a sequence $\{i(p),\, p \ge 1\}$ such that

$$i(p) \to \infty,\, 2^{-i(p)} \log z(p) \to \infty \quad \text{as}\, p \to \infty. \tag{3.1.27}$$

Fix $p \in \mathbb{N}$. Since the intervals $I_i(p) := (z(p)^{2^{-i-1}}, z(p)^{2^{-i}}]$, $0 \le i < i(p)$, are disjoint, there exists an integer $k = k(p)$ with $0 \le k \le i(p)$ such that

$$\sigma_p^{-2} \int_{|S_p|/\sigma_p \in I_k(p)} S_p^2 \, dP \le 1/i(p). \tag{3.1.28}$$

Let

$$g(p) = z(p)^{2^{-k(p)}}. \tag{3.1.29}$$

Then $g(p) \to \infty$ as $p \to \infty$ by (3.1.27). Because of (3.1.26)–(3.1.28), (3.1.23) and (3.1.24) are satisfied. Since $2g(p)^{-1/2} \vee v^2(p) \to 0$ as $p \to \infty$ we can choose $\{c(p)\}$ with $c(p) \downarrow 0$ and satisfying (3.1.25).

With these choices of $\{g(p)\}$ and $\{c(p)\}$, we define $u(p)$, $r(p)$ and $n(p)$ by (3.1.7), (3.1.8) and (3.1.9) respectively. Put $Q = \{n(p),\, p \ge 1\}$ and define $\tau_n^2$, $n \in Q$ by (3.1.9). Since

$$\begin{aligned} \sigma_p^{-1} E|S_p| &= \sigma_p^{-1} \int_{|S_p|/\sigma_p \le g(p)} |S_p| \, dP + \sigma_p^{-1} \int_{|S_p|/\sigma_p > g(p)} |S_p| \, dP \\ &\le \sigma_p^{-1} u(p) + g(p)^{-1}, \end{aligned} \tag{3.1.30}$$

we have, by (3.1.5), for $p$ large enough

$$u(p)/\sigma_p \ge \gamma/2, \tag{3.1.31}$$

where $\gamma = \inf\{E|S_p|/\sigma_p,\, p \ge 1\} > 0$. Lemma 3.1.1 now implies (3.1.21).

Next we show that

$$\lim_{n\to\infty, n \in Q} \tau_n/\rho_n = 1. \tag{3.1.32}$$

To see this we choose a sequence $\{b(m),\, m \ge 1\}$ with

$$\begin{aligned} &\lim_{m\to\infty} b(m) = \infty, \\ &\lim_{m\to\infty} \sup\{|h(tm)/h(m) - 1|,\, 1 \le t \le b(m)\} = 0. \end{aligned} \tag{3.1.33}$$

This is possible. Indeed, by the Karamata theorem (see Appendix Theorem A1) there exists an increasing sequence $\{m_k,\, k \ge 2\}$ such that

$$\Big| \sup_{1 \le t \le k} h(tm)/h(m) - 1 \Big| \le \frac{1}{k}, \quad m \ge m_k.$$

Then $b(\cdot)$ defined by $b(m) = k$ for $m_k < m \le m_{k+1}$ has the desired properties. Of course, we can assume that $\{z(p),\, p \ge 1\}$ is chosen so that

in addition to (3.1.26) we have $z(p) \le b(p)^{1/2}/2$. Then for all large $p$ we have, by (3.1.9) and $\sigma_p^2 \le p^2$,

$$\frac{\sigma^2(n(p))}{r(p)\sigma_p^2} = \frac{r(p)(p+[\sigma_p^{1/4}])h(r(p)(p+[\sigma_p^{1/4}]))}{r(p)ph(p)}$$
$$= (1+O(p^{-1/2}))\frac{h(r(p)(p+[\sigma_p^{1/4}]))}{h(p)} = 1+o(1)$$

by (3.1.33). Thus, by (3.1.9) and (3.1.30) we have for $p$ large enough

$$E(S_{n(p)}/\tau_{n(p)})^2 = \sigma_{n(p)}^2/\tau_{n(p)}^2 \le 2\sigma_p^2/u(p)^2 \le 8/\gamma^2. \tag{3.1.34}$$

Hence $\{S_n/\tau_n,\, n \in Q\}$ is uniformly integrable and thus, by (3.1.21),

$$\lim_{n\to\infty, n\in Q} E|S_n|/\tau_n = E|N| = \sqrt{2/\pi}.$$

This proves (3.1.32).

If $Q = \mathbb{N}$, (3.1.21) and (3.1.32) imply

$$S_n/\rho_n \xrightarrow{d} N(0,1) \quad \text{as}\, n\to\infty. \tag{3.1.35}$$

Consider the case of $Q \subset \mathbb{N}$. Assume that $\{z(p),\, p \ge 1\}$ constructed in the proof of (3.1.21) satisfies

$$z(p) \le \min(\alpha(\sigma_p^{1/4})^{-1/16},\, \sigma_p^{1/16},\, p^{1/4},\, b(p)^{1/8}/2) \tag{3.1.36}$$

and

$$z(p) \le z(q) \le z(p)^{3/2},\, p \le q \le p^2. \tag{3.1.37}$$

Such a sequence can be constructed as follows: First choose an increasing sequence $y(p)$ satisfying (3.1.36). By induction on $k$ define $z(p) = y(p_k) \wedge z(p_{k-1})^{3/2}$ for $p = p_k = 2^{2^k}, p_k+1, \cdots, p_{k+1}-1$. Let $\{l(n),\, n \in Q\}$ and $\{j(n),\, n \in Q\}$ be two arbitrary sequences of real numbers tending to infinity and with $l(n) < j(n)$, $n \in Q$.

Recalling condition (3.1.5) and noting $E|S_n| \le \sigma_n$, we have

$$\sqrt{2/\pi} \le \limsup_{n\to\infty} \sigma_n/\rho_n \le 1. \tag{3.1.38}$$

Moreover, we have shown that $S_n/\rho_n \xrightarrow{d} N(0,1)$, $n\to\infty$, $n \in Q$. Consequently, we obtain, for any $\alpha > 0$

$$\int_{|N|\le\alpha} N^2\, dP = \lim_{n\in Q} \int_{|S_n|/\rho_n \le \alpha} S_n^2/\rho_n^2\, dP$$
$$\le \liminf_{n\in Q} \rho_n^{-2} \int_{|S_n|/\sigma_n \le l(n)} S_n^2\, dP,$$

which implies by letting $\alpha \to \infty$

$$\begin{aligned} 1 &\leq \liminf_{n\in Q} \rho_n^{-2} \int_{|S_n|/\sigma_n \leq l(n)} S_n^2 dP \\ &\leq \limsup_{n\in Q} \rho_n^{-2} \int_{|S_n|/\sigma_n \leq j(n)} S_n^2 dP \leq 1. \end{aligned} \tag{3.1.39}$$

(3.1.38) and (3.1.39) imply

$$\lim_{n\in Q} \sigma_n^{-2} \int_{l(n)<|S_n|/\sigma_n \leq j(n)} S_n^2\, dP = 0. \tag{3.1.40}$$

We shall apply Lemma 3.1.1 again. To prepare for it we set

$$l(n) = l(n(p)) = z(n(p)) \quad \text{if } n = n(p),\ p \geq 1 \tag{3.1.41}$$

and

$$\begin{aligned} j(n) = j(n(p)) = \min(\alpha^{-1/4}(\sigma_n^{1/4}),\ \sigma_n^{1/4},\ b(n)^{1/2}/2) \\ \text{if} n = n(p),\ p \geq 1. \end{aligned} \tag{3.1.42}$$

For $p$ large enough, we have by (3.1.22), (3.1.29) and (3.1.36)

$$n(p) \leq g^2(p)c(p)(p + p^{1/4}) \leq z^2(p)p \leq p^{1/2}p < p^2. \tag{3.1.43}$$

By (3.1.36), (3.1.42) and (3.1.41)

$$j(n) \geq z^4(n) = l^4(n) > l(n), \quad n \in Q. \tag{3.1.44}$$

Since, by (3.1.41)

$$w^2(n) := \sigma_n^{-2} \int_{l^{1/2}(n)<|S_n|/\sigma_n \leq j(n)} S_n^2\, dP \to 0, \quad n \in Q, \tag{3.1.45}$$

we can choose a nonincreasing sequence $\{d(n),\ n \in Q\}$ such that

$$\lim_{n\in Q} d(n) = 0 \quad \text{and} \quad d(n) \geq 2h(n)^{-1/2} \wedge w^2(n),\ n \in Q. \tag{3.1.46}$$

Let $Q = \{n_k,\ k \geq 1\}$ be arranged in increasing order and let $J_k$ be the interval

$$J_k = [n_k l^2(n_k) d(n_k),\ n_k j^2(n_k) d(n_k)].$$

We show that there exists a $k_0$ such that

$$J_k \cap J_{k+1} \neq \emptyset,\ k \geq k_0. \tag{3.1.47}$$

Obviously $n(n_k) = r(n_k)(n_k + [\sigma(n_k)^{1/4}]) \in Q$. As $n_{k+1}$ is the smallest member of $Q$ bigger than $n_k$ we must have for large $k$

$$n_{k+1} \leq n(n_k) \leq n_k z^2(n_k) < n_k^2$$

by (3.1.43). Hence, by (3.1.41) and (3.1.37), the left endpoint of $J_{k+1}$ does not exceed

$$\begin{aligned} n_{k+1} l^2(n_{k+1}) d(n_{k+1}) &\leq n_k z^2(n_k) z^2(n_{k+1}) \\ &\leq n_k z^2(n_k) z^2(n_k^2) \leq n_k z^5(n_k) \end{aligned}$$

for large $k$. On the other hand, by (3.1.46) and (3.1.44), the right endpoint of $J_k$ is bigger than

$$n_k j^2(n_k) d(n_k) \geq n_k j^2(n_k) l^{-1/2}(n_k) \geq n_k z(n_k)^{15/2}$$

for large $k$. Since $z(n_k) \to \infty$ we obtain (3.1.47). Let $m \geq \min\{l,\ l \in J_{k_0}\}$. Then there is a $k \geq k_0$ such that $m \in J_k$. Thus we have for some $g \in [l(n_k),\ j(n_k)]$ and some $|\theta| \leq 2$

$$\begin{aligned} m &= g^2 d(n_k) n_k = [g^2 d(n_k)](n_k + [\sigma^{1/4}(n_k)]) + \theta n_k \\ &=: M_k + \theta n_k. \end{aligned} \tag{3.1.48}$$

Now by (3.1.8), $M_k$ is of the form (3.1.9) and hence we can apply Lemma 3.1.1. Put $p = n_k$ and $d = d(n_k)$. By (3.1.31), $u(n_k)/\sigma(n_k) \geq \frac{1}{2}\gamma > 0$. Now $g \geq l(n_k) \to \infty$ and $\alpha(\sigma^{1/4}(n_k)) \to 0$. Finally, by (3.1.45) and noting $l(n_k) \leq g \leq j(n_k)$, we have

$$v^2(n_k) := \sigma^{-2}(n_k) \int_{g^{1/2} < |S_{n_k}|/\sigma(n_k) \leq g} S_{n_k}^2 \, dP \leq w^2(n_k) \to 0.$$

Hence, by Lemma 3.1.1,

$$S_{M_k}/\tau(M_k) \xrightarrow{d} N(0,1). \tag{3.1.49}$$

Since $|\theta| \leq 2$ we have by (3.1.33) for large $k$

$$\frac{ES^2_{|\theta| n_k}}{\sigma^2(n_k)} \leq \frac{|\theta| n_k h(|\theta| n_k)}{n_k h(n_k)} \leq 4.$$

Consequently

$$E(S_m - S_{M_k})^2 = ES^2_{|\theta| n_k} \leq 4\sigma^2(n_k). \tag{3.1.50}$$

Denoting $r^*(n_k) = [g^2 d(n_k)]$ we obtain by (3.1.33) for large $k$

$$\frac{\sigma^2(M_k)}{r^*(n_k)\sigma^2(n_k)} \geq \frac{r^*(n_k)n_k h(r^*(n_k)(n_k + [\sigma^{1/4}(n_k)]))}{r^*(n_k)n_k h(n_k)} \geq \frac{1}{2}$$

since $r^*(n_k) \leq g^2 d(n_k) \leq j^2(n_k)d(n_k) \leq b(n_k)/2$ by (3.1.41 ). Hence, from (3.1.50) and as $r^*(n_k) \to \infty$, we have

$$E(S_m - S_{M_k})^2/\sigma^2(M_k) \to 0 \quad \text{as } k \to \infty. \tag{3.1.51}$$

In the same way as (3.1.34) one can prove

$$\sigma^2(M_k)/\tau^2(M_k) \leq 8/\gamma^2. \tag{3.1.52}$$

Put

$$\tau_m = \tau(M_k), \quad \text{if } m \text{ and } M_k \text{ are as in (3.1.47).}$$

Then by (3.1.49), (3.1.51) and (3.1.52)

$$S_m/\tau_m \xrightarrow{d} N(0,1) \quad \text{as } m \to \infty. \tag{3.1.53}$$

The sequence $\{S_m/\tau_m,\ m \geq 1\}$ is uniformly integerable in view of (3.1.51) and (3.1.52). We obtain (3.1.35) from (3.1.53) and (3.1.32).

With the help of it and similarly to (3.1.39) we have $\sigma_n/\rho_n \to 1$ as $n \to \infty$. This completes the proof of Theorem 3.1.3.

## 3.2 Sufficient conditions for the CLT and WIP

In the last section, we give some necessary and sufficient conditions for the CLT. But it is not easy to verify them. In this section, we shall give some sufficient conditions for the CLT and *weak invariance principle* (*WIP*). Rosenblatt (1956) first gave sufficient conditions for the CLT. After that many authors (e.g. Ibragimov 1962) have discussed this subject and have obtained some further results. One of the best is due to Herrndorf (1984, 1985). The following theorem is attributed to Gordin (1969) and has been restated and proved by Hall and Heyde (1980) (Corollary 5.3(ii)) via approximating $S_n$ by a naturally related martingale with stationary ergodic differences. Its proof will not been presented here.

**Theorem 3.2.1.** *Suppose that* $EX_1 = 0$, $E|X_1|^{2+\delta} < \infty$ *for some* $\delta > 0$ *and*

$$\sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty. \tag{3.2.1}$$

*Then*

$$\sigma^2 := EX_1^2 + 2\sum_{j=2}^{\infty} EX_1X_j < \infty \tag{3.2.2}$$

*and, if* $\sigma \neq 0$,

$$S_n/\sigma\sqrt{n} \xrightarrow{d} N(0,1). \tag{3.2.3}$$

**Remark 3.2.1.** For the case of bounded variables, i.e. $\delta = \infty$, condition (3.2.1) is reduced to

$$\sum_{n=1}^{\infty} \alpha(n) < \infty. \tag{3.2.4}$$

**Remark 3.2.2.** Davydov (1973) gave two examples which pointed out that the rate of $\alpha(n)$ in Theorem 3.2.1 could not be improved in certain senses.

**Example 3.2.1.** For any $\delta > 0$ and $\varepsilon > 0$, there is a strictly stationary countable-state Markov chain $\{X_n, n \geq 1\}$ with $EX_1 = 0$, $E|X_1|^{2+\delta} < \infty$ such that

(i) $\alpha(n) = o(n^{-(1-\varepsilon)(1+2/\delta)})$ as $n \to \infty$,

(ii) $\mathrm{Var}S_n \approx n^d$ [1] for some $1 < d < 2$,

(iii) $S_n$ is attracted to a symmetric stable law with exponent $\alpha$, $1 < \alpha < 2$.

**Example 3.2.2.** For any $\varepsilon > 0$ there exists a strictly stationary countable-state Markov chain $\{X_n, n \geq 1\}$ with $EX_1 = 0$, $|X_1| < c_0$ a.s. for some $c_0 < \infty$ such that $\alpha(n) = o(n^{-(1-\varepsilon)})$ as $n \to \infty$ and properties (ii) and (iii) in Example 3.2.1 hold.

We now investigate the weak invariance principles. Define random elements on $D[0,1]$ as follows:

$$W_n(t) = S_{[nt]}/\sigma_n, \quad 0 \leq t \leq 1.$$

Convergence theory of probability measures tells us that the key to the proof for weak invariance principle lies in verification of *tightness*. One of the following conditions is sufficient for tightness (cf. Billingsley 1968, Section 16).

(1) For any $\varepsilon > 0$, $\eta > 0$, there exist a $\delta$, $0 < \delta < 1$, and an integer $n_0$ such that, for $0 \leq t \leq 1$

$$\frac{1}{\delta}P\Big\{\sup_{t\leq s\leq t+\delta}|W_n(s) - W_n(t)| \geq \varepsilon\Big\} \leq \eta, \quad n \geq n_0. \tag{3.2.5}$$

[1] $a \approx b$ means $\lim a/b = 1$.

(2) For any $\varepsilon > 0$, there exists a $\lambda > 1$ and an integer $n_0$ such that

$$P\Big\{\max_{1\le i\le n} |S_i| \ge \lambda\sigma_n\Big\} \le \varepsilon/\lambda^2, \quad n \ge n_0. \tag{3.2.6}$$

Davydov (1968) first generalized the CLT to the invariance principle for bounded variables, but condition (3.2.4) was strengthened as $\sum_{n=1}^{\infty} \alpha^{1/2}(n) < \infty$. Oodaira and Yoshihara (1972) sharpened his result and obtained the invariance principle under the conditions for the CLT basically.

**Theorem 3.2.2.** *Suppose that* $EX_1 = 0$, $|X_1| < c_0 < \infty$. *If (3.2.4) is satisfied and* $\alpha(n) \le c/n\log n$. *Then, if* $\sigma > 0$,

$$W_n \Longrightarrow W,$$

*i.e.* $W_n$ *weakly converges to a Wiener process* $W$ *with* $\sigma_n = \sigma\sqrt{n}$.

**Proof.** By Remark 3.2.1, it follows that $W_n(t)$ converges to $W(t)$ in distribution for any $t$, $0 \le t \le 1$. By the Cramer-Wold method, it is easy to see that for any given $0 \le t_1 < t_2 < \cdots < t_k \le 1$, $(W_n(t_1), \cdots, W_n(t_k))$ converges to $(W(t_1), \cdots, W(t_k))$ in distribution.

We now show the tightness of $\{W_n\}$. It is enough by (3.2.6) to prove that

$$P\Big\{\max_{1\le i\le n} |S_i| \ge 3\lambda\sigma\sqrt{n}\Big\} \le \varepsilon/\lambda^2, \quad n \ge n_0. \tag{3.2.7}$$

Put $p = [\sqrt{n}/(\log n)^{3/8}]$, $k = [n/p]$. We have

$$P\{|X_1| + \cdots + |X_{2p}| \ge \lambda\sigma\sqrt{n}\} = 0 \tag{3.2.8}$$

for large $n$ by boundedness of $\{X_n\}$. Moreover, using Lemma 1.2.1 and condition (3.2.4) we have uniform integrability of $\{S_n^2/n,\ n \ge 1\}$. Therefore, for any $\varepsilon > 0$, there exists a $\lambda > 1$ such that for each $i \ge 1$

$$P\{|S_i| \ge \lambda\sigma\sqrt{i}\} \le \varepsilon/3\lambda^2. \tag{3.2.9}$$

Put $E_j = \{\max_{1\le i<j}|S_i| < 3\lambda\sigma\sqrt{n} \le |S_j|\}$. We have

$$\begin{aligned}
&P\Big\{\max_{1\le i\le n}|S_i| \ge 3\lambda\sigma\sqrt{n}\Big\}\\
&\le P\{|S_n| \ge \lambda\sigma\sqrt{n}\} + P\Big(\bigcup_{j=1}^{n}\{E_j \bigcap (|S_n - S_j| \ge 2\lambda\sigma\sqrt{n})\}\Big)\\
&\le P\{|S_n| \ge \lambda\sigma\sqrt{n}\}\\
&\quad + \sum_{i=0}^{k-2} P\Big(\bigcup_{j=1}^{p}\Big\{E_{ip+j} \bigcap (|S_n - S_{ip+j}| \ge 2\lambda\sigma\sqrt{n})\Big\}\Big)\\
&\quad + \sum_{j=(k-1)p+1}^{n} P\Big\{|S_n - S_j| \ge 2\lambda\sigma\sqrt{n}\Big\}\\
&\le P\{|S_n| \ge \lambda\sigma\sqrt{n}\}\\
&\quad + \sum_{i=0}^{k-2} P\Big(\bigcup_{j=1}^{p}\Big\{E_{ip+j} \bigcap (|S_n - S_{(i+2)p}| \ge \lambda\sigma\sqrt{n})\Big\}\Big)\\
&\quad + \sum_{j=1}^{p} P\Big\{|S_{(i+2)p} - S_{ip+j}| \ge \lambda\sigma\sqrt{n}\Big\}\\
&\quad + \sum_{j=(k-1)p+1}^{n} P\Big\{|X_1| + \cdots + |X_{n-j}| \ge 2\lambda\sigma\sqrt{n}\Big\}\\
&\le P\{|S_n| \ge \lambda\sigma\sqrt{n}\}\\
&\quad + \sum_{i=0}^{k-2} P\Big\{\Big(\bigcup_{j=1}^{p} E_{ip+j}\Big) \bigcap \Big(|S_n - S_{(i+2)p}| \ge \lambda\sigma\sqrt{n}\Big)\Big\}\\
&\quad + 2nP\{|X_1| + \cdots + |X_{2p}| \ge \lambda\sigma\sqrt{n}\}\\
&=: I_1 + I_2 + I_3. \qquad (3.2.10)
\end{aligned}$$

By (3.2.8), $I_3 = 0$. (3.2.9) implies that $I_1 \le \varepsilon/3\lambda^2$. Since $\cup_{j=1}^{p} E_{ip+j} \in \mathcal{F}_0^{(i+1)p}$, $(|S_n - S_{(i+2)p}| \ge \lambda\sigma\sqrt{n}) \in \mathcal{F}_{(i+2)p+1}^{\infty}$, we obtain

$$\begin{aligned}
I_2 &\le \sum_{i=o}^{k-2} P\Big\{\bigcup_{j=1}^{p} E_{ip+j}\Big\} P\{|S_n - S_{(i+2)p}| \ge \lambda\sigma\sqrt{n}\} + k\alpha(p)\\
&\le \varepsilon/3\lambda^2 + k\alpha(p)
\end{aligned}$$

by (3.2.9) again. From $\alpha(n) \le c/n\log n$, it follows that

$$k\alpha(p) \le \frac{cn}{n(\log n)^{-3/4}\log(n^{1/2}(\log n)^{-3/8})} \to 0 \quad \text{as} \quad n \to \infty.$$

Inserting these estimations into (3.2.10) yields (3.2.7). The proof of Theorem 3.2.2 is completed.

Some authors have discussed and extended this theorem. The general result was given by Herrndorf (1985). He removed the assumption of stationarity and made the moment condition more flexible. Denote

$$\mathcal{G} = \{g(x) : [0,\infty) \to [0,\infty),\ g(x) \text{ is convex}; g(0) = 0, \\ g(x)/x^2 \text{is non-decreasing}, \lim_{x\to\infty} g(x)/x^2 = \infty\}.$$

For every $g \in \mathcal{G}$ we define the inverse $\operatorname{inv} g : (0, \infty) \to (0, \infty)$ by $g(\operatorname{inv} g(x)) = x$ and $f_g : [0, \infty) \to [0, \infty)$ by $f_g(0) = 0$ and

$$f_g(x) = (\operatorname{inv} g(1/x))^2 x \quad \text{for } x > 0.$$

**Theorem 3.2.3.** *Let $\{X_n,\ n \geq 1\}$ be an $\alpha$-mixing sequence with $EX_n = 0$, $EX_n^2 < \infty$ for all $n \geq 1$ and*

$$ES_n^2/n \to \sigma^2 \quad \text{as } n \to \infty \tag{3.2.11}$$

*for some $\sigma > 0$. If there exists a $g \in G$ such that*

$$\sup_{n\geq 1} Eg(|X_n|) < \infty, \qquad \sum_{n=1}^{\infty} f_g(\alpha(n)) < \infty, \tag{3.2.12}$$

*then $W_n \Longrightarrow W$.*

The proof of Theorem 3.2.3 needs the following lemmas.

**Lemma 3.2.1.** *Let $\xi_1, \cdots, \xi_n$ be random variables. Put*

$$\alpha = \max_{1\leq k\leq n-1} \sup\{|P(AB) - P(A)P(B)| :\ A \in \sigma(\xi_1, \cdots, \xi_k), \\ B \in \sigma(\xi_{k+1}, \cdots, \xi_n)\}.$$

*Then for any $\varepsilon > 0$,*

$$P\Big\{\max_{1\leq k\leq n}\Big|\sum_{j=1}^{k} \xi_j\Big| > 2\varepsilon\Big\} \\ \leq \frac{P\{|\sum_{j=1}^n \xi_j| > \varepsilon\} + n\alpha}{\min_{1\leq k\leq n-1} P\{|\sum_{j=k+1}^n \xi_j| \leq \varepsilon\}}. \tag{3.2.13}$$

**Proof.** Put

$$A_1 = \{|\xi_1| > 2\varepsilon\},$$
$$A_k = \Big\{\Big|\sum_{j=1}^{k} \xi_j\Big| > 2\varepsilon,\ \Big|\sum_{j=1}^{l} \xi_j\Big| \le 2\varepsilon,\ 1 \le l \le k-1\Big\}, \qquad 1 < k \le n.$$
$$B_k = \Big\{\Big|\sum_{j=k+1}^{n} \xi_j\Big| < \varepsilon\Big\},\ 1 \le k < n,\ B_n = \Omega,\ C = \Big\{\Big|\sum_{j=1}^{n} \xi_j\Big| > \varepsilon\Big\}.$$

It is easy to see that $\cup_{k=1}^{n} A_k B_k \subset C$ and

$$|P(A_k B_k) - P(A_k)P(B_k)| \le \alpha.$$

Hence

$$P(C) \ge \sum_{k=1}^{n} P(A_k B_k) \ge \min_{1 \le k \le n} P(B_k) \sum_{k=1}^{n} P(A_k) - n\alpha. \tag{3.2.14}$$

Note that

$$P\Big\{\max_{1 \le k \le n}\Big|\sum_{j=1}^{k} \xi_j\Big| > 2\varepsilon\Big\} = \sum_{k=1}^{n} P(A_k).$$

Combining it with (3.2.14) yields (3.2.13).

**Lemma 3.2.2.** *Let $\{X_n,\ n \ge 1\}$ be as in Theorem 3.2.3, and satisfy*

$$\sup_{n \ge 1,\, m \ge 0} E(S_{m+n} - S_m)^2/n < \infty. \tag{3.2.15}$$

*Assume that there exist positive integers $p = p(n)$, $q = q(n)$ such that as $n \to \infty$*

$$p = o(n),\ q = o(p),\ np^{-1}\alpha(q) = o(1), \tag{3.2.16}$$
$$n^{-1} \sum_{1 \le i,\, j \le n,\, |i-j| > q} |EX_i X_j| \to 0, \tag{3.2.17}$$
$$p^{-1} \max_{0 \le m \le n-p} E\{(S_{m+p} - S_m)^2 I(|S_{m+p} - S_m| > \varepsilon n^{1/2})\} \to 0 \quad \text{for any } \varepsilon > 0. \tag{3.2.18}$$

*Then the CLT holds.*

*If, moreover, for any $\varepsilon > 0$*

$$np^{-1} \max_{0 \le m \le n-p} P\Big\{\max_{1 \le r \le p} |S_{m+r} - S_m| > \varepsilon\sqrt{n}\Big\} \to 0 \tag{3.2.19}$$

*then $W_n \Longrightarrow W$ with $\sigma_n = \sigma\sqrt{n}$.*

**Proof.** Denote $k = [n/(p+q)]$. $k \to \infty$ as $n \to \infty$ by (3.2.16). Put

$$\xi_j = \sum_{i=j(p+q)+1}^{j(p+q)+p} X_i,$$

$$\eta_j = \sum_{i=j(p+q)+p+1}^{(j+1)(p+q)} X_i, \quad 0 \le j \le k-1,$$

$$\eta_k = \sum_{i=k(p+q)+1}^{n} X_i,$$

$$S_n' = \sum_{j=0}^{k-1} \xi_j, \quad S_n'' = \sum_{j=0}^{k} \eta_j.$$

In order to prove the CLT, it suffices to show that as $n \to \infty$

(a) $ES_n''^2/n \to 0$,

(b) $\sum_{0 \le i < j \le k-1} |E\xi_i\xi_j|/n \to 0$,

(c) $\left|E\exp(itS_n' - \prod_{j=0}^{k-1} E\exp(it\xi_j)\right| \to 0$ uniformly in $t \in (-\infty, \infty)$,

(d) $\sum_{j=0}^{k-1} E(\xi_j^2 I(|\xi_j| > \varepsilon\sigma n^{1/2}))/n \to 0$ for any $\varepsilon > 0$.

From Lemma 1.2.3 and (3.2.15) we have

$$\begin{aligned} ES_n''^2 &\le \sum_{j=0}^{k} E\eta_j^2 + 2\sum_{i=1}^{n} \sum_{i+p<j\le n} |EX_iX_j| \\ &\le c(kq+p+q) + 20n \sum_{i=p+1}^{\infty} f_g(\alpha(i)) \sup_{j\le n} \|X_j\|_g^2. \end{aligned}$$

Now (a) follows from (3.2.12) and (3.2.16). By the same way, we obtain (b). As to (c), we have by Lemma 1.2.1 and (3.2.16)

$$\left|E\exp(itS_n') - \prod_{j=0}^{k-1} E\exp(it\xi_j)\right| \le 16k\alpha(q) \to 0 \quad \text{as } n \to \infty.$$

Finally, (d) follows from (3.2.18) immediately. Thus the CLT holds true for $\{X_n\}$.

Let $0 \le t_1 < \cdots < t_k \le 1$ be given. We wish to show

$$(W_n(t_1), \cdots, W_n(t_k)) \to (W(t_1), \cdots, W(t_k)) \quad \text{as} \quad n \to \infty \tag{3.2.20}$$

in distribution. From (3.2.11) and the CLT, $W_n(t)$ converges to $W(t)$ in distribution for any $t$, $0 \le t \le 1$. Therefore $\{W_n(t_1), \cdots, W_n(t_k)\}$ is tight by Prohorov's classical characterization of tightness. Let $Q$ be the

limit distribution of some subsequence of $\{W_n(t_1), \cdots, W_n(t_k)\}$, and $\pi_{t_i}$ the mapping that carries the point $x = (x_1, \cdots, x_k)$ of $R^k$ to the point $x_{t_i}$ of $R$. According to the CLT, the marginal distribution $Q\pi_{t_i}^{-1}$ are normal with variance $t_i$. Take $r_n = q/n$, then $r_n \to 0$ and $\alpha_n(q) \to 0$, where

$$\begin{aligned}\alpha_n(q) = \sup\{|P(AB) - P(A)P(B)| :\\ A \in \sigma(X_i,\, 1 \le i \le m),\\ B \in \sigma(X_i,\, m+q \le i \le n,\, 1 \le m \le n-q\}.\end{aligned}$$

Using (3.2.15) one obtains $E(W_n(t_i + r_n) - W_n(t_i))^2 \to 0$ as $n \to \infty$. Hence $Q(\pi_{t_1}, \pi_{t_2} - \pi_{t_1}, \cdots, \pi_{t_k} - \pi_{t_{k-1}})^{-1}$ is the limit distribution of some subsequence of $(W_n(t_1), W_n(t_2) - W_n(t_1 + r_n), \cdots, W(t_k) - W_n(t_{k-1} + r_n))$. Now $\alpha_n([nr_n] - 1) \to 0$ implies that $\pi_{t_1}, \pi_{t_2} - \pi_{t_1}, \cdots, \pi_{t_k} - \pi_{t_{k-1}}$ are independent under $Q$. Therefore $Q$ is the distribution of $(W(t_1), \cdots, W(t_k))$. This argument proves (3.2.20).

Finally, we have to prove the tightness of the sequence $\{W_n\}$. It suffices to show a version of (3.2.5), i.e., for any $\varepsilon > 0$, $\eta > 0$ there exist a $\delta$, $0 < \delta < 1$, and an integer $n_0$ such that

$$\sum_{k=0}^{[1/\delta]} P\Big\{\max_{[nk\delta]<r\le[n(k+1)\delta]} |S_r - S_{[nk\delta]}| > \varepsilon\sigma\sqrt{n}\Big\} < \eta, \quad n \ge n_0. \tag{3.2.21}$$

Let $k \in \{0, \cdots, [1/\delta]\}$ be fixed, $m = m(n) = [([n(k+1)\delta] - [nk\delta])/(p+q)]$,

$$\xi_j = \sum_{i=[nk\delta]+j(p+q)+1}^{[nk\delta]+j(p+q)+p} X_i,\ \eta_j = \sum_{i=[nk\delta]+j(p+q)+p+1}^{[nk\delta]+(j+1)(p+q)} X_i, \quad j = 0, 1, \cdots, m-1.$$

Then we have

$$\begin{aligned}&P\Big\{\max_{[nk\delta]<r\le[n(k+1)\delta]} |S_r - S_{[nk\delta]}| > \varepsilon\sigma\sqrt{n}\Big\}\\ &\le (m+1) \max_{0\le l\le n-(p+q)} P\Big\{\max_{1\le r\le p+q} |S_{l+r} - S_l| > \varepsilon\sigma\sqrt{n}/3\Big\}\\ &\quad + P\Big\{\max_{0\le r\le m-1} \Big|\sum_{j=0}^{r} \xi_j\Big| > \varepsilon\sigma\sqrt{n}/3\Big\}\\ &\quad + P\Big\{\max_{0\le r\le m-1} \Big|\sum_{j=0}^{r} \eta_j\Big| > \varepsilon\sigma\sqrt{n}/3\Big\}\\ &=: I_1 + I_2 + I_3.\end{aligned} \tag{3.2.22}$$

From (3.2.19), it follows that as $n \to \infty$

$$I_1 \le cnp^{-1} \max_{0\le l\le n-(p+q)} P\Big\{\max_{1\le r\le p+q} |S_{l+r} - S_l| > \varepsilon\sigma\sqrt{n}/3\Big\} \to 0.$$

By Lemma 1.2.3, (3.2.15) and (3.2.12), we obtain

$$\begin{aligned}&\max_{J\subset\{0,\cdots,m-1\}} E\Big(\sum_{j\in J}\eta_j\Big)^2/(\sigma^2 n)\\ &\le \sum_{0\le j\le m-1} E\eta_j^2/(\sigma^2 n)+2\sum_{1\le i\le n}\sum_{i+p\le j\le n}|EX_iX_j|/(\sigma^2 n)\\ &\le c\,mq/(\sigma^2 n)+20\sigma^{-2}\sum_{j>p} f_g(\alpha(j))\sup_{j\le n}\|X_j\|_g^2\to 0.\end{aligned}$$

Similarly

$$\begin{aligned}&\max_{J\subset\{0,\cdots,m-1\}} E\Big(\sum_{j\in J}\xi_j\Big)^2/(\sigma^2 n)\\ &\le \sum_{0\le j\le m-1} E\xi_j^2/(\sigma^2 n)+2\sum_{1\le i\le n}\sum_{i+q\le j\le n}|EX_iX_j|/(\sigma^2 n)\\ &\le c\,mp/(\sigma^2 n)+20\sigma^{-2}\sum_{j>q} f_g(\alpha(j))\sup_{j\le n}\|X_j\|_g^2.\end{aligned}$$

Using the definition of $m$ and (3.2.12), we obtain that the last expression converges to $c\sigma^{-2}\delta$ as $n\to\infty$. Choose $\delta_0(\varepsilon)>0$ such that

$$(18/\varepsilon)^2 c\sigma^{-2}\delta_0(\varepsilon)<\frac{1}{2}.$$

From now on we shall assume $\delta<\delta_0(\varepsilon)$. By the Chebyshev inequality we have

$$\min_{0\le r\le m-2} P\Big\{\Big|\sum_{j=r+1}^{m-1}\xi_j\Big|\le\varepsilon\sigma\sqrt{n}/6\Big\}\ge\frac{1}{2}$$

for large $n$. Now we apply Lemma 3.2.1 and obtain

$$I_2\le 2P\Big\{\Big|\sum_{j=0}^{m-1}\xi_j\Big|>\varepsilon\sigma\sqrt{n}/6\Big\}+2m\alpha_n(q+1).$$

(3.2.16) implies $m\alpha_n(q+1)\to 0$ as $n\to\infty$. Since

$$E\Big(\sum_{j=0}^{m-1}\eta_j\Big)^2/(\sigma^2 n)\to 0,$$

$$E\Big(S_{[n(k+1)\delta]}-S_{[nk\delta]}-\sum_{j=0}^{m-1}(\xi_j+\eta_j)\Big)^2/(\sigma^2 n)\to 0\ \text{ as } n\to\infty,$$

we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} P\Big\{\max_{0\le r\le m-1}\Big|\sum_{j=0}^{r}\xi_j\Big| > \varepsilon\sigma\sqrt{n}/3\Big\} \\
&\le 2\limsup_{n\to\infty} P\Big\{\Big|S_{[n(k+1)\delta]} - S_{[nk\delta]}\Big| > \varepsilon\sigma\sqrt{n}/7\Big\}.
\end{aligned}$$

Applying the Chebyshev inequality and Lamma 3.2.1 again yields

$$\begin{aligned}
&\limsup_{n\to\infty} P\Big\{\max_{0\le r\le m-1}\Big|\sum_{j=0}^{r}\eta_j\Big| > \varepsilon\sigma\sqrt{n}/3\Big\} \\
&\le 2\limsup_{n\to\infty} P\Big\{\Big|\sum_{j=0}^{m-1}\eta_j\Big| > \varepsilon\sigma\sqrt{n}/6\Big\} + 2\limsup_{n\to\infty} m\alpha_n(p+1) = 0.
\end{aligned}$$

Summing up these results for $k = 0, \cdots, [1/\delta]$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\sum_{k=0}^{[1/\delta]} P\Big\{\max_{[nk\delta]<r\le[n(k+1)\delta]}\Big|S_r - S_{[nk\delta]}\Big| > \varepsilon\sigma\sqrt{n}\Big\} \\
&\le 2\sum_{k=0}^{[1/\delta]}\limsup_{n\to\infty} P\Big\{\Big|S_{[n(k+1)\delta]} - S_{[nk\delta]}\Big| > \varepsilon\sigma\sqrt{n}/7\Big\} \\
&\le 2(\frac{1}{\delta}+1)P\{|N(0,\delta)| > \varepsilon/7\} \to 0 \quad \text{as}\,\delta\to 0.
\end{aligned}$$

Thus Lemma 3.2.2 is proved.

**Proof of Theorem 3.2.3.**

Assume that some $g \in G$ satisfies condition (3.2.12). Then $K := \sup_{n\ge1}\|X_n\|_g < \infty$.

We note that the conditions (3.2.18) and (3.2.19) of Lemma 3.2.2 are both implied by

$$p^{-1}\max_{0\le m\le n-p} E\Big(\Big(\sum_{i=m+1}^{m+p}|X_i|\Big)^2 I\Big(\sum_{i=m+1}^{m+p}|X_i| > \varepsilon\sqrt{n}\Big)\Big) \to 0, \quad \text{as} \quad n\to\infty. \tag{3.2.23}$$

From the monotonicity of $g(x)/x^2$, the convexity of $g$ and $Eg(|X|/\|X\|_g) \le 1$ if $0 < \|X\|_g < \infty$, we have

$$\begin{aligned}
&E\Big(\Big(\sum_{i=m+1}^{m+p}|X_i|\Big)^2 I\Big(\sum_{i=m+1}^{m+p}|X_i| > \varepsilon\sqrt{n}\Big)\Big) \\
&\le Eg\Big(\sum_{i=m+1}^{m+p}|X_i|/Kp\Big)\varepsilon^2 n/g(\varepsilon\sqrt{n}/Kp) \\
&\le \varepsilon^2 n/g(\varepsilon\sqrt{n}/Kp).
\end{aligned}$$

Thus (3.2.23) is implied by

$$p^{-1}n/g(\varepsilon\sqrt{n}/Kp) \to 0 \quad \text{as}\, n \to \infty. \tag{3.2.24}$$

From Lemma 1.2.3 we have for $m \geq 0$, $n \geq 1$

$$E(S_{m+n} - S_m)^2 \leq n \sup_{j\geq 1} EX_j^2 + 2nK^2 \sum_{k=1}^{\infty} f_g(\alpha(k)).$$

Since $\sup_{j\geq 1} EX_j^2 < \infty$, we obtain (3.2.15). The proof of Theorem 3.2.3 will be completed by constructing sequences $p(n)$, $q(n)$ such that (3.2.16), (3.2.17) and (3.2.24) are fulfilled. Since $k \to f_g(\alpha(k))$ is non-increasing, the assumption $\sum_{k=1}^{\infty} f_g(\alpha(k)) < \infty$ implies $f_g(\alpha(k))k \to 0$ as $k \to \infty$. Therefore we can choose $a: [0, \infty) \to [0, \infty)$ continuous and strictly decreasing such that

$$f_g(a(x))x \to 0 \quad \text{as}\, x \to \infty, \tag{3.2.25}$$

$$a(x) \geq \alpha(k) \quad \text{for all integers } k \geq x. \tag{3.2.26}$$

Since $x \to x \operatorname{inv} g(1/a(x))$ homeomorphically maps $(0, \infty)$ on $(0, \infty)$, we can define $x(n) \in (0, \infty)$ by

$$x(n) \operatorname{inv} g(1/a(x(n))) = n^{1/4}.$$

Then $x(n) \to \infty$ as $n \to \infty$. (3.2.25) implies that there exists an $L = L(n) \geq n^{-1/4}$ with $L(n) \to 0$ as $n \to \infty$ such that

$$L(n)^{-2} \sup_{t\geq x(n)} (\operatorname{inv} g(1/a(t)))^2 a(t)t \to 0 \quad \text{as}\, n \to \infty. \tag{3.2.27}$$

Define $y = y(n) \in (0, \infty)$ by

$$y(n) \operatorname{inv} g(1/a(y(n))) = L(n)n^{1/2}.$$

Clearly $y(n) \geq x(n)$ and $y(n) = o(n^{1/2})$. Put $q(n) = \min\{j,\, j \geq y(n)\}$ and choose a sequence $p(n)$ such that as $n \to \infty$

$$p(n)/y(n) \to \infty, \quad p(n)/n \to 0, \qquad L(n)p(n)/y(n) \to 0. \tag{3.2.28}$$

Now $p = p(n) = o(n)$, $q = q(n) = o(p)$ hold true. Since $q(n) \to \infty$, (3.2.17) can be obtained from the assumption of the theorem by an application of Lemma 1.2.3. Using (3.2.26), the definition of $y(n)$ and (3.2.27) we obtain

$$\begin{aligned} nq^{-1}\alpha(q) &\leq ny^{-1}\alpha(q) \leq ny^{-1}a(y) \\ &= L^{-2}(\operatorname{inv} g(1/a(y)))^2 a(y)y \to 0 \quad \text{as}\, n \to \infty, \end{aligned} \tag{3.2.29}$$

which and $q = o(p)$ imply $np^{-1}\alpha(q) = o(1)$. For all large $n$ we have $p \geq y$ and $y\varepsilon/KpL \geq 1$ by (3.2.28). For such $n$ we obtain by the convexity of $g$ and the definition of $y(n)$ :

$$\begin{aligned} p^{-1}n/g(\varepsilon\sqrt{n}/Kp) &\leq y^{-1}npLK/(\varepsilon y g(L\sqrt{n}/y)) \\ &= y^{-1}na(y)pLK/(\varepsilon y). \end{aligned}$$

This expression tends to 0 by (3.2.28) and (3.2.29). Hence (3.2.24) holds. The proof of Theorem 3.2.3 is completed.

The following corollaries are immediate. Let $g(x) = x^{2+\delta}$ for some $\delta > 0$ in Theorem 3.2.3.

**Corollary 3.2.1.** *Let $\{X_n,\, n \geq 1\}$ be an $\alpha$-mixing sequence with*

$$EX_n = 0,$$

$$\sup_n E|X_n|^{2+\delta} < \infty, \ \sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty \quad \text{for some} \quad \delta > 0.$$

*Suppose that condition (3.2.11) is satisfied. Then*

$$W_n \Longrightarrow W.$$

**Corollary 3.2.2.** *Let $\{X_n,\, n \geq 1\}$ be an $\alpha$-mixing sequence with $EX_n = 0$. If*

$$\sup_n EX_n^2 |\log|X_n||^a < \infty, \quad \sum_{n=1}^{\infty} |\log \alpha(n)|^{-a} < \infty \tag{3.2.30}$$

*for some $a > 0$, particularly*

$$\sup_n EX_n^2 |\log|X_n||^a < \infty, \ \alpha(n) = O(b^{-n}) \tag{3.2.31}$$

*for some $a > 1$ and $b > 1$, then*

$$W_n \Longrightarrow W.$$

Herrndorf (1985) has given two examples, which show that $a > 1$ cannot be replaced by $a = 1$ in (3.2.31). The constructions of the examples are omitted here.

**Remark 3.2.3.** Doukhan, Massart and Rio (1994) discussed the functional CLT for $\alpha$-mixing sequence, via the $\alpha$-mixing function $\alpha(t)$ and the tail distribution function of $|X_1|$, they gave another sufficient condition as follows:

Let $\{X_n, n \geq 1\}$ be a strictly stationary $\alpha$-mixing sequence of centered random variables satisfying

$$\int_0^1 \text{inv}\alpha(u)[\text{inv}G(u)]^2 du < \infty \tag{3.2.32}$$

where $G(u) = P(|X_1| > u)$. Then the series $\sigma^2 = \sum_{n=1}^{\infty} \text{Cov}(X_1, X_n)$ is absolutely convergent, and

$$Z_n/\sigma \Longrightarrow W,$$

where $Z_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k$.

Particularly, if $|X_1| \leq c < \infty$, condition (3.2.32) is equivalent to (3.2.4). So, in this case, Theorem 3.2.2 is recovered.

## 3.3 The CLT and WIP when the variance is infinite

The CLT for a mixing sequence with possibly infinite variance was first discussed by Lin (1981, 1982). He proved the following theorem under the conditions comparable with that in the case of independent sums. Assume also that $\{X_n,\ n \geq 1\}$ is a strictly stationary $\alpha$-mixing sequence.

**Theorem 3.3.1.** *Suppose that $EX_1 = 0$ and the following conditions are satisfied:*

*(i) There exist two sequences of positive integers $p = p(n)$ and $q = q(n)$ satisfying*

$$p = o(n),\ q = o(p),\ k\alpha(q) \to 0 \quad \text{as } n \to \infty, \tag{3.3.1}$$

*where $k = k(n) = [n/(p+q)]$.*

*(ii) There exists a sequence of constants $\{C_n\}$ with $C_n \uparrow \infty$ and*

$$k \int_{|X_1|>C_k} dP = o\Big(\frac{1}{p}\Big), \tag{3.3.2}$$

$$C_k^2 \Big/ \Big(k \int_{|X_1|<C_k} X_1^2\, dP\Big) = o\Big(\frac{1}{p}\Big), \tag{3.3.3}$$

$$\frac{\int_{|X_1|<C_k,|X_i|<C_k} X_1X_i\,dP}{\int_{|X_1|<C_k} X_1^2\,dP} = o\Big(\frac{1}{p}\Big) \quad \text{as } n\to\infty \tag{3.3.4}$$

*uniformly in i. Then, $\{X_n\}$ obeys the CLT.*

*If moreover there exists a constant $a>0$ such that $p\le k^a$ and*

$$\lim_{n\to\infty} k^{2+a}\alpha(q) = 0, \tag{3.3.5}$$

*then there are constants $B_n>0$ such that $\{W_n(t) = S_{[nt]}/B_n,\ 0\le t\le 1\}_{n\ge 1}$ weakly converges to W.*

**Proof.** Define $\xi_j$, $j=0,1,\cdot\cdot,k-1$, and $\eta_j$, $j=0,1,\cdots,k$ as in Lemma 3.2.2. From (3.3.2) and (3.3.3) we obtain

$$C_k^2\int_{|X_1|>C_k} dP\Big/\int_{|X_1|<C_k} X_1^2\,dP = o(p^{-2}). \tag{3.3.6}$$

Hence

$$\begin{aligned}0 &\le \int_{|\xi_0|<pC_k}\xi_0^2\,dP - \int_{\cap_{j=1}^p(X_j|<C_k)}\xi_0^2\,dP\\ &= \int_{(|\xi_0|<pC_k)\cap(\cup_{j=1}^p(|X_j|\ge C_k))}\xi_0^2\,dP\\ &\le p^3C_k^2\int_{|X_1|\ge C_k} dP = o\Big(p\int_{|X_1|<C_k}X_1^2\,dP\Big),\end{aligned}$$

i.e.,

$$\begin{aligned}\int_{|\xi_0|<pC_k}\xi_0^2\,dP = \int_{\cap_{j=1}^p(|X_j|<C_k)}\Big(\sum_{j=1}^p X_j^2 + 2\sum_{1\le i<j\le p}X_iX_j\Big)\,dP\\ + o\Big(p\int_{|X_1|<C_k}X_1^2\,dP\Big).\end{aligned} \tag{3.3.7}$$

Furthermore, from (3.3.6) again

$$\begin{aligned}0 &\le \int_{|X_j|<C_k} X_j^2\,dP - \int_{\cap_{i=1}^p(|X_i|<C_k)} X_j^2\,dP\\ &= \int_{(|X_j|<C_k)\cap\cup_{j\ne i<p}(|X_i|\ge C_k)} X_j^2\,dP\\ &\le pC_k^2\int_{|X_1|\ge C_k} dP = o\Big(p^{-1}\int_{|X_1|<C_k}X_1^2\,dP\Big).\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{j=1}^p\int_{\cap_{i=1}^p(|X_i|<C_k)} X_j^2\,dP\\ &\qquad = p\int_{|X_1|<C_k}X_1^2\,dP + o\Big(\int_{|X_1|<C_k}X_1^2\,dP\Big).\end{aligned} \tag{3.3.8}$$

Similarly

$$\int_{\cap_{j=1}^{p}(|X_j|<C_k)} X_i X_j \, dP$$
$$= \int_{|X_i|<C_k,\,|X_j|<C_k} X_i X_j \, dP + o\Big(p^{-1} \int_{|X_1|<C_k} X_1^2 \, dP\Big).$$

Using condition (3.3.4), we obtain

$$\sum_{1\le i<j\le p} \int_{\cap_{j=1}^{p}(|X_j|<C_k)} X_i X_j \, dP = o\Big(p \int_{|X_1|<C_k} X_1^2 \, dP\Big). \tag{3.3.9}$$

Now (3.3.7), (3.3.8) and (3.3.9) imply

$$\int_{|\xi_0|<pC_k} \xi_0^2 \, dP = (1+o(1))p \int_{|X_1|<C_k} X_1^2 \, dP. \tag{3.3.10}$$

Moreover,

$$\int_{|\xi_0|>pC_k} dP \le \int_{\cup_{j=1}^{p}(|X_j|>C_k)} dP \le p \int_{|X_1|>C_k} dP. \tag{3.3.11}$$

Then using conditions (3.3.3) and (3.3.2) we obtain

$$\frac{k}{(pC_k)^2} \int_{|\xi_0|<pC_k} \xi_0^2 \, dP \to \infty, \tag{3.3.12}$$

$$k \int_{|\xi_0|>pC_k} dP \to 0. \tag{3.3.13}$$

According to (3.3.10), $\int_{|\xi_0|<pC_k} \xi_0^2 \, dP \to \infty$ as $n \to \infty$. Imitating the proof of Theorem 35.1 in Gnedenko and Kolmogorov (1954), we have

$$\Big(\int_{|\xi_0|<pC_k} \xi_0 \, dP\Big)^2 = o\Big(\int_{|\xi_0|<pC_k} \xi_0^2 \, dP\Big),$$

i.e., (3.3.12) is equivalent to

$$\frac{k}{(pC_k)^2} \Big\{ \int_{|\xi_0|<pC_k} \xi_0^2 \, dP - \Big(\int_{|\xi_0|<pC_k} \xi_0 \, dP\Big)^2 \Big\} \to \infty. \tag{3.3.14}$$

Let $\xi_j'$, $j = 0, 1, \cdots, k-1$, be independent random variables with the same distribution as $\xi_0$. By checking the proof of Theorem 26.4 in Gnedenko and Kolmogorov (1954), under conditions (3.3.13) and (3.3.14), there exists a sequence of positive constants $B_k^{(p)}$ such that

$$\frac{1}{B_k^{(p)}} \sum_{j=0}^{k-1} \xi_j' \xrightarrow{d} N(0,\,1) \quad \text{as } n \to \infty. \tag{3.3.15}$$

In fact, we can take

$$
\begin{aligned}
B_k^{(p)^2} &= k\Big\{\int_{|\xi_0|<pC_k} \xi_0^2\, dP - \Big(\int_{|\xi_0|<pC_k} \xi_0\, dP\Big)^2\Big\} \\
&= (1+o(1))k\int_{|\xi_0|<pC_k} \xi_0^2\, dP \\
&= (1+o(1))kp\int_{|X_1|<C_k} X_1^2\, dP
\end{aligned}
\tag{3.3.16}
$$

by (3.3.10). Using Lemma 1.2.1 and (3.3.1) we have

$$
\Big|E\exp\Big(it\sum_{j=0}^{k-1}\xi_j/B_k^{(p)}\Big) - \prod_{j=1}^{k-1} E\exp(it{\xi'}_j/B_k^{(p)})\Big| \le 4k\alpha(q) \to 0. \tag{3.3.17}
$$

Combining it with (3.3.15) yields

$$
\frac{1}{B_k^{(p)}}\sum_{j=0}^{k-1}\xi_j \xrightarrow{d} N(0,\,1) \quad \text{as}\, n\to\infty. \tag{3.3.18}
$$

Repeat the above discussion for $\eta_j$, $j=0,1,\cdots,k-1$. Then there exists a $B_k^{(q)}>0$ such that

$$
\frac{1}{B_k^{(q)}}\sum_{j=0}^{k-1}\eta_j \xrightarrow{d} N(0,\,1) \quad \text{as}\, n\to\infty. \tag{3.3.19}
$$

Similarly to (3.3.16), $B_k^{(q)^2} = (1+o(1))kq\int_{|X_1|<C_k} X_1^2 dP$. Therefore

$$
\Big(B_k^{(q)}/B_k^{(p)}\Big)^2 = (1+o(1))q/p = o(1). \tag{3.3.20}
$$

Thus

$$
\frac{1}{B_k^{(p)}}\sum_{j=0}^{k-1}\eta_j \xrightarrow{P} 0 \quad \text{as } n\to\infty. \tag{3.3.21}
$$

Moreover, noting $n-k(p+q)\le p+q$ and using (3.3.16) and (3.3.3) we have for any $\varepsilon>0$

$$
\begin{aligned}
P\{|\eta_k/B_k^{(p)}|>\varepsilon\} &\le (p+q)E|X_1|/\varepsilon B_k^{(p)} \\
&\le 2pE|X_1|\Big/\varepsilon\Big(kp\int_{|X_1|<C_k} X_1^2\, dP\Big)^{1/2} = o(1/C_k)\to 0.
\end{aligned}
\tag{3.3.22}
$$

Let $B_n = B_k^{(p)}$. Then (3.3.18), (3.3.21) and (3.3.22) imply

$$
S_n/B_n \xrightarrow{d} N(0,\,1) \quad \text{as}\, n\to\infty.
$$

Now we turn to the weak invariance principle. Put $k_a = [k^{2+a}]$. For any positive integer $N$ there exists an $n$ such that $k_a(p+q) \leq N < (k_a + 1)(p+q)$. Define $\xi_j$ and $\eta_j$ as above, however their indices are extended to $j = 0, 1, \cdots, k_a - 1$. Recalling the proof of (3.3.18), under the condition $k_a\alpha(q) \to 0$, there are constants $B_m^{(p)} > 0$ $(m = k,\ k+1, \cdots, k_a)$ such that

$$\frac{1}{B_m^{(p)}} \sum_{j=0}^{m-1} \xi_j \xrightarrow{d} N(0,\ 1) \quad \text{as}\, n \to \infty, \tag{3.3.23}$$

where $B_m^{(p)}$ are the normed constants related to the independent random variables with the same distribution as $\xi_0$ (for fixed $p$ ). If we denote the sequence of constants corresponding to $\{\xi_j,\ j = 0, 1, \cdots, m\}$ in Theorem 26.4 of Gnedenko and Kolmogorov (1954) by $\{C_m'\}$, then, similarly to (3.3.16),

$$B_m^{(p)^2} = (1 + o(1))mp \int_{|X_1| < C_m'} X_1^2\, dP. \tag{3.3.24}$$

And similarly to (3.3.19),

$$B_{k_a}^{(q)-1} \sum_{j=0}^{k_a - 1} \eta_j \xrightarrow{d} N(0,\ 1) \quad \text{as}\, n \to \infty. \tag{3.3.25}$$

Define $B_N = B_{k_a}^{(p)}$, put

$$S_{[Nt]} = \sum_{j=0}^{[(k_a-1)t]} \xi_j + \sum_{j=0}^{[(k_a-1)t]} \eta_j + \eta_{N,t},$$

where $\eta_{N,t} = X_{([(k_a-1)t]+1)(p+q)+1} + \cdots + X_{[Nt]}$, with the number of terms

$[Nt] - ([(k_a-1)t]+1)(p+q) \leq (k_a+1)(p+q)t - (k_a-1)(p+q)t \leq 2(p+q)$.

Define

$$W_N'(t) = B_N^{-1} \sum_{j=0}^{[(k_a-1)t]} \xi_j, \quad W_N''(t) = B_N^{-1}\Big( \sum_{j=0}^{[(k_a-1)t]} \eta_j + \eta_{Nt}\Big).$$

Recalling (3.3.17), we know that the normed sums of $\xi_j(\eta_j)$ and the normed sums of $\xi_j{}'(\eta_j{}')$, where $\xi_j{}', j = 1, \cdots, k_a - 1, (\eta_j{}', j = 1, \cdots, k_a - 1)$ are independent, with the common normed constants, have the same convergence.

Hence by Theorem 26.2 in Gnedenko and Kolmogorov (1954), (3.3.23) implies that for any $\varepsilon > 0,\ 0 \leq t \leq 1$,

$$[mt] \int_{|x| \geq \varepsilon} dF_m^{(p)}(x) \to 0,$$

$$[mt]\Big\{\int_{|x|<\varepsilon} x^2\,dF_m^{(p)}(x) - \Big(\int_{|x|<\varepsilon} x\,dF_m^{(p)}(x)\Big)^2\Big\} \to t,$$

where $F_m^{(p)}(x)$ is the distribution of $\xi_0/B_m^{(p)}$, i.e.

$$[mt]\int_{|y|\geq\varepsilon/\sqrt{t}} dF_m^{(p)}(\sqrt{t}y) \to 0,$$

$$[mt]\Big\{\int_{|y|<\varepsilon/\sqrt{t}} y^2\,dF_m^{(p)}(\sqrt{t}y) - \Big(\int_{|y|<\varepsilon/\sqrt{t}} y\,dF_m^{(p)}(\sqrt{t}y)\Big)^2\Big\} \to 1.$$

Then by the same theorem cited just now we obtain

$$W_N'(t) \xrightarrow{d} W(t). \tag{3.3.26}$$

Similarly, we can also show

$$W_N'(t) - W_N'(s) \xrightarrow{d} W(t) - W(s) \quad \text{for } 0 \leq s < t \leq 1. \tag{3.3.27}$$

An analogue of (3.3.25) for $\{\eta_j\}$ is

$$\frac{1}{B_{k_a}^{(q)}} \sum_{j=0}^{[(k_a-1)t]} \eta_j \xrightarrow{d} W(t). \tag{3.3.28}$$

With the help of the rusult similar to (3.3.20), (3.3.27) implies

$$\frac{1}{B_N} \sum_{j=0}^{[(k_a-1)t]} \eta_j \xrightarrow{P} 0$$

uniformly in $t$. Imitating (3.3.22) we also have

$$\eta_{N,t}/B_N \xrightarrow{P} 0$$

uniformly in $t$. Consequently

$$W'' \xrightarrow{P} 0.$$

Thus we investigate $W_N'$ instead of $W_N$.

At first we consider convergence of finit-dimensional distributions of $W_N'$. The convergence of one-dimensional distribution has been given by (3.3.26). For the two-dimensional case we need to prove

$$(W_N'(s),\, W_N'(t) - W_N'(s)) \xrightarrow{d} (W(s),\, W(t) - W(s)), \quad 0 \leq s < t \leq 1,$$

in other words, to show that for any Borel sets $A_1$ and $A_2$

$$P\{W'_N(s) \in A_1, W'_N(t) - W'_N(s) \in A_2\} \\ \to P\{W(s) \in A_1\}P\{W(t) - W(s) \in A_2\}. \quad (3.3.29)$$

According to the $\alpha$-mixing property, we have

$$\Big|P\{W'_N(s) \in A_1, W'_N(t) - W'_N(s) \in A_2\} \\ - P\{W'_N(s) \in A_1\}P\{W'_N(t) - W'_N(s) \in A_2\}\Big| \\ \leq \alpha(q).$$

Combining it with (3.3.26) and (3.3.27) yields (3.3.32). Three or more dimension case can be treated in the same way, and hence the finite-dimensional distributions converge properly.

Finally, we prove tightness of $W'_N$. By (3.2.5), it suffices to show that for any $\varepsilon$, $\eta > 0$ there exist a $\delta$, $0 < \delta < 1$ and an integer $n_0$ such that for $0 \leq t \leq 1$

$$P\{\sup_{t\leq s\leq t+\delta} |W'_N(s) - W'_N(t)| \geq \varepsilon\} \leq \delta\eta, \quad n \geq n_0.$$

Equivalently,

$$P\Big\{\max_{0\leq j\leq 2k_a\delta}\Big|\sum_{i=0}^{j} \xi_j\Big| \geq \varepsilon B_N\Big\} \leq \delta\eta, \quad n \geq n_0. \quad (3.3.30)$$

By the CLT, we have

$$P\Big\{\Big|\sum_{i=0}^{[2k_a\delta]} \xi_i\Big| \geq \varepsilon B_N/2\Big\} \to P\{|W(2\delta)| \geq \varepsilon/2\} \\ = P\{|W(1)| \geq \varepsilon/(2\sqrt{2\delta})\} \leq \frac{16\delta}{\varepsilon^3}\sqrt{2\delta}E|W(1)|^3 \leq \frac{\delta\eta}{4}$$

provided that $\delta$ is small enough. Hence, there exists an $n_0$ such that

$$P\Big\{\Big|\sum_{i=0}^{[2k_a\delta]} \xi_i\Big| \geq \varepsilon B_N/2\Big\} \leq \frac{\delta\eta}{2}, \quad n \geq n_0.$$

Therefore, if we prove that for large $N$ and every $j$, $0 \leq j \leq 2k_a\delta - 1$,

$$P\Big\{\Big|\sum_{i=0}^{j} \xi_i\Big| \leq \varepsilon B_N/2\Big\} \geq \frac{1}{2}, \quad (3.3.31)$$

it follows from Lemma 2.2.1 that we have

$$P\Big\{\max_{0\le j\le 2k_a\delta}\Big|\sum_{i=0}^{j}\xi_i\Big|\ge \varepsilon B_N\Big\}\le 2P\Big\{\Big|\sum_{i=0}^{[2k_a\delta]}\xi_i\Big|\ge \varepsilon B_N/2\Big\}\le \delta\eta.$$

Now it remains to show (3.3.31). First we consider the case of $j\le k$. Using (3.3.24) and noting $p\le k^a$, we have

$$P\Big\{\Big|\sum_{i=0}^{j}\xi_i\Big|\ge \varepsilon B_N/2\Big\}\le 2jpE|X_1|/\varepsilon B_N$$

$$\le 3kpE|X_1|/\varepsilon\Big(k_a p\int_{|X_1|<C'_{k_a}}X_1^2\,dP\Big)^{1/2}\to 0.$$

Assume $k<j\le 2k_a\delta$. Recall (3.3.23). By Theorem II1, we can write

$$B_j^{(p)^2}=jh^{(p)}(j).$$

With the help of the property of a slowly varying function,

$$\limsup_{N\to\infty}\frac{B_j^{(p)^2}}{B_N^2}=\limsup_{N\to\infty}\frac{jh^{(p)}(j)}{k_a h^{(p)}(k_a)}\le 2\delta.$$

Therefore, there exists an $n_1$ such that $B_N^2/B_j^{(p)^2}\ge 1/3\delta$ for $n\ge n_1$. Let $\delta$ be small enough so that

$$P\{|W(1)|\ge \varepsilon/2\sqrt{3\delta}\}<1/4.$$

Then

$$P\Big\{\Big|\sum_{i=0}^{j}\xi_i\Big|\ge \varepsilon B_N/2\Big\}\le P\Big\{\Big|\sum_{i=0}^{j}\xi_i\Big|\ge \varepsilon B_{j+1}^{(p)}/2\sqrt{3\delta}\Big\}$$

$$\to P\{|W(1)|\ge \varepsilon/2\sqrt{3\delta}\}<1/4,$$

which implies (3.3.31). The proof of Theorem 3.3.1 is completed.

# Chapter 4 Weak Convergence for $\rho$-mixing Sequences

Ibragimov (1975) first showed that the CLT and the WIP hold true under some conditions for a strictly stationary $\rho$-mixing sequence of random variables, i.e.,

**Theorem 4.0.1.** *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence of random variables with $EX_1 = 0, EX_1^2 < \infty$ and*
*(i)* $\sigma_n^2 = ES_n^2 \to \infty$,
*(ii)* $\sum_{n=1}^{\infty} \rho(2^n) < \infty$.
*Then the distridution of $S_n/\sigma_n$ converges to $\Phi(x)$.*

**Theorem 4.0.2.** *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence of random variables with $EX_1 = 0$,*

$$E|X_1|^{2+\delta} < \infty \quad \textit{for some} \quad \delta > 0$$

*and condition (i) is satisfied. Then with $W_n(t) = S_{[nt]}/\sigma_n$, $0 \leq t \leq 1$,*

$$W_n \Longrightarrow W.$$

Peligrad (1982) gave a functional form under 2-th moment condition, however, the mixing rate is more restrictive, namely $\sum \rho^{1/2}(2^n) < \infty$. Peligrad (1986) suggested five open problems, one of which is to prove the weak invariance principle under the same sufficient conditions for the CLT. Shao (1988b) gave a positive answer, which will be introduced in Section 4.1.

When the moments of order $s > 2$ is finite, Bradley (1984) raised the problem of finding the slowest permissible $\rho$-mixing rates to assure the CLT under more flexible moment assumptions. Suppose that $\{X_n, n \geq 1\}$

is a strictly stationary $\rho$-mixing sequence of random variables with $EX_1 = 0, \sigma_n^2 = ES_n^2 \to \infty$ and $EX_1^2 g(X_1) < \infty$, where $g : [0, \infty) \to [0, \infty)$ satisfies:

both $g(x)$ and $x^\delta / g(x)$ are non-decreasing functions for some $\delta > 0$.

Bradley asked whether the CLT holds true when

$$\exp(d \sum_{i=1}^{[logn]} \rho(2^i)) = O(g(n^{1/2}))$$

for any $d > 0$. Peligrad (1987) proved a more meticulous result.

**Theorem 4.0.3.** *Let $\{X_n, n \geq 1\}$ and $g(x)$ be as above. If*

$$\exp\Big((2 + \varepsilon^*) \sum_{i=1}^{[logn]} \rho(2^i)\Big) = O(g(n^{1/2}))$$

*for some $o < \varepsilon^* < 1$, then $S_n/\sigma_n \to N(0, 1)$ in distribution as $n \to \infty$.*

Peligrad (1987) conjectured that $W_n \Longrightarrow W$ under the same conditions as in Theorem 4.0.3. Shao (1989b) gave a positive answer, which will be discussed in Section 4.2. In Shao (1989a), he gave a more general result for the non-stationary sequence, which will be discussed in Section 4.3. Some invariance principles for stationary $\rho$-mixing sequences with infinite variance will also be introduced in Section 4.4.

In the study of invariance principles for dependent random variables, the following basic result due to Billingsley (1968) is used frequently. Let

$$\mathcal{B} = \sigma\{(-\infty, x), -\infty < x < \infty\}$$

be the Borel $\sigma$-field of $R$.

**Theorem 4.0.4.** *Let $\{W_n, n \geq 1\}$ be a sequence of random elements in $D[0, 1]$ satisfying the following conditions:*

*(i) $\{W_n(t), 0 \leq t \leq 1\}$ has asymptotically independent increments, i.e., for any given $B_i \in \mathcal{B}, i = 1, \cdots, r$ and $0 \leq s_1 \leq t_1 < \cdots < s_r \leq t_r \leq 1$ we have*

$$\lim_{n\to\infty} \{P(W_n(t_i) - W_n(s_i) \in B_i, i = 1, \cdots, r) - \prod_{i-1}^{r} P(W_n(t_i) - W_n(s_i) \in B_i)\} = 0,$$

*(ii)* $\{W_n^2(t), n \geq 1\}$ *is uniformly integrable for each* $t$,

*(iii)* $EW_n(t) \to 0, EW_n^2(t) \to t$ *as* $n \to \infty$,

*(iv)* *for any* $\varepsilon, \eta > 0$ *there exist a* $\delta > 0$ *and a positive integer* $n_0$ *such that*

$$P\{w(W_n, \delta) \geq \varepsilon\} \leq \eta \quad n \geq n_0,$$

*where* $w(x, \delta) = \sup_{|s-t|<\delta} |x(s) - x(t)|$. *Then* $W_n \Longrightarrow W$.

## 4.1 The WIP when the moments of order 2 are finite

Shao (1988b) proved the following theorem.

**Theorem 4.1.1.** *Let* $\{X_n, n \geq 1\}$ *be a* $\rho$*-mixing sequence of random variables with* $EX_n = 0, EX_n^2 < \infty$ *and*

*(i)* $\lim_{n\to\infty} ES_n^2/n = \sigma^2 > 0$,

*(ii)* $\{X_n^2, n \geq 1\}$ *is uniformly integrable,*

*(iii)* $\sum_{n=1}^{\infty} \rho(2^n) < \infty$.

*Then*

$$W_n \Longrightarrow W,$$

*where*

$$W_n(t) = S_{[nt]}/\sigma\sqrt{n}, \qquad 0 \leq t \leq 1.$$

The proof of Theorem 4.1.1 will need the following lemmas.

**Lemma 4.1.1.**(Peligrad 1982) *Let* $\{X_n, n \geq 1\}$ *be an* $\alpha$*-mixing sequence of random variables with* $EX_n = 0, EX_n^2 < \infty$, *satisfying*

*(i)* *of Theorem 4.1.1 and*

*(ii)$'$* *for each* $t \in [0,1], \{W_n^2(t), n \geq 1\}$ *is uniformly integrable,*

*(iv)* *for any* $\varepsilon > 0$ *there exist a real number* $\lambda > 2$ *and a positive integer* $n_0$ *such that for every* $n \geq n_0$ *and all* $k \geq 1$

$$P\{\max_{1\leq i\leq n} |S_k(i)| \geq \lambda\sigma\sqrt{n}\} \leq \varepsilon/\lambda^2. \tag{4.1.1}$$

*Then* $W_n \Longrightarrow W$.

Lemma 4.1.1 is a corollary of Theorem 4.0.4.

**Lemma 4.1.2.**(Moricz 1982) *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables. Suppose that the non-negative function* $f(k, m)$ *satisfies*

$$f(k, m) + f(k + m, l) \leq f(k, m + l) \tag{4.1.2}$$

*for $k \geq 0, m \geq 1, l \geq 1$ and the function $g(t,s)$ is non-decreasing for each arguments. If*

$$E|S_k(m)|^r \leq f(k,m)g^r(f(k,m),m) \qquad r \geq 1, \tag{4.1.3}$$

*then*

$$E \max_{1\leq m\leq n} |S_k(m)|^r \leq \frac{5}{2} f(k,n)\Big\{ \sum_{i=0}^{[\log n]} g\Big(\frac{f(k,n)}{2^i}, \Big[\frac{n}{2^i}\Big]\Big)\Big\}^r. \tag{4.1.4}$$

The proof of Lemma 4.1.2 will not be presented here.

**Proof of Theorem 4.1.1.** We need only to check the conditions $(ii)'$ and $(iv)$ in Lemma 4.1.1.

1) We prove that $\{S_k^2(n)/n, n \geq 1, k \geq 0\}$ is uniformly integrable. Let $N > 0$ be specified later on. Denote

$$\begin{aligned}
X_i^N &= X_i I(|X_i| < N) - EX_i I(|X_i| < N),\\
\overline{X}_i^N &= X_i - X_i^N,\\
S_k^N(n) &= \sum_{i=k+1}^{k+n} X_i^N,\\
\overline{S}_k^N(n) &= \sum_{i=k+1}^{k+n} \overline{X}_i^N,\\
E_\alpha U &= \int_{U>\alpha} U dP.
\end{aligned}$$

It is obvious that

$$S_k(n) = S_k^N(n) + \overline{S}_k^N(n),$$

$$E_\alpha S_k^2(n)/n \leq 4E_{\alpha/4}(S_k^N(n))^2/n + 4E(\overline{S}_k^N(n))^2/n. \tag{4.1.5}$$

From Lemma 2.2.2 it follows that for every $n$

$$\sup_{k\geq 1} E(\overline{S}_k^N(n))^2/n \leq K \sup_{k\geq 1} E(\overline{X}_k^N)^2$$

for some $K > 0$. Since $\{X_n^2, n \geq 1\}$ is uniformly integrable, for any given $\varepsilon > 0$ we have $N$ such that $\sup_{k\geq 1} E(\overline{X}_k^N)^2 \leq \varepsilon/8K$. Thus for each $n \geq 1$ and $k \geq 1$

$$E(\overline{S}_k^N(n))^2/n \leq \varepsilon/8. \tag{4.1.6}$$

On the other hand, from Lemma 2.2.4, there exists a constant $K_1 = K_1(N,\delta,\rho) > 0$ such that for every $n$

$$\sup_{k\geq 1} E|S_k^N(n)|^{2+\delta}/n^{1+\delta/2} \leq K_1.$$

Then for large $\alpha$ we obtain

$$E_{\alpha/4}(S_k^N(n))^2/n \leq \frac{c}{\alpha^{\delta/2}} E_{\alpha/4}|S_k^N(n)|^{2+\delta}/n^{1+\delta/2} < \frac{\varepsilon}{8}. \tag{4.1.7}$$

Inserting (4.1.6), (4.1.7) into (4.1.5) we prove that $\{S_k^2(n)/n, n \geq 1, k \geq 1\}$ is uniformly integrable.

**2)** We show (4.1.1). Let

$$p = [\exp(2C \log^{1/3} n)], \qquad r = [n/p],$$

where the constant C is specified in Lemma 2.2.4 corresponding to $\delta = 1$. Denote

$$\begin{aligned}
&X_i^n = X_i I(|X_i| < n^{1/2}/p) - EX_i I(|X_i| < n^{1/2}/p), \\
&\overline{X}_i^n = X_i - X_i^n, \\
&S_k^n(l) = \sum_{j=k+1}^{k+l} X_j^n, && \overline{S}_k^n(l) = \sum_{j=k+1}^{k+l} \overline{X}_j^n, \\
&Y_i = \sum_{j=k+1+(2i-1)r}^{k+2ir} \overline{X}_j^n, && i = 1, 2, \cdots, p_1 := [p/2], \\
&Z_i = \sum_{j=k+1+2ir}^{k+(2i+1)r} \overline{X}_j^n, && i = 0, 1, \cdots, p_2 := [(p-1)/2], \\
&T_l(i) = \sum_{j=l+1}^{l+i} Y_j, && \overline{T}_l(i) = \sum_{j=l}^{l+i} Z_j, \\
&T(i) = T_0(i), && \overline{T}(i) = \overline{T}_0(i).
\end{aligned}$$

It is obvious that $S_k(i) = S_k^n(i) + \overline{S}_k^n(i)$. Without loss of generality assume that $\sigma = 1$.

We first prove

$$P\Big\{\max_{1\leq i\leq n} |S_k^n(i)| \geq \lambda n^{1/2}\Big\} \leq \varepsilon/6\lambda^2 \tag{4.1.8}$$

for large $n$. Form Lemma 2.2.4 with $\delta = 1$ it follows that

$$\begin{aligned}
\max_{1\leq k\leq n} |ES_k^n(i)|^3 &\leq C\{n^{3/2}\sigma_0^3 + n\exp(C\log^{1/3} n)a_0\} \\
&\leq cn^{3/2},
\end{aligned}$$

where $\sigma_0^2 = \sup_n E(X_i^{(n)})^2$, $a_0 = \sup_n E|X_i^{(n)}|^3$. Using Lemma 4.1.2 we have

$$\sup_k E \max_{1\leq i\leq n} |S_k^n(i)|^3 = O(n^{3/2}).$$

That is to say, $\{\max_{1\le i\le n}|S_k^n(i)|^2/n, n\ge 1, k\ge 1\}$ is uniformly integrable. Therefore for any $\varepsilon>0$ there exist $\lambda\ge 2$ and large $n_0$ such that (4.1.8) is satisfied for $n\ge n_0$.

Next, we show

$$P\{\max_{1\le i\le n}|\overline{S}_k^n(i)|\ge 5\lambda n^{1/2}\}\le 5\varepsilon/6\lambda^2. \tag{4.1.9}$$

Note that the left hand side of (4.1.9) does not exceed

$$\begin{aligned}&P\Big\{\max_{1\le i\le p_1}|T(i)|\ge 2\lambda n^{1/2}\Big\}+P\Big\{\max_{1\le i\le p_2}|\overline{T}(i)|\ge 2\lambda n^{1/2}\Big\}\\&\quad+\sum_{j=0}^{p}P\Big\{\max_{1\le i\le r}|\overline{S}_{k+jr}^n(i)|\ge \lambda n^{1/2}\Big\}\\&=: I_1+I_2+I_3.\end{aligned} \tag{4.1.10}$$

Since $\{X_n^2, n\ge 1\}$ is uniformly integrable, without loss of generality we can assume that $\sup_{n\ge 1}\|X_n\|_2\le 1$. Thus we have

$$\begin{aligned}&P\Big\{\max_{1\le i\le r}|\overline{S}_{k+jr}^n(i)|\ge \lambda n^{1/2}\Big\}\\&\le P\Big\{\sum_{i=k+1+jr}^{k+(j+1)r}(|\overline{X}_i^n|-E|\overline{X}_i^n|)\ge \lambda n^{1/2}/2\Big\}\\&=O\Big(\frac{r}{n\lambda^2}\sup_m E(\overline{X}_m^n)^2\Big).\end{aligned}$$

Then by condition (ii) again

$$I_3=O\Big(\frac{pr}{\lambda^2 n}\sup_m E(\overline{X}_m^n)^2\Big)\le\frac{\varepsilon}{6\lambda^2}. \tag{4.1.11}$$

Now we estimate $I_1$. Denote

$$\begin{aligned}&\mathcal{G}_i=\sigma(Y_1,\cdots,Y_i), && u_i=E(Y_i|\mathcal{G}_{i-1}),\\&U_l(i)=\textstyle\sum_{j=l+1}^{l+i}u_j, && T_l^*(i)=T_l(i)-U_l(i),\\&U(i)=U_0(i), && T^*(i)=T(i)-U(i).\end{aligned}$$

It is easy to see that

$$\begin{aligned}I_1&\le P\Big\{\max_{1\le i\le p_1}|T^*(i)|\ge \lambda n^{1/2}\Big\}+P\Big\{\max_{1\le i\le p_1}|U(i)|\ge \lambda n^{1/2}\Big\}\\&=: I_{11}+I_{12}.\end{aligned} \tag{4.1.12}$$

Since $\{T^*(i), i = 1, \cdots, p_1\}$ is a martingale, for large $n$

$$I_{11} \le \varepsilon/6\lambda^2. \tag{4.1.13}$$

In order to estimate $I_{12}$, we prove that there exists a constant $C_0$, which does not depend on $l, i, k$ and $n$, such that

$$EU_l^2(i) \le C_0 ir\rho^2(r)(\log 2i)^2. \tag{4.1.14}$$

From Lemma 2.2.2 there exists a constant $C_1 > 0$, which does not depend on $l, i, k$ and $n$, such that

$$ET_l^2(i) \le C_1 ir. \tag{4.1.15}$$

Using induction for $i$, we can show that (4.1.14) holds for $C_0 = C_1/(\log \frac{3}{2})^2$. From the definition of $\rho$-mixing, we have

$$EU_l^2(1) = Eu_{l+1}^2 = EY_{l+1}u_{l+1} \le \rho(r)\|Y_{l+1}\|_2\|u_{l+1}\|_2.$$

Combining it with (4.1.15) implies (4.1.14) for $i = 1$.

For $i \ge 2$, assume that (4.1.14) holds for $j < i$. Put $i_1 = [i/2], i_2 = i - i_1$. We have

$$\begin{aligned} EU_l^2(i) &= EU_l^2(i_1) + EU_{l+i_1}^2(i_2) + 2EU_l(i_1)U_{l+i_1}(i_2) \\ &\le EU_l^2(i_1) + EU_{l+i_1}^2(i_2) + 2\rho(r)\|U_l(i_1)\|_2\|T_{l+i_1}(i_2)\|_2. \end{aligned}$$

By the assumption of induction and (4.1.15), we obtain

$$\begin{aligned} EU_l^2(i) &\le C_0 i_1 r\rho^2(r)(\log(2i_1))^2 + C_0 i_2 r\rho^2(r)(\log(2i_2))^2 \\ &\quad + 2\rho^2(r)ri_1^{1/2}i_2^{1/2}C_0^{1/2}C_1^{1/2}\log(2i_1) \\ &\le C_0 ir\rho^2(r)\Big((\log(2i_2))^2 + 2\Big(\log\frac{3}{2}\Big)\log(2i_2)\Big) \\ &\le C_0 ir\rho^2(r)(\log(2i))^2. \end{aligned}$$

This proves that (4.1.14) holds for every $i$. From (4.1.14) and Lemma 4.1.2 we obtain

$$E\max_{1\le i\le p_1} U^2(i) \le c\, p_1 r(\log(2p_1))^4\rho^2(r) \le c\, n(\log n)^{-1/2}.$$

Thus for large $n$ there exists a $\lambda \ge 2$ such that

$$I_{12} \le \varepsilon/6\lambda^2. \tag{4.1.16}$$

Combining it with (4.1.13) yields

$$I_1 \leq \varepsilon/3\lambda^2. \tag{4.1.17}$$

By the same way we have

$$I_2 \leq \varepsilon/3\lambda^2. \tag{4.1.18}$$

From (4.1.11), (4.1.17) and (4.1.18) it follows that (4.1.9) holds true. This proves Theorem 4.1.1.

From Theorem 4.1.1 and Theorem 2.1.5, we have the following corollary immediately.

**Corollary 4.1.1.** *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence of random variables with $EX_1 = 0, EX_1^2 < \infty$ and $\sum_{n=1}^{\infty} \rho(2^n) < \infty$. If $\sigma_n^2 = ES_n^2 \to \infty$, then*

$$\lim_{n\to\infty} \sigma_n^2/n = \sigma^2.$$

*Moreover if $\sigma > 0$, then*

$$W_n \Longrightarrow W,$$

*where $W_n(t) = S_{[nt]}/\sigma\sqrt{n}, \quad 0 \leq t \leq 1$.*

## 4.2 The WIP when moments of higher than two orders

Let $\{X_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence of random variables with $EX_1 = 0, EX_1^2 < \infty$ and $\sigma_n^2 = ES_n^2 \to \infty (n \to \infty)$. Let $g : [0, \infty) \to [0, \infty)$ be a non-decreasing function and for some $0 < \delta < 1, x^\delta/g(x)$ be also a non-decreasing function.

**Theorem 4.2.1.**(Peligrad 1987, Shao 1989b) *Let $\{X_n, n \geq 1\}$ be as above and satisfy*

*(i)* $EX_1^2 g(|X_1|) < \infty$,

*(ii)* $\exp((2+\varepsilon^*)\sum_{k=1}^{[\log n]} \rho(2^k)) = O(g(n^{1/2}))$

*for some $0 < \varepsilon^* < 1$. Then*

$$W_n \Longrightarrow W.$$

By some simple computations we have the following corollaries:

**Corollary 4.2.1.** *Let $\{X_n, n \geq 1\}$ be as above. Suppose that for some $\varepsilon > 0$ and $a > 0$*

$$EX_1^2(\log^+ |X_1|)^{2a/(1-\varepsilon)} < \infty$$

*and*

$$\rho(n) \leq a/\log n$$

*for every $n$ sufficiently large. Then*

$$W_n \Longrightarrow W.$$

**Corollary 4.2.2.** *Let $\{X_n, n \geq 1\}$ be as above. Suppose that for some $0 < \beta < 1, \varepsilon > 0$ and $a > 0$*

$$EX_1^2 \exp\Big(\frac{2a(1+\varepsilon)}{1-\beta}(2\log^+ |X_1|)^{1-\beta}\Big) < \infty$$

*and*

$$\rho(n) \leq a/(\log n)^\beta$$

*for every $n$ sufficiently large. Then*

$$W_n \Longrightarrow W.$$

**Corollary 4.2.3.** *Let $\{X_n, n \geq 1\}$ be as above. Soppose that for some $r > 0, \varepsilon > 0$ and $a > 0$*

$$EX_1^2 \exp\Big(\frac{4a\log^+ |X_1|}{(1-\varepsilon)(\log^+\log^+ |X_1|)^r}\Big) < \infty$$

*and*

$$\rho(n) \leq a/(\log\log n)^r$$

*for every $n$ sufficiently large. Then*

$$W_n \Longrightarrow W.$$

The proof of Theorem 4.2.1 will need the following lemma, which is an immediate consequence of Theorem 3.1.2 and Theorem 8.4 of Billingsley (1968).

**Lemma 4.2.1.** *Let $\{X_n, n \geq 1\}$ be as above. In order that $W_n$ weakly converges to $W$ it is necessary and sufficient that $\{S_n^2/\sigma_n^2, n \geq 1\}$ is uniformly integrable and for any given $\varepsilon > 0$ there exists a $\lambda > 1$ such that*

$$P\Big\{\max_{1\leq i\leq n} |S_i| > \lambda\sigma_n\Big\} \leq \varepsilon/\lambda^2. \tag{4.2.1}$$

**Lemma 4.2.2.**(Peligrad 1987) *Let $\{X_n, n \geq 1\}$ be as above with $EX_1^4 < \infty$. Then for any given $\varepsilon > 0$ there exists a $C = C(\varepsilon, \rho(\cdot))$ such that for every $n \geq 1$*

$$ES_n^4 \leq C(n^{1+\varepsilon}EX_1^4 + \sigma_n^4).$$

**Proof.** Denote $a_m = \|S_m\|_4$. It is obvious that

$$a_{2m} \leq \|S_m + S_{k+m}(m)\|_4 + 2ka_1.$$

Using the Schwarz inequality and by the definition of $\rho$-mixing we have

$$\begin{aligned}
&E|S_m + S_{k+m}(m)|^4 \\
&\quad\leq 2a_m^4 + 6E|S_mS_{k+m}(m)|^2 + 8a_m^2(E|S_mS_{k+m}(m)|^2)^{1/2} \\
&\quad\leq 2(1 + 7\rho^{1/2}(k))a_m^4 + 8a_m^2\sigma_m^2 + 6\sigma_m^4 \\
&\quad\leq (2^{1/4}(1 + 7\rho^{1/2}(k))^{1/4}a_m + 2\sigma_m)^4.
\end{aligned}$$

It follows that

$$a_{2m} \leq 2^{1/4}(1 + 7\rho^{1/2}(k))^{1/4}a_m + 2\sigma_m + 2ka_1.$$

Let $0 < \varepsilon < 1/3$ and $k$ be large enough such that $1 + 7\rho^{1/2}(k) \leq 2^\varepsilon$. By the recurrence method for every integer $r \geq 1$ we have

$$a(2^r) \leq 2^{r(1+\varepsilon)/4}a_1 + 2\sum_{i=1}^{r} 2^{(i-1)(1+\varepsilon)/4}(\sigma(2^{r-i}) + ka_1).$$

Whence

$$a(2^r) \leq c(2^{r(1+\varepsilon)/4}a_1 + \sigma(2^r)).$$

This implies the conclusion of the lemma.

**Proof of Theorem 4.2.1.**

If $\sum\rho(2^n) < \infty$, it follows from Theorem 2.1.4 that the conditions of Theorem 4.1.1 are satisfied. Therefore the conclusion of Theorem 4.2.1 holds true. We shall treat here the case when $\sum\rho(2^n) = \infty$. At this time

we have that $g(x) \to \infty$ as $x \to \infty$ by condition (ii). Without loss of generality, we can assume that

$$\rho(n) \geq (\log n)^{-1}(\log\log n)^{-2} \tag{4.2.2}$$

for every large $n$.

**1**) We first prove that $\{S_n^2/\sigma_n^2, n \geq 1\}$ is uniformly integrable. It is easy to see that the condition (ii) implies

$$g(n^{1/2}) > \exp\Big(2\sum_{i=1}^{[\log n]} \rho(2^i)/(1-\varepsilon^*)\Big) \tag{4.2.3}$$

for large $n$. Put

$$T = \operatorname{inv} g\Big(\exp\Big(2\sum_{i=1}^{[(1-\varepsilon)\log n]} \rho(2^i)/(1-\varepsilon^*)\Big)\Big), \tag{4.2.4}$$

where $0 < \varepsilon < \varepsilon^* < 1$. Denote

$$\begin{aligned}
&X_{i1} = X_i I(|X_i| \leq T) - EX_i I(|X_i| \leq T),\\
&X_{i2} = X_i I(|X_i| > T) - EX_i I(|X_i| > T),\\
&S_{n1} = \sum_{i=1}^n X_{i1}, \quad S_{n2} = \sum_{i=1}^n X_{i2},\\
&\sigma_{n1}^2 = \operatorname{Var} S_{n1}, \quad \sigma_{n2}^2 = \operatorname{Var} S_{n2}.
\end{aligned}$$

By Lemma 2.2.2 and noting that the function $g(x)$ is non-decreasing, we have

$$\begin{aligned}
\sigma_{n2}^2 \leq & C_1 \frac{n}{g(T)} EX_1^2 g(|X_1|) I(|X_1| > T)\\
& \times \exp\Big(\sum_{i=1}^{[(1-\varepsilon)\log n]} \rho(2^i)/(1-\varepsilon)\Big),
\end{aligned}$$

where $C_1 = C_1(\varepsilon)$. From the definition of $T$ and Lemma 2.2.3 we get

$$\sigma_{n2}^2 \leq \frac{C_1}{C'} \sigma_n^2 EX_1^2 g(|X_1|) I(|X_1| > T). \tag{4.2.5}$$

Obviously $\operatorname{inv} g(x) \to \infty$ $(x \to \infty)$. It follows from (4.2.4) and (4.2.5) that

$$\sigma_{n2} = o(\sigma_n), \tag{4.2.6}$$

which implies

$$\sigma_{n1}/\sigma_n \to 1 \qquad \text{as } n \to \infty. \tag{4.2.7}$$

By Lemma 4.2.2 for the sequence $\{X_{n1}, n \geq 1\}$, there exists a constant $K_1 = K_1(\rho(\cdot), \varepsilon)$ such that for every $n \geq 1$

$$E|S_{n1}|^4 \leq K_1(n^{1+\varepsilon/2}T^2EX_1^2 + \sigma_{n1}^4). \tag{4.2.8}$$

From Lemma 2.2.3 and noting that $\exp(d\sum_{i=1}^{[\log n]}\rho(2^i))$ is a slowly varying function ($n \to \infty$), it follows that there exists a $C = C(\rho(\cdot), \varepsilon)$ such that for every $n \geq 1$

$$\sigma_n^4 \geq Cn^{2-\varepsilon/2}.$$

Thus by (4.2.7) and (4.2.8) we have

$$E(|S_{n1}|/\sigma_n)^4 \leq K_2(T^2/n^{1-\varepsilon} + 1),$$

where $K_2 = K_2(\rho(\cdot), \varepsilon)$.

From (4.2.3) it follows that for large $n$

$$g([n^{1-\varepsilon}]^{1/2}) \geq \exp\Big(2\sum_{i=1}^{[(1-\varepsilon)\log n]}\rho(2^i)/(1-\varepsilon^*)\Big).$$

Combining it with (4.2.4) we obtain that $T/n^{(1-\varepsilon)/2}$ is bounded by 1. Therefore

$$\sup_n E(|S_{n1}|/\sigma_n)^4 < \infty. \tag{4.2.9}$$

Then, from (4.2.6) and (4.2.9), we prove that $\{S_n^2/\sigma_n^2, n \geq 1\}$ is uniformly integrable.

**2**) Next we show that for any $\varepsilon > 0$ there exists a $\lambda > 1$ such that

$$P\Big\{\max_{1\leq i\leq n}|S_i| \geq 6\lambda\sigma_n\Big\} \leq 6\varepsilon/\lambda^2. \tag{4.2.10}$$

Denote

$$T_1 = \exp\Big(\frac{40}{\delta}\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big), \qquad J = n^{1/2}/T_1, \tag{4.2.11}$$

$$X_{i1} = X_iI(|X_i| \leq J) - EX_iI(|X_i| \leq J),$$

$$X_{i2} = X_iI(|X_i| > J) - EX_iI(|X_i| > J),$$

$$S_{n1}(k) = \sum_{i=1}^k X_{i1}, \qquad S_{n2}(k) = \sum_{i=1}^k X_{i2},$$

$$\sigma_{n1}^2(k) = ES_{n1}^2(k), \quad \sigma_{n2}^2(k) = ES_{n2}^2(k).$$

Obviously $S_k = S_{n1}(k) + S_{n2}(k)$ and

$$P\Big\{\max_{1\le i\le n}|S_i| \ge 6\lambda\sigma_n\Big\} \le P\Big\{\max_{1\le i\le n}|S_{n1}(i)| \ge \lambda\sigma_n\Big\} + P\Big\{\max_{1\le i\le n}|S_{n2}(i)| \ge 5\lambda\sigma_n\Big\}.$$

We first note that

$$\begin{aligned}\log T_1 &= \frac{40}{\delta}\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\\ &\le \frac{40}{\delta}\rho^{-\delta/(2+\delta)}\Big(\frac{n}{T_1^2}\Big)\sum_{i=1}^{[\log n/T_1^2]}\rho(2^i)\\ &\quad+\frac{90}{\delta}\rho^{2/(2+\delta)}\Big(\frac{n}{T_1^2}\Big)\log T_1.\end{aligned}$$

Hence we have for every $n$ sufficiently large

$$\log T_1 \le \frac{50}{\delta}\rho^{-\delta/(2+\delta)}\Big(\frac{n}{T_1^2}\Big)\sum_{i=1}^{[\log(n/T_1^2)]}\rho(2^i) \tag{4.2.12}$$

and

$$\sum_{i=1}^{[\log n]}\rho(2^i) \le \Big(1+\frac{\varepsilon^*}{12}\Big)\sum_{i=1}^{[\log(n/T_1^2)]}\rho(2^i). \tag{4.2.13}$$

From this and by condition (ii) and the fact that $g(x)$ is non-decreasing we have

$$g(J) \ge \exp\Big(\frac{2+\varepsilon^*}{1+\varepsilon^*/12}\sum_{i=1}^{[\log n]}\rho(2^i)\Big), \tag{4.2.14}$$

and by Lemma 2.2.2, Lemma 2.2.3 and (4.2.14) for every $k \le n$ and $n$ sufficiently large

$$\begin{aligned}\sigma_{n2}^2(k) &\le CkEX_1^2I(|X_1|>J)\exp\Big(\sum_{i=1}^{[\log n]}\Big(1+\frac{\varepsilon^*}{4}\Big)\rho(2^i)\Big)\\ &\le \frac{CC'^{-1}\sigma_k^2EX_1^2g(|X_1|)}{g(J)}\exp\Big(\Big(2+\frac{3\varepsilon^*}{4}\Big)\sum_{i=1}^{[\log n]}\rho(2^i)\Big)\\ &\le CC'^{-1}\sigma_k^2EX_1^2g(|X_1|)\exp\Big(\frac{-\varepsilon^*}{52}\sum_{i=1}^{[\log n]}\rho(2^i)\Big).\end{aligned}$$

From this and noting that $\sum \rho(2^i) = \infty$, we deduce that

$$\max_{1\le k\le n} \frac{\sigma_{n2}(k)}{\sigma_k} = o(1) \quad \text{as } n \to \infty.$$

Whence it is easy to see that for $k = 1, 2, \cdots, n$ and $n$ sufficiently large

$$\sigma_{n1}^2(k) \le 2\sigma_k^2. \tag{4.2.15}$$

By Lemma 2.2.4 and (4.2.15)

$$E|S_{n1}(k)|^{2+\delta} \le c\Big(\sigma_k^{2+\delta} + kE|X_1|^{2+\delta} I(|X_1| < J) \exp\Big(30 \sum_{i=1}^{[\log n]} \rho(2^i)\Big)\Big).$$

From this and by Lemma 2.2.2, conditions (i), (ii), (4.2.11), (4.2.14) and Lemma 4.1.2 we see

$$\begin{aligned}
&E \max_{1\le k\le n} |S_{n1}(k)|^{2+\delta} \\
&\le c\Big(\sigma_n^{2+\delta} + n(\log n)^{2+\delta} E|X_1|^{2+\delta} I(|X_1| < J) \\
&\quad \times \exp\Big(30 \sum_{i=1}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big)\Big) \\
&\le c\Big(\sigma_n^{2+\delta} + \frac{n^{2+\delta}(\log n)^{2+\delta} EX_1^2 g(|X_1|)}{g(J)T_1^\delta} \\
&\quad \times C \exp\Big(35 \sum_{i=1}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big)\Big) \\
&\le c\sigma_n^{2+\delta}(1 + EX_1^2 g(|X_1|)).
\end{aligned}$$

Thus there exists a constant $\lambda > 1$ such that for every $n$ sufficiently large

$$P\Big\{\max_{1\le k\le n} |S_{n1}(k)| \ge \lambda\sigma_n\Big\} \le \varepsilon/\lambda^2. \tag{4.2.16}$$

We now estimate $P\{\max_{1\le k\le n} |S_{n2}(k)| \ge 5\lambda\sigma_n\}$. Let

$$p = \exp\Big(\frac{50}{\delta} \sum_{i=1}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big),$$

$$r = \Big[\frac{n}{p}\Big], \quad p_1 = \Big[\frac{p}{2}\Big], \quad p_2 = \Big[\frac{p-1}{2}\Big].$$

Put

$$
\begin{aligned}
&Y_i = \textstyle\sum_{j=1+(2i-1)r}^{2ir} X_{j2}, && i = 1, 2, \cdots, p_1; \\
&Z_i = \textstyle\sum_{j=1+2ir}^{(2i+1)r} X_{j2}, && i = 0, 1, \cdots, p_2; \\
&T_1(i) = \textstyle\sum_{j=1}^{i} Y_j, && T_2(i) = \textstyle\sum_{j=0}^{i} Z_j.
\end{aligned}
$$

Noting that $\{X_{j2}, 1 \le j \le n\}$ is stationary we have

$$
\begin{aligned}
&P\Big\{\max_{1\le k\le n} |S_{n2}(k)| \ge 5\lambda\sigma_n\Big\} \\
&\le P\Big\{\max_{1\le k\le p_1} |T_1(k)| \ge 2\lambda\sigma_n\Big\} \\
&\quad + P\Big\{\max_{1\le k\le p_2} |T_2(k)| \ge 2\lambda\sigma_n\Big\} \\
&\quad + (p+1)P\Big\{\max_{1\le k\le r} |S_{n2}(k)| \ge \lambda\sigma_n\Big\} \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

In terms of Lemmas 2.2.2 and 2.2.3 we have for every $n$ sufficiently large

$$
\begin{aligned}
&P\Big\{\max_{1\le k\le r} |S_{n2}(k)| \ge \lambda\sigma_n\Big\} \\
&\le P\Big\{\sum_{i=1}^{r}(|X_{n2}(i)| - E|X_{n2}(i)|) \ge \lambda\sigma_n - 2\sum_{i=1}^{r} E|X_{n2}(i)|\Big\} \\
&\le P\Big\{\sum_{i=1}^{r}(|X_{n2}(i)| - E|X_{n2}(i)|) \ge \lambda\sigma_n - 2r\frac{EX_1^2 g(|X_1|)}{g(J)J}\Big\} \\
&\le P\Big\{\sum_{i=1}^{r}(|X_{n2}(i)| - E|X_{n2}(i)|) \ge \lambda\sigma_n/2\Big\} \\
&\le 4Cr\sigma_n^{-2} \exp\Big((1+\varepsilon^*/4) \sum_{i=1}^{[\log n]} \rho(2^i)\Big) EX_1^2 I(|X_1| > J) \cdot \lambda^{-2}.
\end{aligned}
$$

Whence by (4.2.14)

$$
\begin{aligned}
I_3 &\le 4Cn\sigma_n^{-2} \exp\Big((1+\varepsilon^*/4) \sum_{i=1}^{[\log n]} \rho(2^i)\Big) EX_1^2 I(|X_1| > J) \cdot \lambda^{-2} \\
&\le \frac{4C}{C'\lambda^2 g(J)} \exp\Big((2+3\varepsilon^*/4) \sum_{i=1}^{[\log n]} \rho(2^i)\Big) EX_1^2 g(|X_1|) \\
&\le \frac{4C}{C'\lambda^2} \exp\Big(-\frac{\varepsilon^*}{52} \sum_{i=1}^{[\log n]} \rho(2^i)\Big) EX_1^2 g(|X_1|) \\
&\le \varepsilon/\lambda^2. \qquad\qquad (4.2.17)
\end{aligned}
$$

In order to establish the estimation of $I_1$, let

$$\mathcal{F}_0 = (\Omega, \emptyset), \quad \mathcal{F}_k = \sigma(X_i, 1 \le i \le 2rk);$$
$$u_k = E(Y_k|\mathcal{F}_{k-1}), \quad k = 1, 2, \cdots, p_1;$$
$$U_i(k) = \sum_{j=i+1}^{i+k} u_j, \quad T^*(k) = T_1(k) - U_0(k).$$

Obviously

$$I_1 \le P\Big\{\max_{1\le i\le p_1} |T^*(i)| \ge \lambda\sigma_n\Big\} + P\Big\{\max_{1\le i\le p_1} |U_0(i)| \ge \lambda\sigma_n\Big\}$$
$$=: I_{11} + I_{12}.$$

Because $\{T^*(i), i = 1, 2, \cdots, p_1\}$ is a martingale, we have

$$I_{11} \le \frac{16}{\lambda^2\sigma_n^2} \sum_{i=1}^{p_1} EY_i^2.$$

In a way somewhat similar to the estimation of $I_3$ we also have for any $\lambda > 1$ and for every $n$ sufficiently large

$$I_{11} \le \epsilon/\lambda^2. \tag{4.2.18}$$

Finally, we shall prove by induction on $k$ that for every $i, k, n$,

$$EU_i^2(k) \le Ck\rho^2(r)\log^2(2k)EX_1^2 I(|X_1| > J) \cdot r \cdot \exp\Big((1 + \varepsilon^*/4) \sum_{i=1}^{[\log n]} \rho(2^i)\Big). \tag{4.2.19}$$

When $k = 1$, by the definition of $\rho$-mixing

$$EU_i^2(1) = EE^2(Y_{i+1}|\mathcal{F}_i) = E(Y_{i+1}E(Y_{i+1}|\mathcal{F}_i))$$
$$\le \rho(r)\|Y_{i+1}\|_2 \cdot \|E(Y_{i+1}|\mathcal{F}_i)\|_2,$$

thus (4.2.19) is true for $k = 1$ and for every $i + 1 \le p_1$ by Lemma 2.2.2. When $k \ge 2$, assume (4.2.19) holds for every integer less than $k$. Put $k_1 = [k/2], k_2 = k - k_1$, then

$$EU_i^2(k) = EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1)U_{i+k_1}(k_2)$$
$$= EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1) \sum_{j=i+k_1+1}^{i+k} Y_j$$
$$\le EU_i^2(k_1) + EU_{i+k_1}^2(k_2)$$
$$+ 2\|U_i(k_1)\|_2 \cdot \|\sum_{j=i+k_1+1}^{i+k} Y_j\|_2\rho(r).$$

By induction hypothesis and Lemma 2.2.2

$$
\begin{aligned}
EU_i^2(k) &\le C(k_1 \log^2 2k_1 + k_2 \log^2 2k_2 + 2(k_1k_2)^{1/2}\log 2k_1) \\
&\quad \cdot \rho^2(r) \cdot r \cdot \exp\Big(\Big(1+\frac{\varepsilon^*}{4}\Big)\sum_{j=1}^{[\log n]} \rho(2^j)\Big) EX_1^2 I(|X_1| > J) \\
&\le Ck(\log 2k)^2 r \cdot \exp\Big(\Big(1+\frac{\varepsilon^*}{4}\Big)\sum_{j=1}^{[\log n]} \rho(2^j)\Big) \\
&\quad \cdot EX_1^2 I(|X_1| > J) \cdot \rho^2(r),
\end{aligned}
$$

which proves that (4.2.19) holds.

From (4.2.19) we obtain by Lemma 4.1.2

$$
\begin{aligned}
&E \max_{1\le i\le p_1} U_0^2(i) \\
&\le 3Crp_1\rho^2(r)\log^4(2p_1) \\
&\quad \cdot \exp\Big((1+\varepsilon^*/4)\sum_{j=1}^{[\log n]} \rho(2^j)\Big) EX_1^2 I(|X_1| > J) \\
&\le \frac{3C\sigma_n^2\rho^2(n/p_1)\log^4(2p_1)}{C'g(J)} \\
&\quad \cdot \exp\Big((2+3\varepsilon^*/4)\sum_{j=1}^{[\log n]} \rho(2^j)\Big) EX_1^2 g(|X_1|) \\
&\le \frac{3C\sigma_n^2\rho^2(n/p_1)\log^4(2p_1)}{C'} \\
&\quad \cdot \exp\Big(-\frac{\varepsilon^*}{52}\sum_{j=1}^{[\log n]} \rho(2^j)\Big) EX_1^2 g(|X_1|).
\end{aligned}
$$

By (4.2.12)

$$
\rho^2\Big(\frac{n}{p_1}\Big)\log^4(2p_1) \le \Big(\frac{50}{\delta}\Big)^4 \rho^{2/3}\Big(\frac{n}{T_1^2}\Big)\sum_{j=1}^{[\log n]} \rho(2^j),
$$

hence we finally get that for any $\lambda > 1$ and for every $n$ sufficiently large

$$
I_{12} \le \varepsilon/\lambda^2.
$$

Therefore

$$
I_1 \le 2\varepsilon/\lambda^2. \tag{4.2.20}
$$

Similarly, we have

$$I_2 \leq 2\varepsilon/\lambda^2. \tag{4.2.21}$$

(4.2.10) now follows from (4.2.16), (4.2.17) and (4.2.20), (4.2.21). Theorem 4.2.1 is proved.

## 4.3 A generalized result when moments of higher than two orders

Shao (1989a) gave a generalized result of Theorem 4.2.1, where the condition of strict stationarity was removed.

**Theorm 4.3.1.** *Let $\{X_n, n \geq 1\}$ be a $\rho$-mixing sequence of random variables with $EX_n = 0, EX_n^2 \geq c > 0$. Suppose that $g : [0,\infty) \to [0,\infty)$ is a non-decreasing function and $x^\delta/g(x), 0 < \delta < 1$, is also non-decreasing. If the following conditions are satisfied:*

*(i)* $\{X_n^2 g(|X_n|), n \geq 1\}$ *is uniformly integrable,*

*(ii)* $\sigma_n^2 := ES_n^2 = nh(n)$, *where $h(n)$ is a slowly varying function,*

*(iii)* $\sup_{m\geq 0, n\geq 1} ES_m^2(n)/\sigma_n^2 < \infty$,

*(iv)* $\lim_{n\to\infty} \inf_{m\geq 0} ES_m^2(n) = \infty$,

*(v)* $\exp\Big((2+\varepsilon^*)\sum_{k=1}^{[\log n]} \rho(2^k)\Big) = O(g(n^{1/2}))$, for some $0 < \varepsilon^* < 1$,

*then $W_n$ weakly converges to $W$.*

The proof of Theorem 4.3.1 needs the following lemmas.

**Lemma 4.3.1 .** *Let $f(k,m)$ be a non-negative function satisfying (4.1.2). Suppose that there exist $\alpha > 0, r \geq 1$ such that*

$$E|S_k(l)|^r \leq f(k,l)\{f^\alpha(k,l)w_1(f(k,l)) + w_2(f(k,l))\}, \tag{4.3.1}$$

*where $w_2$ is a non-negative non-decreasing function, $w_1$ is a non-negative function with*

$$\max_{0\leq s\leq t} s^\beta w_1(s) \leq at^\beta w_1(t) \tag{4.3.2}$$

*for some $a > 0, 0 < \beta < \alpha$ and any $t > 0$. Then we have*

$$\begin{aligned} &E\max_{1\leq i\leq n}|S_k(i)|^r \\ &\leq 2^{r+1}f(k,n)\{a(1-(\frac{1}{2})^{(\alpha-\beta)/r})^{-r}f^\alpha(k,n)w_1(f(k,n)) \\ &\quad + w_2(f(k,n))\log^r(2n)\}. \end{aligned} \tag{4.3.3}$$

**Proof.** From (4.1.2) and (4.3.2), we have for $l \le n$

$$f^{\beta}(k,l)w_1(f(k,l)) \le af^{\beta}(k,n)w_1(f(k,n)).$$

Therefore, by (4.3.1), for every $k \ge 0, 1 \le l \le n$

$$E|S_k(l)|^r \le f(k,l)\{f^{\alpha-\beta}(k,l)\cdot af^{\beta}(k,n)w_1(f(k,n)) + w_2(f(k,n))\}.$$

It follows from Lemma 4.1.2 and the monotonicity of $w_2(\cdot)$ that

$$\begin{aligned}
&E\max_{1\le l\le n}|S_k(l)|^r \\
&\le \frac{5}{2}f(k,n)\Big\{\sum_{i=0}^{[\log n]} w_2^{1/r}(f(k,n)) \\
&\quad + (af^{\beta}(k,n)w_1(f(k,n)))^{1/r}\sum_{i=0}^{[\log n]}\Big(\frac{f(k,n)}{2^i}\Big)^{(\alpha-\beta)/r}\Big\}^r \\
&\le \frac{5}{2}f(k,n)\Big\{w_2^{1/r}(f(k,n))\log(2n) \\
&\quad + (af^{\alpha}(k,n)w_1(f(k,n)))^{1/r}\Big(1-\Big(\frac{1}{2}\Big)^{(\alpha-\beta)/r}\Big)^{-1}\Big\}^r \\
&\le 5\cdot 2^{r-2}f(k,n)\Big\{af^{\alpha}(k,n)w_1(f(k,n))\Big(1-\Big(\frac{1}{2}\Big)^{(\alpha-\beta)/r}\Big)^{-r} \\
&\quad + w_2(f(k,n))\log^r(2n)\Big\}.
\end{aligned}$$

This completes the proof of Lemma 4.3.1.

**Lemma 4.3.2.** *Let $\{X_n, n\ge 1\}$ be a $\rho$-mixing sequence with $EX_n = 0$, and $q_1, q_2 \ge 2$. Suppose that the non-negative function $h(n)$ satisfies:*

$$\max\Big(h\Big(\Big[\frac{n}{2}\Big]\Big), h\Big(n-\Big[\frac{n}{2}\Big]\Big)\Big) \le \theta h(n) \tag{4.3.4}$$

*for every $n \ge 1$ and some $0 < \theta < 2^{1-\frac{2}{q_1\wedge 3}}$, and if $q_1 > 3$*

$$h(n) \ge \frac{1}{a}\exp\Big\{-a\sum_{i=0}^{[\log n]}\rho^{2/q_1}(2^i)\Big\}, \tag{4.3.5}$$

$$\max_{1\le i\le n} i^{\alpha}h(i) \le an^{\alpha}h(n)$$

*for some $a > 0$ and some $\alpha$, $0 < \alpha < q_1 - 2$. And suppose that integers $1 \le k \le n$, $l \ge 0$ and numbers $x > 0, 0 < B \le A < \infty$ satisfy*

$$4n \max_{l<i\le l+n} E|X_i|I(|X_i| \ge A) \le x, \tag{4.3.6}$$

$$48k \max_{l<i\le l+n} E|X_i|I(|X_i| \ge B) \le x, \tag{4.3.7}$$

$$\operatorname{Var}\Big(\sum_{i=j+1}^{j+n} X_i I(|X_i| \le B)\Big) \le nh(n) \max_{j<i\le j+n} EX_i^2 I(|X_i| \le B). \tag{4.3.8}$$

*Then for any given $\varepsilon > 0$, there exists a $K = K(\varepsilon, q_1, q_2, a, \theta, \alpha, \rho(\cdot))$ such that*

$$\begin{aligned}
&P\Big\{\max_{1\le i\le n} |S_l(i)| \ge x\Big\} \\
&\quad \le \sum_{i=l+1}^{l+n} P(|X_i| \ge A) \\
&\qquad + K\Big\{x^{-q_1}\Big[(nh(n) \max_{l<i\le l+n} EX_i^2 I(|X_i| \le B))^{q_1/2} \\
&\qquad + n\exp\Big\{K \sum_{i=0}^{[\log n]} \rho^{2/q_1}(2^i)\Big\} \log^{q_1}(2n) \\
&\qquad \times \max_{l<i\le l+n} E|X_i|^{q_1} I(|X_i| \le B)\Big] \\
&\qquad + x^{-q_2}\Big[\Big(n\exp\Big\{(1+\varepsilon) \sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \\
&\qquad \times \max_{l<i\le l+n} EX_i^2 I(B < |X_i| < A)\Big)^{q_2/2} \\
&\qquad + n\exp\Big\{K \sum_{i=0}^{[\log n]} \rho^{2/q_2}(2^i)\Big\} \\
&\qquad \times \max_{l<i\le l+n} E|X_i|^{q_2} I(B < |X_i| < A)\Big] \\
&\qquad + x^{-2} n\rho^2(k) \log^4[\frac{n}{k}] \cdot \exp\Big\{(1+\varepsilon) \sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \\
&\qquad \times \max_{l<i\le l+n} EX_i^2 I(B < |X_i| < A)\Big\}.
\end{aligned}$$

**Proof.** For simplicity, we assume that $X_n, n \ge 1$ have a common

distribution. Denote

$$\begin{aligned}
X_{i1} &= X_i I(|X_i| \le B) - EX_i I(|X_i| \le B),\\
X_{i2} &= X_i I(B < |X_i| < A) - EX_i I(B < |X_i| < A),\\
X_{i3} &= X_i I(|X_i| \ge A) - EX_i I(|X_i| \ge A),
\end{aligned}$$

$$S_{lm}(i) = \sum_{j=l+1}^{l+i} X_{jm}, \qquad m = 1, 2, 3.$$

It is easy to see that

$$\begin{aligned}
&P\{\max_{1\le i\le n} |S_l(i)| \ge x\}\\
&\le P\{\max_{1\le i\le n} |S_{l1}(i)| \ge \frac{x}{4}\} + P\{\max_{1\le i\le n} |S_{l2}(i)| \ge \frac{x}{4}\}\\
&\quad + P\{\max_{1\le i\le n} |S_{l3}(i)| \ge \frac{x}{2}\}\\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{4.3.9}$$

By (4.3.6), we have

$$\begin{aligned}
I_3 &\le P\Big\{\sum_{i=l+1}^{l+n} |X_i| I(|X_i| \ge A) \ge \frac{x}{2} - \sum_{i=l+1}^{l+n} E|X_i| I(|X_i| \ge A)\Big\}\\
&\le P\Big\{\sum_{i=l+1}^{l+n} |X_i| I(|X_i| \ge A) \ge \frac{x}{4}\Big\}\\
&\le \sum_{i=l+1}^{l+n} P(|X_i| \ge A).
\end{aligned} \tag{4.3.10}$$

By Lemma 2.2.6, there exists a $K_1 = K_1(q_1, a, \theta, \rho(\cdot))$ such that

$$\begin{aligned}
&E\Big|\sum_{i=j+1}^{j+n} X_{i1}\Big|^{q_1}\\
&\qquad \le K_1\Big\{(nh(n)EX_1^2 I(|X_1| \le B))^{q_1/2}\\
&\qquad\quad + n\exp\Big[K_1 \sum_{i=0}^{[\log n]} \rho^{2/q_1}(2^i)\Big] E|X_1|^{q_1} I(|X_1| \le B)\Big\}.
\end{aligned}$$

It follows from Lemma 4.3.1 that there exists a $K_2 = K_2(K_1, \alpha, q_1)$ such

that

$$
\begin{aligned}
E & \max_{1\le i\le n} |S_{l1}(i)|^{q_1} \\
& \le K_2\Big\{(nh(n)EX_1^2 I(|X_1|\le B))^{q_1/2} \\
& \quad + n\exp\Big\{K_2 \sum_{i=0}^{[\log n]} \rho^{2/q_1}(2^i)\Big\} \\
& \quad \cdot (\log(2n))^{q_1} E|X_1|^{q_1} I(|X_1|\le B)\Big\}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
I_1 \le{} & K_2 x^{-q_1}\Big\{(nh(n)EX_1^2 I(|X_1|\le B))^{q_1/2} \\
& + n\exp\Big\{K_2 \sum_{i=0}^{[\log n]} \rho^{2/q_1}(2^i)\Big\}(\log(2n))^{q_1} \\
& \cdot E|X_1|^{q_1} I(|X_1|\le B)\Big\}. \qquad (4.3.11)
\end{aligned}
$$

In order to estimate $I_2$, put

$$
\begin{aligned}
Y_i &= \sum_{j=2ik+l+1}^{(2i+1)k+l} X_{j2}, \quad i=0,1,\cdots,p_1, \\
Z_i &= \sum_{j=(2i+1)k+l+1}^{2(i+1)k+l} X_{j2}, \quad i=0,1,\cdots,p_2,
\end{aligned}
$$

where $p_1 = [(\frac{n}{k}-1)/2], p_2 = [(\frac{n}{k}-2)/2]$. Denote

$$
W_i = \sum_{j=0}^{i} Y_j, \qquad W_i^* = \sum_{j=0}^{i} Z_j.
$$

It is easy to see that

$$
\begin{aligned}
I_2 \le{} & P\Big\{\max_{0\le i\le p_1} |W_i| \ge \frac{x}{12}\Big\} + P\Big\{\max_{0\le i\le p_2} |W_i^*| \ge \frac{x}{12}\Big\} \\
& + P\Big\{\max_{0\le i\le [n/k]}\ \max_{ik+1\le j<(i+1)k} \Big|\sum_{\nu=l+ik+1}^{l+j} X_{\nu 2}\Big| \ge \frac{x}{12}\Big\} \\
=:{} & I_{21}+I_{22}+I_{23}. \qquad (4.3.12)
\end{aligned}
$$

From the condition (4.3.7) and Lemma 2.2.5 it follows that

$$
\begin{aligned}
I_{23} \le\ & 2\Big[\frac{n}{k}\Big] \max_{0\le i\le[n/k]} P\Big\{ \sum_{j=l+ik+1}^{l+(i+1)k} (|X_j|I(B<|X_j|<A) \\
& - E|X_j|I(B<|X_j|<A)) \ge \frac{x}{12} \\
& - 2\sum_{j=l+ik+1}^{l+(i+1)k} E|X_j|I(B<|X_j|<A)\Big\} \\
\le\ & \frac{2n}{k} \max_{0\le i\le[n/k]} P\Big\{ \sum_{j=l+ik+1}^{l+(i+1)k} (|X_j|I(B<|X_j|<A) \\
& - E|X_j|I(B<|X_j|<A)) \ge x/24\Big\} \\
\le\ & C\frac{n}{k}x^{-q_2}\Big\{\Big(k\exp\Big((1+\varepsilon)\sum_{i=0}^{[\log k]}\rho(2^i)\Big)EX_1^2I(B<|X_1|<A)\Big)^{q_2/2} \\
& + k\exp\Big\{C\sum_{i=0}^{[\log k]}\rho^{2/q_2}(2^i)\Big\}E|X_1|^{q_2}I(B<|X|<A)\Big\} \\
\le\ & Cx^{-q_2}\Big\{\Big(n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}EX_1^2I(B<|X_1|<A)\Big)^{q_2/2} \\
& + n\exp\Big\{C\sum_{i=0}^{[\log n]}\rho^{2/q_2}(2^i)\Big\}E|X_1|^{q_2}I(B<|X_1|<A)\Big\}.
\end{aligned}
$$

Next, we estimate $I_{21}$. The estimation of $I_{22}$ can be obtained by the same way. Denote $\mathcal{F}_{-1}=\{\emptyset,\Omega\}$.

$$
\mathcal{F}_i=\sigma(X_j: j\le l+(2i+1)k), \qquad i=0,1,\cdots,p_1.
$$

Put

$$
\begin{aligned}
&U_i=Y_i-E(Y_i|\mathcal{F}_{i-1}), && G_i=\textstyle\sum_{j=0}^{i}U_j, \\
&H_i=\textstyle\sum_{j=1}^{i}E(Y_j|\mathcal{F}_{j-1}), && i=0,1,\cdots,p_1.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
I_{21} &\le P\{\max_{1\le i\le p_1}|G_i|\ge x/24\}+P\{\max_{1\le i\le p_1}|H_i|\ge x/24\} \\
&=: I_{21}(1)+I_{21}(2). \qquad (4.3.13)
\end{aligned}
$$

Since $\{U_i,\mathcal{F}_i,i\ge 1\}$ is a martingale difference sequence, by the maximum

value inequality of Brown (1971), we have

$$\begin{aligned} I_{21}(1) &\le \frac{24}{x} E|G_{p_1}|I(|G_{p_1}| \ge \frac{x}{48}) \\ &\le \frac{48}{x}\Big\{E|W_{p_1}|I\Big(|W_{p_1}| \ge \frac{x}{96}\Big) \\ &\quad + E|H_{p_1}|I\Big(|H_{p_1}| \ge \frac{x}{96}\Big)\Big\} \\ &\le (96/x)^{q_2} E|W_{p_1}|^{q_2} + (96/x)^2 E|H_{p_1}|^2. \end{aligned} \tag{4.3.14}$$

From Lemma 2.2.5 it follows that

$$\begin{aligned} &E|W_{p_1}|^{q_2} \\ &\le C\Big\{\Big(n \exp\Big[(1+\varepsilon)\sum_{i=0}^{[\log n]} \rho(2^i)\Big] EX_1^2 I(B < |X_1| < A)\Big)^{q_2/2} \\ &\quad + n \exp\Big[C \sum_{i=0}^{[\log n]} \rho^{2/q_2}(2^i)\Big] E|X_1|^{q_2} I(B < |X_1| < A)\Big\}. \end{aligned} \tag{4.3.15}$$

We prove below by induction that there exists a constant $K'$ such that

$$\begin{aligned} &E\Big(\sum_{j=i+1}^{i+m} E(Y_i|\mathcal{F}_{j-1})\Big)^2 \\ &\quad \le K' mk\rho^2(k)(\log(2m))^2 \\ &\qquad \times \exp\Big[(1+\varepsilon)\sum_{j=0}^{[\log m]} \rho(2^j)\Big] EX_1^2 I(B < |X_1| < A). \end{aligned} \tag{4.3.16}$$

Indeed, it follows from Lemma 2.2.1 that there exists a constant $C'$ such that for $mk \le n$

$$\begin{aligned} E\Big(\sum_{j=i+1}^{i+m} Y_j\Big)^2 &\le C' mk \exp\Big[(1+\varepsilon)\sum_{j=0}^{[\log n]} \rho(2^j)\Big] \\ &\quad \times EX_1^2 I(B < |X_1| < A). \end{aligned} \tag{4.3.17}$$

When $m = 1$, by the definition of $\rho$-mixing

$$E(E(Y_{i+1}|\mathcal{F}_i))^2 = E(Y_{i+1}E(Y_{i+1}|\mathcal{F}_i)) \le \rho(k)\|Y_{i+1}\|_2 \|E(Y_{i+1}|\mathcal{F}_i)\|_2,$$

that is

$$E(E(Y_{i+1}|\mathcal{F}_i))^2 \le \rho^2(k) EY_{i+1}^2.$$

Let $K' = C'/\log^2(3/2)$, it follows that (4.3.16) holds true for $m = 1$.

Suppose that (4.3.16) holds true for the integers less than $m$. Now let us show that (4.3.16) holds true for $m$. Denote $m_1 = [m/2], m_2 = m - m_1$, we have

$$\begin{aligned}
&E\Big(\sum_{j=i+1}^{i+m} E(Y_j|\mathcal{F}_{j-1})\Big)^2 \\
&\quad = E\Big(\sum_{j=i+1}^{i+m_1} E(Y_j|\mathcal{F}_{j-1})\Big)^2 + E\Big(\sum_{j=i+m_1+1}^{i+m} E(Y_j|\mathcal{F}_{j-1})\Big)^2 \\
&\qquad + 2E\Big(\sum_{j=i+1}^{i+m_1} E(Y_j|\mathcal{F}_{j-1})\Big)\Big(\sum_{j=i+m_1+1}^{i+m} E(Y_j|\mathcal{F}_{j-1})\Big) \\
&\quad \le E\Big(\sum_{j=i+1}^{i+m_1} E(Y_j|\mathcal{F}_{j-1})\Big)^2 + E\Big(\sum_{j=i+m_1+1}^{i+m} E(Y_j|\mathcal{F}_{j-1})\Big)^2 \\
&\qquad + 2\rho(k)E\Big\|\sum_{j=i+1}^{i+m_1} E(Y_j|\mathcal{F}_{j-1})\Big\|_2\Big\|\sum_{j=i+m_1+1}^{i+m} Y_j\Big\|_2,
\end{aligned}$$

which, by the inductive assumption and (4.3.17), implies

$$\begin{aligned}
&E\Big(\sum_{j=i+1}^{i+m} E(Y_j|\mathcal{F}_{j-1})\Big)^2 \\
&\quad \le K'\{m_1\log^2(2m_1) + m_2\log^2(2m_2) \\
&\qquad + 2\Big(\log\frac{3}{2}\Big)(m_1m_2)^{1/2}\log(2m_1)\} \\
&\qquad \times \rho^2(k)k\exp\Big\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\Big\}EX_1^2I(B<|X_1|<A) \\
&\quad \le K'mk\rho^2(k)\log^2(2m) \\
&\qquad \times \exp\Big\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\Big\}EX_1^2I(B<|X_1|<A).
\end{aligned}$$

This proves (4.3.16). Moreover, it follows from (4.3.16) and Lemma 4.3.1 that there exists a constant $K''$ such that

$$\begin{aligned}
E\max_{0\le i\le p_1} H_i^2 \le K''p_1k\rho^2(k)(\log(2p_1))^4\exp\Big\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\Big\} \\
\cdot EX_1^2I(B<|X_1|<A). \qquad (4.3.18)
\end{aligned}$$

Thus we have

$$I_{21}(2) \le 288K''x^{-2}n\rho^2(k)\log^4[n/k] \cdot \exp\Big\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\Big\}EX_1^2I(B<|X_1|<A). \quad (4.3.19)$$

From (4.3.13), (4.3.14), (4.3.15), (4.3.18) and (4.3.19) it follows that

$$\begin{aligned} I_{21} \le K_3x^{-q_2}\Big\{&\Big(n\exp\Big\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\Big\}EX_1^2I(B<|X_1|<A)\Big)^{q_2/2} \\ &+ n\exp\Big\{K_3\sum_{j=0}^{[\log n]}\rho^{2/q_2}(2^j)\Big\}E|X_1|^{q_2}I(B<|X_1|<A)\Big\} \\ &+ K_3x^{-2}n\rho^2(k)\log^4\Big[\frac{n}{k}\Big] \\ &\times\exp\Big\{(1+\varepsilon)\sum_{j=0}^{[\log n]}\rho(2^j)\Big\}EX_1^2I(B<|X_1|<A) \end{aligned}$$

for some $K_3 > 0$. The proof of Lemma 4.3.2 is completed.

**Lemma 4.3.3.** *Let $0<\delta\le 1$. Suppose that the non-negative function $h(n)$ satisfies the following conditions: there exist integer $n_0>0$, $0<\theta<2^{\delta/(2+\delta)}$, $0<\delta'<\delta$ and $a>0$ such that for any $n\ge n_0$*

$$h\Big(\Big[\frac{n}{2}\Big]\Big)\vee h\Big(n-\Big[\frac{n}{2}\Big]\Big)\le\theta h(n), \quad (4.3.20)$$

$$\max_{1\le i\le n} i^{\delta'}h(i)\le an^{\delta'}h(n). \quad (4.3.21)$$

*Let $\{X_n, n\ge 1\}$ be a $\rho$-mixing sequence with $EX_n=0, EX_n^2<\infty$ and for every $k\ge 0, n\ge 1$*

$$ES_k^2(n)\le nh(n)\max_{k<i\le k+n}EX_i^2. \quad (4.3.22)$$

*Then for any given $\varepsilon>0$ there exists a $K=K(\varepsilon,\delta,\delta',n_0,a,\rho(\cdot))$ such that*

*for any* $n \geq k \geq 0, l \geq 0$ *and* $B > 0$

$$
\begin{aligned}
&E \max_{1\le i\le n} |S_l(i)|^2 I(\max_{1\le i\le n} |S_l(i)| \geq x) \\
&\quad \leq K\Big\{x^{-\delta}\Big\{(nh(n) \max_{l<i\le l+n} EX_i^2 + n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \\
&\quad\quad \times \max_{l<i\le l+n} EX_i^2 I(|X_i| \geq B))^{\frac{2+\delta}{2}} \\
&\quad\quad + n\exp\Big\{K \sum_{i=0}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big\}\log^{2+\delta}(2n) \\
&\quad\quad \times \max_{l<i\le l+n} E|X_i|^{2+\delta} I(|X_i| < B)\Big\} \\
&\quad\quad + n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\}\Big(1+\rho^2(k)\log^4\Big[\frac{n}{k}\Big]\Big) \\
&\quad\quad \max_{l<i\le l+n} EX_i^2 I(|X_i| \geq B) \\
&\quad\quad + nk \max_{\iota<i\le l+n} (E|X_i| I(|X_i| \geq B))^2\Big\}. \qquad (4.3.23)
\end{aligned}
$$

**Proof.** Denote

$$
X_{i1} = X_i I(|X_i| < B) - EX_i I(|X_i| < B), \qquad S_{l1}(i) = \sum_{j=l+1}^{l+i} X_{j1},
$$

$$
X_{i2} = X_i I(|X_i| \geq B) - EX_i I(|X_i| \geq B), \qquad S_{l2}(i) = \sum_{j=l+1}^{l+i} X_{j2}.
$$

It is easy to see that

$$
\begin{aligned}
&E \max_{1\le i\le n} S_l^2(i) I\Big(\max_{1\le i\le n} |S_l(i)| \geq x\Big) \\
&\leq 4E \max_{1\le i\le n} S_{l1}^2(i) I(\max_{1\le i\le n} |S_{l1}(i)| \geq x/2) + 4E \max_{1\le i\le n} S_{l2}^2(i) \\
&\leq 8x^{-\delta} E \max_{1\le i\le n} |S_{l1}(i)|^{2+\delta} + 4E \max_{1\le i\le n} S_{l2}^2(i) \\
&=: 8I_1 + 4I_2. \qquad (4.3.24)
\end{aligned}
$$

From Lemma 2.2.2 , for every $n \geq 1, l \geq 0$ we have

$$
ES_{l2}^2(n) \leq Cn\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \max_{l\le i\le l+n} EX_i^2 I(|X_i| \geq B),
$$

where $C = C(\varepsilon)$. Therefore, by (4.3.22) one obtains

$$\begin{aligned}ES_{l1}^2(n) &\le 2ES_l^2(n) + 2ES_{l2}^2(n)\\ &\le 2nh(n)\max_{l\le i\le l+n} EX_i^2\\ &\quad + 2Cn\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\\ &\quad \times \max_{l\le i\le l+n} EX_i^2 I(|X_i|\ge B). \end{aligned}\tag{4.3.25}$$

It follows from Lemma 2.2.6 that there exists a $K_1$ such that for every $l\ge 0, n\ge 1$

$$\begin{aligned}&E|S_{l1}(n)|^{2+\delta}\\ &\quad\le K_1\Big\{\Big(nh(n)\max_{l<i\le l+n} EX_i^2\\ &\qquad + n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\max_{l<i\le l+n} EX_i^2 I(|X_i|\ge B)\Big)^{(2+\delta)/2}\\ &\qquad + n\exp\Big\{K_1\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big\}\max_{l<i\le l+n} E|X_i|^{2+\delta} I(|X_i|\le B)\Big\}.\end{aligned}$$

By Lemma 4.3.1 we get

$$\begin{aligned}I_1 &\le x^{-\delta}c\Big\{\Big(nh(n)\max_{l<i\le l+n} EX_i^2\\ &\quad + n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}\max_{l<i\le l+n} EX_i^2 I(|X_i|> B)\Big)^{\frac{2+\delta}{2}}\\ &\quad + n\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big\}\\ &\quad \cdot \max_{l<i\le l+n} E|X_i|^{2+\delta} I(|X_i|\le B)\log^{2+\delta}(2n)\Big\}.\end{aligned}\tag{4.3.26}$$

We estimate $I_2$ below. Denote

$$\begin{aligned}Y_i &= \textstyle\sum_{j=l+2ik+1}^{l+(2i+1)k} X_{j2}, \qquad & W_i &= \textstyle\sum_{j=0}^{i} Y_j, \quad i = 0,1,\cdots,p_1,\\ Z_i &= \textstyle\sum_{j=l+(2i+1)k+1}^{l+2(i+1)k} X_{j2}, \qquad & W_i^* &= \textstyle\sum_{j=0}^{i} Z_j, \quad i = 0,1,\cdots,p_2,\end{aligned}$$

where $p_1 = [(\frac{n}{k} - 1)/2]$, $p_2 = [\frac{n}{2k}] - 1$. We have

$$\begin{aligned} I_2 &\leq 8\Big\{E \max_{0\leq i\leq p_1} W_i^2 \\ &\quad + E \max_{0\leq i\leq p_2} W_i^{*2} + E \max_{0\leq i\leq [n/k]} \max_{ik+1\leq j\leq (i+1)k} \Big| \sum_{v=l+ik+1}^{l+j} X_{v2}\Big|^2\Big\} \\ &=: 8(I_{21} + I_{22} + I_{23}). \end{aligned} \tag{4.3.27}$$

It is easy to see from the proof of Lemma 4.3.2 that

$$\begin{aligned} I_{21} + I_{22} &\leq c\, n \exp\Big\{(1+\varepsilon) \sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \\ &\quad \cdot \Big(1 + \rho^2(k) \log^4 \Big[\frac{n}{k}\Big]\Big) \max_{l<i\leq l+n} EX_i^2 I(|X_i| > B). \end{aligned} \tag{4.3.28}$$

And from Lemma 2.2.2

$$\begin{aligned} I_{23} &\leq \sum_{o\leq i\leq [n/k]} E \max_{ik+1\leq j\leq (i+1)k} \Big| \sum_{v=l+ik+1}^{l+j} X_{v2}\Big|^2 \\ &\leq 32 \sum_{0\leq i\leq [n/k]} \Big(E\Big| \sum_{v=l+ik+1}^{l+(i+1)k} \{|X_v| I(|X_v| \geq B) \\ &\quad - E|X_v| I(|X_v| \geq B)\}\Big|^2 \\ &\quad + k^2 \max_{l+ik+1\leq j\leq l+(i+1)k} (E|X_j| I(|X_j| \geq B)^2\Big) \\ &\leq c\Big\{n \exp\Big\{(1+\varepsilon) \sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \max_{l<i\leq l+n} EX_i^2 I(|X_i| \geq B) \\ &\quad + nk \max_{l<i\leq l+n} (E|X_i| I(|X_i| \geq B))^2\Big\}. \end{aligned} \tag{4.3.29}$$

Inserting (4.3.26)-(4.3.29) into (4.3.24) we prove (4.3.23), as desired.

**Proof of Theorem 4.3.1.**

By the condition (ii), we need only to show that

(a) $\{S_{[nt]}^2/\sigma_n^2, n \geq 1\}$ is uniformly integrable for any $0 \leq t \leq 1$.

(b) For any given $\varepsilon > 0$, there exist a $\lambda > 1$ and an integer $n_0$ such that for $n \geq n_0, 0 \leq k \leq n\lambda^2/\varepsilon$ we have

$$P\Big\{\max_{1\leq i\leq n} \Big| \sum_{j=k+1}^{k+i} X_j\Big| \geq \lambda\sigma_n\Big\} \leq \varepsilon/\lambda^2. \tag{4.3.30}$$

To this end, we need only to show that for any given $\varepsilon > 0$ there exist a $\lambda > 1$ and an integer $n_0$ such that for every $n \geq n_0, l \geq 0$

$$E \max_{1\leq i\leq n} S_l^2(i) I(\max_{1\leq i\leq n} |S_l(i)| \geq \lambda (nh(n))^{1/2})/nh(n) \leq \varepsilon. \tag{4.3.31}$$

Without loss of generality we assume that for $n \geq 16$

$$\rho(n) \geq 1/(\log n (\log\log n)^2). \tag{4.3.32}$$

In fact, if put $\rho^*(n) = \rho(n) \vee (\log n)^{-1}(\log\log n)^{-2}$, it is easy to check that $\rho^*(n)$ satisfies condition (v) also.

By conditions (i), (ii), (iii) and Lemma 4.3.3, there exists a constant $K$ such that for every $n, 1 \leq k \leq n, B > 0$ and $\lambda > 0$, we have

$$\begin{aligned}
&E \max_{1\leq i\leq n} S_l^2(i) I(\max_{1\leq i\leq n} |S_l(i)| \geq \lambda (nh(n))^{1/2})/nh(n) \\
&\leq K\Big\{\Big(\{nh(n) + n\exp\Big(\Big(1+\frac{\varepsilon}{4}\Big)\sum_{i=0}^{[\log n]} \rho(2^i)\Big) \\
&\quad \times \max_{l<i\leq l+n} EX_i^2 I(|X_i| \geq B)\}^{\frac{2+\delta}{2}} \\
&\quad + n\exp\Big(K\sum_{i=0}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big)(\log(2n))^{2+\delta} \\
&\quad \times \max_{l<i\leq l+n} E|X_i|^{2+\delta} I(|X_i| < B)\Big)\Big/\big(\lambda^\delta (nh(n))^{\frac{2+\delta}{2}}\big) \\
&\quad + n\exp\Big((1+\frac{\varepsilon}{4})\sum_{i=0}^{[\log n]} \rho(2^i)\Big) \max_{l<i\leq l+n} EX_i^2 I(|X_i| \geq B) \\
&\quad \times \Big(1 + \rho^2(k)\log^4\Big[\frac{n}{k}\Big]\Big)/nh(n) \\
&\quad + nk \max_{l<i\leq l+n} (E|X_i| I(|X_i| \geq B))^2/nh(n)\Big\} \\
&=: K(I_1 + I_2 + I_3).
\end{aligned} \tag{4.3.33}$$

Put

$$T = \exp\Big\{\frac{3K}{\delta}\sum_{i=0}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big\},$$
$$B = n^{1/2}/T, \qquad k = [n/T^2] + 1.$$

We first estimate $I_1$. Note that

$$\begin{aligned}
\log T &= \frac{3K}{\delta}\sum_{i=0}^{[\log n]} \rho^{2/(2+\delta)}(2^i) \\
&\leq \frac{3K}{\delta}\rho^{-\delta/(2+\delta)}(\frac{n}{T^2})\sum_{i=0}^{[\log nT^{-2}]} \rho(2^i) \\
&\quad + \frac{3K}{\delta}\rho^{2/(2+\delta)}(\frac{n}{T^2})\sum_{i=[\log nT^{-2}]+1}^{[\log n]} 1 \\
&\leq \frac{3K}{\delta}\rho^{-\delta/(2+\delta)}(\frac{n}{T^2})\sum_{i=0}^{[\log nT^{-2}]} \rho(2^i) \\
&\quad + \frac{7K}{\delta}\rho^{2/(2+\delta)}(\frac{n}{T^2})\log T.
\end{aligned}$$

Therefore for large $n$ we have

$$\log T \leq \frac{3K}{\delta}\Big(1+\frac{\varepsilon}{24}\Big)\rho^{-\delta/(2+\delta)}\Big(\frac{n}{T^2}\Big)\sum_{i=0}^{[\log nT^{-2}]} \rho(2^i) \tag{4.3.34}$$

and

$$\sum_{i=0}^{[\log n]} \rho(2^i) \leq \Big(1+\frac{\varepsilon}{12}\Big)\sum_{i=0}^{[\log nT^{-2}]} \rho(2^i). \tag{4.3.35}$$

By condition (v), there exists a $C_1 > 0$ such that for every $n \geq 1$

$$g(n^{1/2}) \geq C_1 \exp\Big\{(2+\varepsilon)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\}. \tag{4.3.36}$$

Combining it with (4.3.35) and condition (v), we obtain

$$g(B) \geq C_1 \exp\Big\{\frac{2+\varepsilon}{1+\varepsilon/12}\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \tag{4.3.37}$$

for large $n$. By Lemma 2.2.3 and condition (iii), there exists a constant $C_2 > 0$ such that for large $n$

$$ES_n^2 \geq C_2 n \exp\Big\{-\Big(1+\frac{\varepsilon}{4}\Big)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\},$$

and hence, by condition (ii), we have

$$h(n) \geq C_2 \exp\Big\{-\Big(1+\frac{\varepsilon}{4}\Big)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\}. \tag{4.3.38}$$

By the monotonicity of $g(x)$ and $x^\delta/g(x)$, we obtain

$$\max_{i\geq 1} EX_i^2 I(|X_i|\geq B) \leq \frac{1}{g(B)} \max_{i\geq 1} EX_i^2 g(|X_i|) I(|X_i|\geq B), \tag{4.3.39}$$

$$\max_{i\geq 1} E|X_i|^{2+\delta} I(|X_i|\leq B) \leq \frac{B^\delta}{g(B)} \max_{i\geq 1} EX_i^2 g(|X_i|). \tag{4.3.40}$$

Combining (4.3.37), (4.3.39),(4.3.40) and (4.3.32) together implies

$$\begin{aligned} & n\exp\Big\{\Big(1+\frac{\varepsilon}{4}\Big)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \max_{i\geq 1} EX_i^2 I(|X_i|\geq B) \\ & \leq \frac{nh(n)}{C_1C_2} \max_{i\geq 1} EX_i^2 g(|X_i|) I(|X_i|\geq B), \end{aligned} \tag{4.3.41}$$

$$\begin{aligned} & n\exp\Big\{K\sum_{i=0}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big\}\log^{2+\delta}(2n) \max_{i\geq 1} E|X_i|^{2+\delta} I(|X_i|<B) \\ & \leq n\exp\Big\{2K\sum_{i=0}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big\} B^\delta \max_{i\geq 1} EX_i^2 g(|X_i|)/g(B) \\ & \leq n^{(2+\delta)/2} \max_{i\geq 1} EX_i^2 g(|X_i|) \exp\Big\{-K\sum_{i=0}^{[\log n]} \rho^{2/(2+\delta)}(2^i)\Big\} \\ & \leq (nh(n))^{(2+\delta)/2} \max_{i\geq 1} EX_i^2 g(|X_i|) \end{aligned} \tag{4.3.42}$$

for large $n$. We have $\max_{i\geq 1} EX_i^2 g(|X_i|) < \infty$ by condition (i). Therefore we obtain

$$I_1 \leq \varepsilon/(3K) \tag{4.3.43}$$

for large $n$ provided that $\lambda$ is large enough.

We now estimate $I_2$. From (4.3.34) and the definition of $k$, we have

$$\begin{aligned} & (\log[\frac{n}{k}])^4 \rho^2(k) \leq 2(\log T)^4 \rho^2(k) \\ & \leq \Big(\frac{3K}{\delta}\Big)^4 \Big(1+\frac{\varepsilon}{24}\Big)^4 \Big(\rho(\frac{n}{T^2})\Big)^{2-\frac{4\delta}{2+\delta}} \Big(\sum_{i=0}^{[\log n]} \rho(2^i)\Big)^4 \\ & \leq \Big(\frac{3K}{\delta}\Big)^4 \Big(\sum_{i=0}^{[\log n]} \rho(2^i)\Big)^4. \end{aligned} \tag{4.3.44}$$

By (4.3.44),(4.3.39),(4.3.37) and (4.3.38), we get

$$\begin{aligned}
& n \exp\Big\{\Big(1+\frac{\varepsilon}{4}\Big)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \\
& \max_{i\ge 1} EX_i^2 I(|X_i|\ge B)\Big(1+\rho^2(k)\log^4\Big[\frac{n}{k}\Big]\Big) \\
& \quad \le \frac{nh(n)}{g(B)h(n)} \exp\Big\{\Big(1+\frac{\varepsilon}{4}\Big)\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \\
& \qquad \cdot \Big(1+(\frac{3K}{\delta})^4\Big(\sum_{i=0}^{[\log n]} \rho(2^i)\Big)^4\Big) \max_{i\ge 1} EX_i^2 g(|X_i|) I(|X_i|\ge B) \\
& \quad \le \frac{nh(n)}{C_1C_2} \exp\Big\{-\frac{\varepsilon}{8}\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} \\
& \qquad \cdot \Big(1+\Big(\frac{3K}{\delta}\Big)^4\Big(\sum_{i=0}^{[\log n]} \rho(2^i)\Big)^4\Big) \max_{i\ge 1} EX_i^2 g(|X_i|) I(|X_i|\ge B) \\
& \quad \le cnh(n) \max_{i\ge 1} EX_i^2 g(|X_i|) I(|X_i|\ge B), \qquad (4.3.45)
\end{aligned}$$

where the following result is used:

$$\Big(\sum_{i=0}^{[\log n]} \rho(2^i)\Big)^4 \exp\Big\{-\frac{\varepsilon}{8}\sum_{i=0}^{[\log n]} \rho(2^i)\Big\} = O(1).$$

Combining (4.3.45) with condition (i), we have

$$I_2 \le \varepsilon/3K. \qquad (4.3.46)$$

Finally, we consider $I_3$. Note that

$$\max_{i\ge 1} E|X_i| I(|X_i|\ge B) \le \frac{1}{Bg(B)} \max_{i\ge 1} EX_i^2 g(|X_i|) I(|X_i|\ge B).$$

Hence it follows from (4.3.37) and (4.3.38) that

$$\begin{aligned}
& kn \max_{i\ge 1}(E|X_i| I(|X_i|\ge B))^2 \\
& \quad \le \frac{kn}{B^2 g^2(B)} \max_{i\ge 1}(EX_i^2 g(|X_i|) I(|X_i|\ge B))^2 \\
& \quad \le \frac{2nh(n)}{C_1^2 C_2} \max_{i\ge 1}(EX_i^2 g(|X_i|) I(|X_i|\ge B))^2.
\end{aligned}$$

By condition (i), we have

$$I_3 \le \varepsilon/3K \tag{4.3.47}$$

for large $n$. Inserting (4.3.43),(4.3.46) and (4.3.47) into (4.3.33) we prove (4.3.31). The proof of Theorem 4.3.1 is completed.

From Theorem 4.3.1, we have the following corollary immediately.

**Corollary 4.3.1.** *Let $\{X_n, n \ge 1\}$ be a strictly stationary $\rho$-mixing sequence of random variables with $EX_1 = 0, EX_1^2 < \infty, \sigma_n^2 = ES_n^2 \to \infty$. If one of the following conditions is statisfied:*

*(i)* $EX_1^2 g(|X_1|) < \infty$, *and for some* $0 < \varepsilon < 1$,

$$\exp\Big((2+\varepsilon)\sum_{k=1}^{[\log n]} \rho(2^j)\Big) = O(g(n^{1/2})),$$

*(ii) for some* $\varepsilon > 0, a > 0, EX_1^2(\log|X_1|)^{2a/(1-\varepsilon)} < \infty$ *and x* $\rho(n) \le a/\log n$,

*(iii) for some* $0 < \beta < 1, \varepsilon > 0, a > 0$,

$$EX_1^2 \exp\Big\{\frac{2a(1+\varepsilon)}{1-\beta}(2\log|X_1|)^{1-\beta}\Big\} < \infty$$

*and*

$$\rho(n) \le a/(\log n)^\beta,$$

*(iv) for some* $r > 0, \varepsilon > 0, a > 0$,

$$EX_1^2 \exp\Big(\frac{4a(1+\varepsilon)\log|X_1|}{(\log\log|X_1|)^r}\Big) < \infty$$

$$\rho(n) \le a(\log\log n)^{-r},$$

*then*

$$W_n \Longrightarrow W.$$

## 4.4 The WIP when the variance is infinite

Bradley (1988) established a CLT for a strictly stationary $\rho$-mixing sequence with infinite variance. Shao (1993a) showed a WIP under the same hypothesis.

**Theorem 4.4.1.** *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence of non-degenerate random variables with $EX_1 = 0$. Suppose that*

*(i) $H(x) := EX_1^2 I(|X_1| \leq x)$ is slowly varying as $x \to \infty$,*

*(ii) $\rho(1) < 1$,*

*(iii) $\sum_{n=1}^{\infty} \rho(2^n) < \infty$.*

*Then there exists a sequence $\{A_n, n \geq 1\}$ of positive numbers with $A_n \to \infty$ as $n \to \infty$, such that*

$$W_n \Longrightarrow W$$

*where $W_n(t) = S_{[nt]}/A_n, \quad 0 \leq t \leq 1$.*

**Remark 4.4.1.** In fact, Shao (1993a) showed a more general result as follows: Let $g : (-\infty, \infty) \to [0, \infty)$ be a non-decreasing continuous even function and $x^\delta/g(x)$ is non-decreasing for any $\delta > 0$ and $x$ large enough. Let

$$e(x, \varepsilon) = \exp\Big\{\varepsilon \sum_{i=1}^{[\log x]} \rho(2^i)\Big\}, \quad x(\delta) = \exp\Big\{\sum_{i=0}^{[\log x]} \rho^{1-\delta}(2^i)\Big\}.$$

Suppose that conditions (i) and (ii) in Theorem 4.4.1 are satisfied and

(i)′ $G(x) := EX_1^2 g(X_1) I(|X_1| \leq x)$ is slowly varying as $x \to \infty$,

(iv) $G(x)e(x^2, 2+\varepsilon) = O(H(x)g(x))$ for some $0 < \varepsilon < 1$,

(v) $g(x) = O(g(x/x(\delta)))$ or $G(x) = O(G(x/x(\delta)))$ for some $0 < \delta < 1$

as $x \to \infty$.

Then we also have the conclusion of Theorem 4.4.1.

Obviously, conditions (i) and (iii) imply condition (i)′, (iv) and (v) by taking $g(x) \equiv 1$. It is well-known that the mixing rate (iii) is essentially sharp, even in the case of a finite second moment. However the following example is interesting: Let $X_1$ have the density function $p(x) = a(1 + |x|^3)^{-1}$ for $x \in R^1$, where $a^{-1} = \int_{-\infty}^{\infty}(1+|x|^3)^{-1}dx$. Let $g(x) = \exp(\log(1+|x|^3)^\alpha)$ for some $0 < \alpha < 1$. It is easy to see that as $x \to \infty$

$$H(x) \sim 2a \log(1 + |x|^3)/3,$$
$$G(x) \sim 2a(\log(1 + |x|^3))^{1-\alpha} g(x)/3\alpha.$$

If $\rho(n) \leq \alpha/(5 \log n)$, we can easily verify that the conditions in Remark 4.4.1 are satisfied but the condition (iii) in Theorem 4.4.1 fails. Hence condition (iii) may be not essentially sharp in some particular case of infinite variance, even of finite variance.

In order to prove Theorem 4.4.1, we introduce some notations. Let $M^*$ be a positive integer such that

$$\sup_{x>0} H(x)/x^2 > 1/M^*. \tag{4.4.1}$$

For each $n \geq M^*$ define

$$t_n = \sup\{x > 0 : H(x)/x^2 \geq 1/n\}. \tag{4.4.2}$$

It is clear that $t_n \to \infty$ monotonically as $n \to \infty$. Note by a trivial argument that

$$t_n^2 = nH(t_n) \qquad \text{for} \quad n \geq M^*.$$

By condition (i), for any $0 < \varepsilon < 1/2$ and large $n$

$$n^{1-\varepsilon} \leq t_n^2 \leq n^{1+\varepsilon}. \tag{4.4.3}$$

We need a few properties of these $t_n$'s.

**Lemma 4.4.1.** *If condition (iii) is satisfied, for any* $0 < a < 1$,

$$t_{[na]}^2/t_n^2 \longrightarrow a \qquad \text{as } n \to \infty. \tag{4.4.4}$$

**Proof.** (4.4.2) implies

$$t_n^2 H(t_{[na]})/(t_{[na]}^2 H(t_n)) \longrightarrow 1/a \qquad \text{as } n \to \infty. \tag{4.4.5}$$

We show that there is a $M > 0$ such that

$$\limsup_{n\to\infty} t_n^2/t_{[na]}^2 \leq M. \tag{4.4.6}$$

In fact, if not, there is a subsequence $n_k$ such that $\lim_{k\to\infty} t_{n_k}^2/t_{[n_k a]}^2 = \infty$. Then using Property A5 of a slowly varying function (see Appendix) we obtain

$$\lim_{k\to\infty} t_{n_k}^2 H(t_{[n_k a]})/(t_{[n_k a]}^2 H(t_{[n_k a]} t_{n_k}/t_{[n_k a]})) = \infty,$$

which is contrary to (4.4.5). By (4.4.2) and (4.4.6), it follows that

$$n/[na] \leq t_n^2/t_{[na]}^2 \leq nH(Mt_{[na]})/([na]H(t_{[na]})),$$

which implies (4.4.4).

**Lemma 4.4.2.**

$$\lim_{n\to\infty} nP(|X_1| > t_n) = 0$$

*and*

$$\lim_{n\to\infty} (n/H(t_n))^{1/2} E|X_1| I(|X_1| > t_n) = 0.$$

The proof of this lemma can be found in Bradley (1988) and will be not presented here.

**Proof of Theorem 4.4.1.**
For some $\varepsilon > 0$, put $C_1 = C\exp\Big\{(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}$, where $C$ is defined in Lemma 2.2.2 and $C_2 = C'\exp\Big\{-(1+\varepsilon)\sum_{i=0}^{[\log n]}\rho(2^i)\Big\}$, where $C'$ is defined in Lemma 2.2.3. For $n \geq M^*$ define

$$X_k^{(n)} = X_k I(|X_k| \leq t_n) - EX_k I(|X_k| \leq t_n), \qquad k \geq 1 \tag{4.4.7}$$

and

$$S_m^{(n)} = X_1^{(n)} + \cdots + X_m^{(n)}, \qquad m \geq 1.$$

By the definitions of $C_1$ and $C_2$ we have

$$C_2 m E{X_1^{(n)}}^2 \leq E(S_m^{(n)})^2 \leq C_1 m E(X_1^{(n)})^2. \tag{4.4.8}$$

Put $A_n(m) = \|S_m^{(n)}\|_2$ and $A_n = A_n(n)$. Note that $EX_1 = 0$, it is easy to verify that

$$E(X_1^{(n)})^2 \sim H(t_n) \qquad \text{as} \quad n \to \infty. \tag{4.4.9}$$

Therefore there exist $0 < C_2' < C_1' < \infty$ such that

$$C_2' n H(t_n) \leq A_n^2 \leq C_1' n H(t_n). \tag{4.4.10}$$

Next we formulate the proof in two steps.

**Step 1.** We prove that $S_n^{(n)}/A_n \xrightarrow{d} N(0,1)$ in distribution as $n \to \infty$. It suffices to show that

$$\lim_{n\to\infty} E\{\exp(itS_n^{(n)}/A_n)\} = \exp(-t^2/2) \qquad \text{for any} \quad t \in R. \tag{4.4.11}$$

Since the case of $t = 0$ is trivial, it needs only to prove the above equality for $t \neq 0$. Fix $t \neq 0$. Let $J$ be a positive integer specified later on. Define $p^*$ and $L^*$ to be positive integers such that

$$\frac{4C_1}{C_2 p^*}\sum_{j=1}^{\infty} 2^{J-j/2} \leq \frac{\varepsilon^2}{4}, \tag{4.4.12}$$

$$|(1 - t^2/(2j))^j - \exp(-t^2/2)| \leq \varepsilon/3 \qquad \text{for each} \quad j \geq 2^{L^*}. \tag{4.4.13}$$

Let $N^* \geq M^*$ be a positive integer such that

$$N^* \geq 2p^* \cdot 2^{L^*}, \tag{4.4.14}$$

$$E(X_1^{(n)})^2 > 0 \qquad \text{for each} \quad n \geq N^*, \tag{4.4.15}$$

and

$$\begin{aligned}&\frac{128p^{*2}t^2}{C_2nE(X_1^{(n)})^2}(EX_1I(|X_1|\le t_n))^2\\&\quad+\frac{400p^{*4}(t^2\vee|t|^3)}{C_2\wedge C_2^{3/2}}E\Big\{\frac{X_1^2I(|X_1|\le t_n)}{nE(X_1^{(n)})^2}\wedge\frac{|X_1|^3I(|X_1|\le t_n)}{(nE(X_1^{(n)})^2)^{3/2}}\Big\}\\&\le\varepsilon/(3n)\qquad\text{for each}\quad n\ge N^*.\end{aligned}\tag{4.4.16}$$

Here, (4.4.16) can be justified by (4.4.9).

Let $N\ge N^*$ be an arbitrary but fixed integer. Then to prove (4.4.11) it suffices to show that

$$|E\exp(itS_N^{(N)}/A_N)-\exp(-t^2/2)|\le\varepsilon.\tag{4.4.17}$$

Referring to (4.4.14), let $L$ be the positive integer such that

$$p^*\le N/2^L\le 2p^*.$$

Note that $L\ge L^*$. Let $p$ be the positive integer such that

$$p2^L\le N<(p+1)2^L.\tag{4.4.18}$$

It is easy to see that

$$p^*\le p<2p^*.\tag{4.4.19}$$

Let us partition $\mathbb{N}$ into disjoint blocks of consecutive integers, leaving no gaps between the blocks. The order of the blocks is $G(1),G(2),\cdots$, with

$$\text{Card}G(j)=\begin{cases}p, & \text{if}\quad j\quad\text{is odd;}\\ [2^{J+l/2}], & \text{where}\quad l\quad\text{is such integer that}\quad j/2^l\\ & \quad\text{is an odd integer if } j \text{ is even.}\end{cases}\tag{4.4.20}$$

Henceforth we shall deal only with the blocks $G(1),G(2),\cdots,G(2^{L+1}-1)$. For each $l=1,\cdots,L$, there are exactly $2^{L-l}$ integers $j\in\{1,2,\cdots,2^{L+1}-1\}$ such that $j/2^l$ is an odd integer. Therefore

$$\text{Card}\{G(2)\cup G(4)\cup\cdots\cup G(2^{L+1}-2)\}=\sum_{l=1}^{L}2^{L-l}[2^{J+l/2}].\tag{4.4.21}$$

Hence by (4.4.18)

$$\begin{aligned}N&\le 2^Lp+\sum_{l=1}^{L}2^{L-l}\le 2^Lp+\sum_{l=1}^{L}2^{L-l}[2^{J+l/2}]\\&=\text{Card}\{G(1)\cup G(2)\cup\cdots\cup G(2^{L+1}-1)\}\\&\le N+\sum_{l=1}^{L}2^{L-l}[2^{J+l/2}].\end{aligned}\tag{4.4.22}$$

For each $j = 1, 2, \cdots, 2^{L+1} - 1$ define

$$U(j) = \sum_{k \in G(j)} X_k^{(N)}.$$

And further, for even integers $j_1$ and $j_2$ such that $0 \le j_1 < j_2 \le 2^{L+1}$, define

$$V(j_1, j_2) = U(j_1 + 1) + U(j_1 + 3) + \cdots + U(j_2 - 1).$$

If $1 \le j \le 2^l - 1$, then for the integer $m$ such that $j/2^m$ is an odd integer, we have that $m < l$ and hence $(2^l + j)/2^m$ is an odd integer, and hence $\mathrm{Card}G(2^l + j) = \mathrm{Card}G(j)$ by (4.4.20). Consequently, if we denote $u = \mathrm{Card}\{G(1) \cup G(2) \cup \cdots \cup G(2^l)\}$ and use the notation $u + G = \{u + g : g \in G\}$ for sets $G$ of positive integers, we have (by induction on $j$) that $G(2^l + j) = u + G(j)$ for $j = 1, 2, \cdots, 2^l - 1$. In particular, if we denote

$$G = G(1) \cup G(3) \cup \cdots \cup G(2^l - 1)$$

and

$$G^* = G(2^l + 1) \cup G(2^l + 3) \cup \cdots \cup G(2^{l+1} - 1),$$

then $G^* = u + G, V(2^l, 2^{l+1}) = \sum_{k \in G^*} X_k^{(N)} = \sum_{k \in G} X_{k+u}^{(N)}$ and $V(0, 2^l) = \sum_{k \in G} X_k^{(N)}$. The stationarity of the sequence $\{X_k^{(N)}, k \ge 1\}$ implies the following useful property:

For each $l = 1, \cdots, L$, $V(0, 2^l)$ and $L(2^l, 2^{l+1})$ have the same distribution. (4.4.23)

Hence by a simple calculation for each $l = 1, \cdots, L$,

$$\begin{aligned} &2(1 - \rho([2^{J+l/2}]))EV(0, 2^l)^2 \\ &\le EV(0, 2^{l+1})^2 \\ &\le 2(1 + \rho([2^{J+l/2}]))EV(0, 2^l)^2. \end{aligned} \tag{4.4.24}$$

Moreover for each $l = 1, \cdots, L$

$$\begin{aligned} &|E\exp\{itV(0, 2^{l+1})\} - (E\exp\{itV(0, 2^l)\})^2| \\ &\le \rho([2^{J+l/2}])E|\exp\{itV(0, 2^l)\} - 1|^2 \\ &\le \rho([2^{J+l/2}])t^2 EV(0, 2^l)^2 \\ &\le \rho([2^{J+l/2}])t^2 C_1 2^{l-1} EU(1)^2. \end{aligned} \tag{4.4.25}$$

In what follows, it should be kept in mind that $E(S_m^{(N)})^2 > 0$ for all $m \geq 1$ by (4.4.15), the fact $N \geq N^*$ and (4.4.8). By (4.4.24) and induction

$$2^L\Big\{\prod_{l=1}^{L}(1-\rho([2^{J+l/2}]))\Big\}EU_1^2 \\ \leq EV(0,2^{L+1})^2 \leq 2^L\Big\{\prod_{l=1}^{L}(1+\rho([2^{J+l/2}]))\Big\}EU_1^2$$

and hence

$$1-\varepsilon/2 \leq \|V(0,2^{L+1})\|_2/(2^{L/2}\|U_1\|_2) \leq 1+\varepsilon/2 \tag{4.4.26}$$

provided $J$ is large enough. By (4.4.21) and (4.4.8)

$$E(U(2)+U(4)+\cdots+U(2^{L+1}-2))^2 \\ \leq C_1\Big(\sum_{l=1}^{L}2^{L+J-l/2}\Big)E(X_1^{(n)})^2.$$

Also by (4.4.22), $U(1)+U(2)+\cdots+U(2^{L+1}-1)-S_N^{(N)}$ is the sum of at most $[\sum_{l=1}^{L}2^{L+J-l/2}]$ distinct $X_k^{(N)}$'s, and hence

$$E(U(1)+U(2)+\cdots+U(2^{L+1}-1)-S_N^{(N)})^2 \\ \leq C_1\Big(\sum_{l=1}^{L}2^{L+J-l/2}\Big)E(X_1^{(N)})^2.$$

Consequently, using (4.4.8), (4.4.12) and (4.4.19)

$$\begin{aligned}
&\|V(0,2^{L+1})-S_N^{(N)}\|_2 \\
&\leq \|U(1)+U(2)+\cdots+U(2^{L+1}-1)-S_N^{(N)}\|_2 \\
&\quad + \|U(2)+U(4)+\cdots+U(2^{L+1}-2)\|_2 \\
&\leq 2C_1^{1/2}\Big(\sum_{l=1}^{L}2^{L+J-l/2}\Big)^{1/2}\|X_1^{(N)}\|_2 \\
&\leq 2C_1^{1/2}2^{L/2}\Big(\sum_{l=1}^{L}2^{J-l/2}\Big)^{1/2}\|U(1)\|_2/(C_2p)^{1/2} \\
&\leq 2^{L/2}\varepsilon\|U(1)\|_2/2.
\end{aligned} \tag{4.4.27}$$

We now come back to (4.4.17). By (4.4.26) and (4.4.27) we have

$$\begin{aligned}
&\left|1-\frac{A_N}{2^{L/2}\|U(1)\|_2}\right| \\
&\leq \left|1-\frac{\|V(0,2^{L+1})\|_2}{2^{L/2}\|U(1)\|_2}\right| + \left|\frac{\|V(0,2^{L+1})\|_2 - \|S_N^{(N)}\|_2}{2^{L/2}\|U(1)\|_2}\right| \\
&\leq \frac{\varepsilon}{2} + \frac{\|V(0,2^{L+1}) - S_N^{(N)}\|_2}{2^{L/2}\|U(1)\|_2} \\
&\leq \varepsilon. \qquad (4.4.28)
\end{aligned}$$

(4.4.28) and (4.4.27) together imply that (4.4.17) is equivalent to

$$D := |E\exp\{itV(0,2^{L+1})/(2^{L/2}\|U(1)\|_2\} - \exp(-t^2/2)| \leq \varepsilon. \qquad (4.4.29)$$

Obviously

$$\begin{aligned}
D \leq\ & |E\exp\{itV(0,2^{L+1})/(2^{L/2}\|U(1)\|_2)\} \\
& - (E\exp\{i(t/2^{L/2})U(1)/\|U(1)\|_2\})^{2^L}| \\
& + |(E\exp\{i(t/2^{L/2})U(1)/\|U(1)\|_2\})^{2^L} - (1-(1/2)t^2/2^L)^{2^L}| \\
& + |(1-(1/2)t^2/2^L)^{2^L} - \exp(-t^2/2)| \\
=:\ & e_1 + e_2 + e_3.
\end{aligned}$$

Using (4.4.25) and the elementary inequality

$$\left|\prod_{k=1}^m y_k - \prod_{k=1}^m z_k\right| \leq \sum_{k=1}^m |y_k - z_k|, \qquad (4.4.30)$$

where $y_1, \cdots, y_m, z_1 \cdots, z_m$ are complex numbers in the closed unit disc, we have

$$\begin{aligned}
&\left|\Big(E\exp\{iTV(0,2^{l+1})\}\Big)^{2^{L-l}} - \Big(E\exp\{iTV(0,2^l)\}\Big)^{2^{L-l+1}}\right| \\
&\quad \leq 2^{L-l}\rho([2^{J+l/2}])T^2C_1 2^{l-1}EU(1)^2
\end{aligned}$$

for any $T$. Hence by induction

$$\begin{aligned}
&|E\exp\{iTV(0,2^{L+1})\} - (E\exp\{iTV(0,2)\})^{2^L}| \\
&\quad \leq 2^L T^2 C_1 EU(1)^2 \sum_{l=1}^{L} \rho([2^{J+l/2}]).
\end{aligned}$$

Letting $T = t/(2^{L/2}\|U(1)\|_2)$ and keeping in mind that $U(1) = V(0,2)$, we have

$$e_1 \le t^2 C_1 \sum_{l=1}^{L} \rho([2^{J+l/2}]) \le \varepsilon/3 \tag{4.4.31}$$

provided the constant $J$ is large enough. In order to estimate $e_2$, define the event

$$F_k = \{|X_k^{(N)}| = \max_{1\le j\le p} |X_j^{(N)}|\}, k = 1, \cdots, p.$$

Put $s = t/(2^{L/2}\|U(1)\|_2)$ for simplicity. By (4.4.8) and (4.4.18),

$$s^2 \le t^2/(2^L C_2 p E(X_1^{(N)})^2) \le 2t^2/(C_2 N E(X_1^{(N)})^2).$$

Now we need a fact that for any real numbers $x$ and $r$,

$$|x-r|^2 \wedge |x-r|^3 \le 4r^2 + 8(x^2 \wedge |x|^3).$$

Using this fact and (4.4.19), (4.4.16) and (4.4.18) we have

$$\begin{aligned}
&E(|sU(1)|^2 \wedge |sU(1)|^3) \\
&\le \sum_{k=1}^{p} EI(F_k)(|spX_k^{(N)}|^2 \wedge |spX_k^{(N)}|^3) \\
&\le p^4 E(|sX_1^{(N)}|^2 \wedge |sX_1^{(N)}|^3) \\
&\le p^4\{4s^2(EX_1 I(|X_1| \le t_N))^2 \\
&\quad + 8E\{(s^2X_1^2 I(|X_1| \le t_N)) \wedge (|s|^3|X_1|^3 I(|X_1| \le t_N))\}\} \\
&\le (2p^*)^4\Big\{\frac{8t^2}{C_2NE(X_1^{(N)})^2}(EX_1I(|X_1| \le t_N))^2 \\
&\quad + 8E\Big\{\frac{2t^2X_1^2I(|X_1| \le t_N)}{C_2NE(X_1^{(N)})^2} \wedge \frac{2^{3/2}|t|^3|X_1|^3I(|X_1| \le t_N)}{C_2^{3/2}N^{3/2}(E(X_1^{(N)})^2)^{3/2}}\Big\}\Big\} \\
&\le \frac{128p^{*4}t^2}{C_2NE(X_1^{(N)})^2}(EX_1I(|X_1| \le t_n))^2 \\
&\quad + \frac{400p^{*4}(t^2 \vee |t|^3)}{C_2 \wedge C_2^{3/2}} E\Big\{\frac{X_1^2 I(|X_1| \le t_N)}{NE(X_1^{(N)})^2} \wedge \frac{|x_1|^3 I(|X_1| \le t_N)}{N^{3/2}(E(X_1^{(N)})^2)^{3/2}}\Big\} \\
&\le \varepsilon/(3N) \le \varepsilon/(3\cdot 2^L).
\end{aligned}$$

Hence noting (4.4.30) we obtain

$$\begin{aligned}
e_2 &\le 2^L|E\exp(isU(1)) - (1 - \frac{1}{2}s^2EU(1)^2)| \\
&\le 2^L E(|sU(1)|^2 \wedge |sU(1)|^3) \le \varepsilon/3.
\end{aligned} \tag{4.4.32}$$

Here we use the inequality about a characteristic function (cf. p.331 in Bradley 1988).

As for $e_3$, by (4.4.13) and noting $L \geq L^*$, it is clear that

$$e_3 \leq \varepsilon/3. \tag{4.4.33}$$

(4.4.31), (4.4.32) and (4.4.33) together imply (4.4.29). This completes the proof of the CLT.

**Step 2.** Now we show that

$$W_n \Longrightarrow W \qquad \text{as} \quad n \to \infty. \tag{4.4.34}$$

By Lemma 4.4.2 and (4.4.10) we have

$$\lim_{n\to\infty} P\{\sup_{0\leq t\leq 1} |S_{[nt]} - S_{[nt]}^{(n)}| > \varepsilon A_n\} = 0 \qquad \text{for any} \quad \varepsilon > 0.$$

Hence, in order to prove the theorem it suffices (cf. Theorem 4.1 in Billingsley 1968) to show that

$$W_n^* \Longrightarrow W \qquad \text{as} \quad n \to \infty \tag{4.4.35}$$

where $W_n^*(t) = S_{[nt]}^{(n)}/A_n$. By Theorem 4.0.4, it is enough to prove that for any $0 < t \leq 1$

$$A_n^2([nt])/A_n^2 \longrightarrow t, \qquad \text{as} \quad n \to \infty \tag{4.4.36}$$

$$\{W_n^*(t)^2, n \geq 1\} \quad \text{is uniformly integrable,} \tag{4.4.37}$$

and there exists a constant $\lambda > 1$ for any $\varepsilon > 0$ such that for all large $n$

$$P\Big\{\max_{0\leq i\leq n} |S_i^{(n)}| \geq \lambda A_n\Big\} < \varepsilon/\lambda^2. \tag{4.4.38}$$

We prove that

$$A_n^2([mt])/A_n^2(m) \longrightarrow t \tag{4.4.39}$$

as $m \to \infty$ uniformly in $n \geq M^*$, which implies (4.4.36). At first consider the case of $t = 1/p$ where $p \geq 2$ is an integer. Let $q = [m/p]$,

$$Y_i = \sum_{j=iq+1}^{(i+1)q} X_j^{(n)}, \qquad i = 0, 1, \cdots, p-1,$$

$$Y_p = \sum_{j=pq+1}^{m} X_j^{(n)}.$$

Note that

$$A_n^2(m) = pA_n^2(q) + \sum_{i \neq j} EY_iY_j + EY_p^2.$$

For $i \neq j$ and integer $k \geq 1$, by the Minkowski inequality

$$|EY_iY_j| \leq 2\|Y_i\|_2\|S_k^{(n)}\|_2 + \rho(k)\|Y_i\|_2\|Y_j\|_2.$$

Hence by (4.4.8)

$$\begin{aligned}|A_n^2(m)/A_n^2(q) - p| &\leq (p+1)^2(\|S_k^{(n)}\|_2/A_n(q) + \rho(k) + \|Y_p\|_2^2/A_n^2(q)) \\ &\leq (p+1)^2((k^2+p^2)m^{-1/3} + \rho(k)) \qquad (4.4.40)\end{aligned}$$

for every $m$ large enough. Choose $k$ such that $(p+1)^2\rho(k) < \varepsilon/2$. Then for $m$ large enough

$$|A_n^2(m)/A_n^2([m/p]) - p| < \varepsilon$$

uniformly in $n \geq M^*$. Therefore as $m \to \infty$ uniformly in $n \geq M^*$

$$A_n^2(m)/A_n^2([m/p]) \longrightarrow p. \qquad (4.4.41)$$

If $t$ is a rational number, that is, $t = q/p$ for some integers $p$ and $q$ with $q < p$, then

$$\begin{aligned}A_n^2([mq/p])/A_n^2(m) &= \frac{A_n^2([mq/p])}{A_n(mq)} \cdot \frac{A_n^2(mq)}{A_n^2(m)} \\ &\longrightarrow q/p = t \qquad (4.4.42)\end{aligned}$$

as $m \to \infty$ uniformly in $n \geq M^*$ by (4.4.41).

If $t$ is a irrational number, then for any given $0 < \varepsilon < 1/2$, take rational number $t_1 > 0$ such that

$$\varepsilon/4 < t - t_1 < \varepsilon/2.$$

By the Minkowski inequality

$$|A_n([mt]) - A_n([mt_1])| \leq A_n([mt] - [mt_1]). \qquad (4.4.43)$$

Let $p = [m/([mt] - [mt_1])]$. Then

$$\begin{aligned}\frac{1}{2}\varepsilon^{-1} &< m/(mt - mt_1 + 1) - 1 \\ &\leq p \leq m/(mt - mt_1 - 1) < 5\varepsilon^{-1}\end{aligned}$$

for $m > 20/\varepsilon$. Similarly to the proof of (4.4.40),

$$\begin{aligned}(p-(p+1)^2\rho(k))A_n^2([mt]-[mt_1]) \\ \le A_n^2(m) + (p+1)^2 A_n^2([mt]-[mt_1])A_n^2(k) \\ + (p+1)^2 E(X_1^{(n)})^2.\end{aligned}$$

Take $k$ such that $\rho(k) < \varepsilon/24$. Then

$$\begin{aligned}&A_n^2([mt]-[mt_1])/A_n^2(m) \\ &\le 6p^{-1}(1+3(p+1)^4(A_n^2(k)+E(X_1^{(n)})^2)/A_n^2(m)) \\ &\le 13\varepsilon \qquad (4.4.44)\end{aligned}$$

provided $m$ is large enough. Combining (4.4.44) with (4.4.43) and (4.4.42) yields (4.4.39). Hence (4.4.36) is proved.

Now turn to (4.4.37). We have proved in Step 1 that

$$S_{[nt]}^{(n)}/A_{[nt]} \xrightarrow{d} N(0,1) \qquad \text{as } n \to \infty$$

for any $0 < t \le 1$. From (4.4.8) and (4.4.4) we have

$$\begin{aligned}&E(S_{[nt]}^{(n)} - S_{[nt]}^{([nt])})^2 A_{[nt]}^{-2} \\ &= A_{[nt]}^{-2}\mathrm{Var}\Big(\sum_{i=1}^{[nt]} X_i I(t_{[nt]} < |X_i| \le t_n)\Big) \\ &\le C_2 C_1^{-1} E(X_1^2 I(t_{[nt]} < |X_1| \le t_n))/E(X_1^2 I(|X_1| \le t_{[nt]})) \\ &= C_2 C_1^{-1}(H(t_n) - H(t_{[nt]}))/H(t_{[nt]}) \\ &\longrightarrow 0, \qquad \text{as} \quad n \to \infty.\end{aligned}$$

It follows that

$$A_n([nt])/A_{[nt]} \longrightarrow 1, \qquad \text{as } n \to \infty.$$

Therefore $S_{[nt]}^{(n)}/A_n([nt]) \xrightarrow{d} N(0,1)$ as $n \to \infty$. By a well-known result on uniform integrability (e.g. cf. Theorem 5.4 of Billingsley 1968), $(S_{[nt]}^{(n)}/A_n([nt]))^2$ is uniformly integrable and so is $(S_{[nt]}^{(n)}/A_n)^2$ by (4.4.36).

Finally we prove (4.4.38).

Let $l_n = \exp\Big\{\sum_{i=0}^{[\log n]} \rho(2^i)^{2/3}\Big\}$. Define

$$\begin{aligned}X_{i1}^{(n)} &= X_i I(|X_i| \le t_n/l_n) - EX_i I(|X_i| \le t_n/l_n), \\ X_{i2}^{(n)} &= X_i I(t_n/l_n < |X_i| \le t_n) - EX_i I(t_n/l_n < |X_i| \le t_n), \\ S_{k1}^{(n)} &= \sum_{i=1}^k X_{i1}^{(n)} \quad \text{and} \quad S_{k2}^{(n)} = \sum_{i=1}^k X_{i2}^{(n)}.\end{aligned}$$

Obviously.

$$\begin{aligned}
&P\Big\{\max_{1\le i\le n}|S_i^{(n)}|\ge 6\lambda A_n\Big\}\\
&\le P\Big\{\max_{1\le i\le n}|S_{i1}^{(n)}|\ge \lambda A_n\Big\}+P\Big\{\max_{1\le i\le n}|S_{i2}^{(n)}|\ge 5\lambda A_n\Big\}\\
&=: p_1+p_2. \qquad (4.4.45)
\end{aligned}$$

Note that for any integer $K>0$

$$\sum_{i=1}^{[\log n]}\rho^{4/5}(2^i)\le K+\rho(2^K)^{2/15}\sum_{i=1}^{[\log n]}\rho(2^i)^{2/3}.$$

Hence, when $l_n\to\infty$ as $n\to\infty$,

$$\sum_{i=1}^{[\log n]}\rho^{4/5}(2^i)=o\Big(\sum_{i=1}^{[\log n]}\rho^{2/3}(2^i)\Big).$$

Therefore, from Lemma 2.2.5, (4.4.2) and Property $A4$, it follows that

$$\begin{aligned}
E|S_{k1}^{(n)}|^{5/2}&\le C\Big\{k^{5/4}(E(X_{11}^{(n)})^2)^{5/4}\\
&\quad+k\exp\Big\{C\sum_{i=1}^{[\log k]}\rho^{4/5}(2^i)\Big\}E|X_{11}^{(n)}|^{5/2}\Big\}\\
&\le C\Big\{(kH(t_n/l_n))^{5/4}\\
&\quad+k\exp\Big\{C\sum_{i=1}^{[\log k]}\rho^{4/5}(2^i)\Big\}(t_n/l_n)^{1/2}H(t_n/l_n)\Big\}\\
&\le c\{(kH(t_n)/l_n)^{5/4}+kn^{1/4}H(t_n)^{5/4}/l_n\}.
\end{aligned}$$

Using Lemma 4.1.2 we have

$$E\max_{1\le k\le n}|S_{k1}^{(n)}|^{5/2}\le c\{(nH(t_n))^{5/4}/l_n\}(\log n)^{5/2}.$$

Without loss of generality we assume that $\rho(2^i)\ge 1/(i\log^2 i)$. Then, recalling (4.4.10) we obtain

$$E\max_{1\le k\le n}|S_{k1}^{(n)}|^{5/2}\le c(nH(t_n))^{5/4}\le cA_n^{5/2}. \qquad (4.4.46)$$

Whence there exists a $\lambda>1$ such that for each $n$ large enough

$$p_1\le\varepsilon/\lambda^2. \qquad (4.4.47)$$

We now estimate $p_2$. Let

$$\begin{aligned}
r_1 &= [n/l_n], \quad r_2 = [n/l_n^2], \quad r = r_1 + r_2, \\
d_1 &= [(n - r_1)/r], \qquad d_2 = [n/r], \\
Y_i &= \sum_{j=ir+1}^{ir+r_1} X_{j2}^{(n)}, \quad i = 0, 1, \cdots, d_1, \\
Z_i &= \sum_{j=ir+r_1+1}^{(i+1)r} X_{j2}^{(n)}, \quad i = 0, 1, \cdots, d_2, \\
T_i(1) &= \sum_{j=0}^{i} Y_j \qquad \text{and} \qquad T_i(2) = \sum_{j=0}^{i} Z_j, \\
Y_i^* &= \sum_{j=ir+1}^{ir+r_1} (|X_j| I(t_n/l_n < |X_j| \le t_n) \\
&\quad - E|X_j| I(t_n/l_n < |X_j| \le t_n)), \qquad i = 0, 1, \cdots, d_1.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&P\Big\{ \max_{1 \le i \le n} |S_{i2}^{(n)}| \ge 5\lambda A_n \Big\} \\
&\le P\Big\{ \max_{1 \le i \le d_1} |T_i(1)| \ge 2\lambda A_n \Big\} + P\Big\{ \max_{1 \le i \le d_2} |T_i(2)| \ge 2\lambda A_n \Big\} \\
&\quad + P\Big\{ \max_{1 \le i \le d_1} \sum_{j=ir+1}^{ir+r_1} |X_{j2}^{(n)}| \ge \frac{1}{2}\lambda A_n \Big\} \\
&\quad + 2l_n P\Big\{ \max_{1 \le i \le r_2} |S_{i2}^{(n)}| \ge \frac{1}{2}\lambda A_n \Big\} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.4.48}$$

By (4.4.2) and (4.4.10), there is a $\lambda_0 > 0$ such that

$$\begin{aligned}
&r_1 E|X_1| I(t_n/l_n < |X_1| \le t_n) \\
&\qquad \le r_1 l_n t_n^{-1} H(t_n) \le \lambda_0 A_n.
\end{aligned}$$

Hence for $\lambda > 8\lambda_0$

$$I_3 \le P\Big\{ \max_{1 \le i \le d_1} |Y_i^*| \ge \frac{1}{4}\lambda A_n \Big\} \tag{4.4.49}$$

and by Lemma 2.2.2 and (4.4.10)

$$\begin{aligned}
I_4 &\le 2l_n P\Big\{\sum_{i=1}^{r_2}\Big(|X_i|I(t_n/l_n < |X_i| \le t_n) \\
&\qquad - E|X_i|I(t_n/l_n < |X_i| \le t_n)\Big) \ge \frac{1}{4}\lambda A_n\Big\} \\
&\le c\, l_n(\lambda A_n)^{-2} r_2 E X_i^2 I(t_n/l_n < |X_i| \le t_n) \\
&\le c\,(\lambda A_n)^{-2} n l_n^{-1} H(t_n) \\
&\le c\,\lambda^{-2} l_n^{-1} \le \varepsilon/\lambda^2
\end{aligned} \tag{4.4.50}$$

provided $n$ is large enough.

In order to estimate $I_1$, let

$$\begin{aligned}
&G_{-1} = (\Omega, \emptyset), \qquad G_k = \sigma(X_i, 1 \le i \le r_1 + kr), \\
&U_0(0) = 0, \quad U_i(k) = \sum_{j=1}^{k} E(Y_{j+i}|G_{j+i-1}), \quad k = 0, 1, \cdots, d_1, \\
&T^*(k) = T_k(1) - U_0(k).
\end{aligned}$$

Obviously

$$\begin{aligned}
I_1 &\le P\Big\{\max_{1\le i\le d_1} |T^*(i)| \ge \lambda A_n\Big\} + P\Big\{\max_{1\le i\le d_1} |U_0(i)| \ge \lambda A_n\Big\} \\
&=: I_1^{(1)} + I_1^{(2)}.
\end{aligned} \tag{4.4.51}$$

Noting that $\{T^*(i), G_i, i = 0, 1, \cdots, d_1\}$ is a martingale and using the maximal inequality of martingale, we have

$$I_1^{(1)} \le 4(\lambda A_n)^{-2} E T^*(d_1)^2 I(|T^*(d_1)| \ge \lambda A_n). \tag{4.4.52}$$

We prove below that for every $i, k$ and $n$, by induction on $k$

$$EU_i^2(k) \le C_1 k r_1 \rho(r_2)^2 \log^2(2k) E X_1^2 I(t_n/l_n < |X_1| \le t_n). \tag{4.4.53}$$

If $k = 1$, from the definition of $\rho$-mixing

$$EU_i^2(1) = E(Y_{i+1} E(Y_{i+1}|G_i)) \le \rho(r_2)\|Y_{i+1}\|_2 \|E(Y_{i+1}|G_i)\|_2.$$

Thus (4.4.53 ) is true for $k = 1$ by a version of (4.4.8). If $k \ge 2$, assume that (4.4.53) holds for every integer less than $k$. Put $k_1 = [k/2], k_2 = k - k_1$.

Then

$$\begin{aligned}EU_i^2(k) &= EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1)U_{i+k_1}(k_2)\\ &\le EU_i^2(k_1) + EU_{i+k_1}^2(k_2)\\ &\quad + 2\rho(r_2)\|U_i(k_1)\|_2\Big\|\sum_{j=k_1+1}^{k} Y_{j+i}\Big\|_2\\ &\le C_1\{k_1\log^2(2k_1) + k_2\log^2(2k_2) + 2k_1^{1/2}k_2^{1/2}\log(2k_2)\}\\ &\quad \cdot r_1\rho(r_2)^2 EX_1^2 I(t_n/l_n < |X_1| \le t_n)\\ &\le C_1 k r_1 \rho(r_2)^2\log^2(2k) EX_1^2 I(t_n/l_n < |X_1| \le t_n)\end{aligned}$$

by induction assumption. This proves (4.4.53). From it and Lemma 4.1.2, we have

$$\begin{aligned}E\max_{1\le i\le d_1} U_0^2(i) &\le 3C_1 d_1 r_1\rho(r_2)^2\log^4(2d_1)H(t_n)\\ &\le cA_n^2\rho(r_2)^2\log^4(2l_n)\\ &\le cA_n^2\rho(r_2)^2\Big(\sum_{i=1}^{[\log n]}\rho(2^i)^{2/3}\Big)^4.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{i=1}^{[\log n]}\rho(2^i)^{2/3} &\le \sum_{i=1}^{[\log r_2]}\rho(2^i)^{2/3} + \rho(r_2)^{2/3}\log(n/r_2)\\ &\le \rho(r_2)^{-1/3}\sum_{i=1}^{[\log r_2]}\rho(2^i) + 2\rho(r_2)^{2/3}\sum_{i=1}^{[\log n]}\rho(2^i)^{2/3}\end{aligned}$$

from which it follows

$$\sum_{i=1}^{[\log n]}\rho(2^i)^{2/3} = O(\rho(r_2)^{-1/3}) \qquad \text{as} \quad n\to\infty.$$

Therefore we obtain

$$E\max_{1\le i\le d_1} U_0^2(i) \le cA_n^2\rho(r_2)^{2/3} \tag{4.4.54}$$

and further

$$I_1^{(2)} \le \varepsilon/\lambda^2 \tag{4.4.55}$$

provided $n$ is large enough.

For $I_2$, having analogue to (4.4.52) and (4.4.55), we can get that for large $n$

$$\begin{aligned} I_2 &\le \varepsilon/\lambda^2 + 4(\lambda A_n)^{-2} ET_{d_2}^2(2) \\ &\le \varepsilon/\lambda^2 + 4C_1(\lambda A_n)^{-2} d_2 r_2 EX_1^2 I(t_n/l_n < |X_1| \le t_n) \\ &\le \varepsilon/\lambda^2 + 4C_1(\lambda A_n)^{-2} n l_n^{-1} H(t_n) \\ &\le \varepsilon/\lambda^2 + c\lambda^{-2} l_n^{-1} \le 2\varepsilon/\lambda^2. \end{aligned} \tag{4.4.56}$$

Now we can come back to $I_1^{(1)}$.

$$\begin{aligned} &ET^*(d_1)^2 I(|T^*(d_1)| \ge \lambda A_n) \\ &\quad \le 4ET_{d_1}^2(1) I(|T_{d_1}(1)| \ge \frac{1}{2}\lambda A_n) + 4EU_0^2(d_1) \\ &\quad \le 36\Big(E(S_{n2}^{(n)})^2 I(|S_{n2}^{(n)}| \ge \lambda \frac{A_n}{6}) \\ &\qquad + E\Big(\sum_{i=d_2 r}^{n} X_{i2}^{(n)}\Big)^2 + EU_0^2(d_1)\Big) \\ &\quad \le 144\Big(E(S_n^{(n)})^2 I(|S_n^{(n)}| \ge \lambda \frac{A_n}{12}) \\ &\qquad + E(S_{n1}^{(n)})^2 I(|S_{n1}^{(n)}| \ge \lambda \frac{A_n}{12}) \\ &\qquad + ET_{d_2}^2(2) + EU_0^2(d_1) + E\Big(\sum_{i=d_2 r}^{n} X_{i2}^{(n)}\Big)^2\Big). \end{aligned}$$

By (4.4.10), (4.4.54) and a version of (4.4.8)

$$\begin{aligned} &A_n^{-1}\Big(ET_{d_2}^2(2) + EU_0^2(d_1) + E\Big(\sum_{i=d_2 r}^{n} X_{i2}^{(n)}\Big)^2\Big) \\ &\quad \le c(n^{-1} d_2 r_2 + \rho(r_2)^{2/3}) \le c(l_n^{-1} + \rho(n^{1/2})^{2/3}) \\ &\quad \le \varepsilon/2000 \end{aligned}$$

for large $n$. Using the uniform integrability of $\{(S_n^{(n)})^2/A_n^2, n \ge 1\}$, we find that

$$A_n^{-2} E(S_n^{(n)})^2 I(|S_n^{(n)}| \ge \lambda A_n/12) \le \varepsilon/2000$$

for each $n \ge 1$ provided $\lambda$ is large enough. Moreover, by (4.4.46)

$$A_n^{-2} E(S_{n1}^{(n)})^2 I(|S_{n1}^{(n)}| \ge \lambda A_n/12) \le 4\lambda^{-1/2} A_n^{-5/2} E|S_{n1}^{(n)}|^{5/2} \le c\lambda^{-1/2}.$$

Whence we obtain that there is a constant $\lambda_1$ such that for any $\lambda > \lambda_1$ and large $n$

$$I_1^{(1)} \le \varepsilon/\lambda^2. \tag{4.4.57}$$

(4.4.55) and (4.4.57) together yield

$$I_1 \leq 2\varepsilon/\lambda^2. \tag{4.4.58}$$

Proceeding exactly as the proof of (4.4.58), we also have

$$I_3 \leq 2\varepsilon/\lambda^2 \tag{4.4.59}$$

for any large $\lambda$ and $n$.

It follows from (4.4.45), (4.4.47), (4.4.48), (4.4.50), (4.4.56), (4.4.58), (4.4.59) that (4.4.38) holds, as desired. This completes the proof of Theorem 4.4.1.

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# Chapter 5 Weak Convergence for $\varphi$-mixing Sequences

The CLT of a $\varphi$-mixing sequence is one of the earlier results for the dependent random variables. Ibragimov (1959) gave the following two Propositions.

**Proposition 5.0.1.** Let $\{X_n, n \geq 1\}$ be a strictly stationary $\varphi$-mixing sequence of random variables with $EX_1 = 0$, $EX_1^2 < \infty$. If

$$\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty,$$

then

$$\sigma^2 = EX_1^2 + 2\sum_{j=2}^{\infty} EX_1X_j$$

converges absolutely, and if the condition $\sigma > 0$ is added, then $S_n/\sigma\sqrt{n}$ converges in distribution to $N(0,1)$.

**Proposition 5.0.2.** Let $\{X_n, n \geq 1\}$ be a strictly stationary $\varphi$-mixing sequence of random variables with $EX_1 = 0, E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$ and $\sigma_n^2 = ES_n^2 \to \infty$. Then $S_n/\sigma_n$ converges in distribution to $N(0,1)$.

Since then, the CLT and the WIP for a $\varphi$-mixing sequence have ever been discussed by many authors. Ibragimov-Linnik and Iosifescu have raised the following conjectures:

**Conjecture 1** (Ibragimov and Linnik 1971). Let $\{X_n, n \geq 1\}$ be a strictly stationary $\varphi$-mixing sequence with $EX_1 = 0$, $EX_1^2 < \infty$. If $\sigma_n^2 \to \infty$ as $n \to \infty$, then the CLT holds.

**Conjecture 2** (Iosifescu 1977). Let $\{X_n, n \geq 1\}$ be as above. Then $W_n$ weakly converges to $W$, where $W_n(t) = S_{[nt]}/\sigma_n$.

Since 1970's, some mathematicians have obtained many beneficial results around these conjectures. Herrndorf (1983b) showed that there exists a strictly stationary $\varphi$-mixing sequence with $\sigma_n^2 \to \infty, \liminf_{n\to\infty} \sigma_n^2/n = 0$, conjecture 2 does not hold. Peligrad (1985) proved that these two conjectures hold true under the additional assumption $\liminf_{n\to\infty} \sigma_n^2/n > 0$. Thus, we can reduce the study of the above conjectures to that of the variances of the partial sums. In those two papers they have also given some sufficient conditions for the CLT and the WIP of a $\varphi$-mixing sequence when the moment of order 2 is finite. We shall introduce these in Section 5.1.

In Section 5.2, we will discuss the above two conjectures and a general conjecture which was raised by Peligrad (1990).

## 5.1 The WIP when the moments of order 2 are finite

Since $\rho(n) \le 2\varphi^{1/2}(n)$, we have the same conclusion as in Theorem 4.1.1 with $\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty$ instead of $\sum_{n=1}^{\infty} \rho(2^n) < \infty$.

Herrndorf (1983) dropped the condition on the mixing rate.

**Theorem 5.1.1.** *Let $\{X_n, n \ge 1\}$ be a $\varphi$-mixing sequence with $EX_n = 0$, $EX_n^2 < \infty$ satisfying*

*(i) $\sigma_n^2 = nh(n)$, where $h(n)$ is slowly varying,*

*(ii) $\lim_{n\to\infty} P\{\max_{1\le i\le n} |X_i| \ge \varepsilon\sigma_n\} = 0$ for any $\varepsilon > 0$,*

*(iii) $\{S_m^2(n)/\sigma_n^2, m \ge 0, n \ge 1\}$ is uniformly integrable.*

*Then*

$$W_n \Longrightarrow W.$$

**Proof.** We are going to verify the conditions in Theorem 4.0.4. By the definition of $\varphi$-mixing,

$$\Big|P\Big\{\bigcap_{i=1}^{r} E_i\Big\} - \prod_{i=1}^{r} P\{E_i\}\Big| \le r\varphi([n\delta]) \to 0 \quad \text{as } n \to \infty,$$

where $E_i = \{W_n(t_i) - W_n(s_i) \in B_i\}, i = 1, \cdots, r$ are defined in Theorem 4.0.4, $\delta = \min_{2\le i\le r}(s_i - t_{i-1}) > 0$. And hence, $\{W_n, n \ge 1\}$ has asymptotically independent increments. Moreover, conditions (i) and (iii) imply the uniform integrability of $\{W_n^2(t), n \ge 1\}$ for each $t \ge 0$. Obviously $EW_n(t) = 0$ and $EW_n^2(t) \to t$ as $n \to \infty$ by condition (i). In order to show the tightness, we need the following lemma.

**Lemma 5.1.1.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence. For any given positive integer $q$ and $a > 0$, $m \geq 0$, $r \geq q+1$, we have*

$$\begin{aligned}\Big(1-\varphi(q)-\max_{q\le j\le r}P\{|S_{m+r}-S_{m+j}|>a\}\Big)&\\ \times P\Big\{\max_{1\le j\le r}|S_{m+j}-S_m|>3a\Big\}&\\ \le P\{|S_{m+r}-S_m|>a\}&\\ +P\Big\{(q-1)\max_{1\le j\le r}|X_{m+j}|>a\Big\}.&\qquad(5.1.1)\end{aligned}$$

**Proof.** Denote $A_1 = \{|X_{m+1}| > 3a\}$,

$$\begin{aligned}&A_j=\{|S_{m+j}-S_m|>3a,\ |S_{m+i}-S_m|\le 3a,\ 1\le i\le j-1\},\quad 2\le j\le r,\\ &B_j=\{|S_{m+r}-S_{m+j+q-1}|\le a\},\quad 1\le j\le r-q,\\ &B_j=\Omega,\quad r-q+1\le j\le r,\qquad C=\{|S_{m+r}-S_m|>a\}.\end{aligned}$$

It is clear that

$$\bigcup_{j=1}^{r}A_jB_j\subset C\bigcup\Big\{(q-1)\max_{1\le j\le r}|X_{m+j}|>a\Big\}.$$

Hence

$$\begin{aligned}&P(C)+P\Big\{(q-1)\max_{1\le j\le r}|X_{m+j}|>a\Big\}\\ &\quad\ge P\Big\{\bigcup_{j=1}^{r}A_jB_j\Big\}=\sum_{j=1}^{r}P(A_jB_j)\\ &\quad\ge\Big\{\min_{1\le j\le r-q}P(B_j)-\varphi(q)\Big\}\sum_{j=1}^{r}P(A_j).\qquad(5.1.2)\end{aligned}$$

Note that

$$\sum_{j=1}^{r}P(A_j)=P\Big\{\max_{1\le j\le r}|S_{m+j}-S_m|>3a\Big\},$$

$$\min_{1\le j\le r-q}P(B_j)+\max_{q\le j\le r}P\{|S_{m+r}-S_{m+j}|>a\}\ge 1.$$

Inserting these into (5.1.2) implies (5.1.1), as desired.

In order to prove that $\{W_n\}$ is tight, it needs only to show that

$$\lim_{\delta\downarrow 0,\,1/\delta\in\mathbb{N}}\frac{1}{\delta}\max_{0\le k\le 1/\delta}\limsup_{n\to\infty}P\Big\{\sup_{k\delta\le s\le(k+1)\delta}|W_n(s)-W_n(k\delta)|>\varepsilon\Big\}=0.\tag{5.1.3}$$

Choose a positive integer $q$ so large that $\varphi(q) < 1$. For any given $\varepsilon > 0$ and $\delta > 0$, by condition (iii) we have

$$\begin{aligned}
&\sup_{m\geq 0}\sup_{1\leq j\leq n\delta} P\{|S_m(j)| > \varepsilon\sigma_n/3\} \\
&\quad \leq 9\varepsilon^{-2} \sup_{m\geq 0,\, j\geq 1} E(S_m^2(j)/\sigma_j^2) \sup_{1\leq j\leq n\delta} \sigma_j^2/\sigma_n^2 \\
&\quad = C(\varepsilon) \sup_{1\leq j\leq n\delta} \sigma_j^2/\sigma_n^2.
\end{aligned}$$

It follows from (i) and Property A4, that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{1\leq j\leq n\delta} \sigma_j^2/\sigma_n^2 = 0.$$

Therefore, there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ such that for any $0 < \delta \leq \delta_0(\varepsilon)$

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{m\geq 0}\sup_{1\leq j\leq n\delta} P\Big\{|S_m(j)| > \frac{1}{3}\varepsilon\sigma_n\Big\} \\
&\qquad \leq C(\varepsilon)\delta \leq \frac{1}{2}(1-\varphi(q)). \qquad (5.1.4)
\end{aligned}$$

Applying Lemma 5.1.1 for $m = [nk\delta]$, $r = [n(k+1)\delta] - [nk\delta]$, $3a = \varepsilon\sigma_n$, from (5.1.4) we obtain that for any $0 < \delta \leq \delta_0$ with $1/\delta \in \mathbb{N}$, $0 \leq k \leq 1/\delta$,

$$\begin{aligned}
&\frac{1}{2}(1-\varphi(q))\limsup_{n\to\infty} P\Big\{\sup_{k\delta\leq s\leq (k+1)\delta} |W_n(s) - W_n(k\delta)| > \varepsilon\Big\} \\
&\quad \leq \limsup_{n\to\infty} P\{|W_n((k+1)\delta) - W_n(k\delta)| > \varepsilon/3\} \\
&\qquad + \limsup_{n\to\infty} P\Big\{(q-1)\max_{1\leq j\leq n} |X_j| > \frac{1}{3}\varepsilon\sigma_n\Big\} \\
&\quad =: I_1 + I_2 \qquad (5.1.5)
\end{aligned}$$

For fixed $q$, it follows from (ii) that $I_2 = 0$. And from (i)

$$\lim_{n\to\infty} \sigma_{[n(k+1)\delta]-[nk\delta]}/\sigma_n = \delta^{1/2}.$$

Thus we have

$$\begin{aligned}
I_1 &\leq \limsup_{n\to\infty} P\{|S_{[n(k+1)\delta]} - S_{[nk\delta]}| > \varepsilon\sigma_{[n(k+1)\delta]-[nk\delta]}/4\sqrt{\delta}\} \\
&\leq 16\varepsilon^{-2}\delta \sup_{m\geq 0,\, n\geq 1} ES_m^2(n) I(|S_m(n)| > \varepsilon\sigma_n/(4\sqrt{\delta}))/\sigma_n^2.
\end{aligned}$$

Combining it with (5.1.5) and condition (iii) implies (5.1.3). The proof of Theorem 5.1.1 is completed.

**Corollary 5.1.1.** *Let $\{X_n\, n \geq 1\}$ be a $\varphi$-mixing sequence with $EX_n = 0$, $EX_n^2 < \infty$. If the conditions (i) and (ii) of Theorem 5.1.1 and*

*(iv)* $\sup\{ES_m^2(n)/\sigma_n^2,\ m \geq 0,\ n \geq 1\} < \infty$

*are satisfied, and if $\{X_n\, n \geq 1\}$ obey the CLT, then $W_n \Longrightarrow W$.*

**Proof.** From the CLT and (i), it follows that $S_{[nt]}/\sigma_n$ converges in distribution to $W(t)$ for every $t > 0$. Now let $0 < s < t$ be given. We shall show

$$(S_{[nt]} - S_{[ns]})/\sigma_n \xrightarrow{d} W(t) - W(s) \quad \text{as } n \to \infty. \tag{5.1.6}$$

It is obvious that $\{(S_{[ns]}/\sigma_n,\ S_{[nt]}/\sigma_n),\ n \geq 1\}$ is tight (see Billingsley 1968 p.41). Hence, it follows from Helly's theorem that there exists a probability measure $Q$ on $R^2$ and a subsequence $\{n_k\}$ such that

$$(S_{[n_k s]}/\sigma_{n_k},\ S_{[n_k t]}/\sigma_{n_k}) \xrightarrow{d} Q \quad \text{as } k \to \infty.$$

Let $\pi_i : R^2 \to R$, $i = 1, 2$, denote the projections. Then

$$(S_{[n_k s]}/\sigma_{n_k},\ (S_{[n_k t]} - S_{[n_k s]})/\sigma_{n_k}) \xrightarrow{d} Q(\pi_1,\ \pi_2 - \pi_1)^{-1} \quad \text{as } k \to \infty.$$

Taking $p = p(n) \in \{0, 1, \dots, [ns]\}$ such that $p(n) \to \infty$ and $p(n)/\sigma_n \to 0$ as $n \to \infty$, we have

$$\begin{aligned}&(ES^2_{[ns]-p(n)}(p(n)))^{1/2}/\sigma_n \\ &\qquad \leq p(n)\sigma_n^{-1} \sup_j (EX_j^2)^{1/2} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence it follows that

$$\begin{aligned}&(S_{[n_k s]-p(n_k)}/\sigma_{n_k}, (S_{[n_k t]} - S_{[n_k s]})/\sigma_{n_k}) \\ &\qquad \xrightarrow{d} Q(\pi_1,\ \pi_2 - \pi_1)^{-1} \quad \text{as } k \to \infty.\end{aligned}$$

For any Borel sets $A, B \subset R$ with $Q(\pi_1,\ \pi_2 - \pi_1)^{-1}(\partial(A \times B)) = 0$, we obtain by an application of the mixing condition

$$\begin{aligned}&|Q(\pi_1,\ \pi_2 - \pi_1)^{-1}(A \times B) - Q\pi_1^{-1}(A)Q(\pi_2 - \pi_1)^{-1}(B)| \\ &\quad = \lim_{k\to\infty} |P\{S_{[n_k s]-p(n_k)}/\sigma_{n_k} \in A,\ (S_{[n_k t]} - S_{[n_k s]})/\sigma_{n_k} \in B\} \\ &\qquad - P\{S_{[n_k s]-p(n_k)}/\sigma_{n_k} \in A\}P\{(S_{[n_k t]} - S_{[n_k s]})/\sigma_{n_k} \in B\}| \\ &\quad = 0.\end{aligned}$$

Hence $\pi_1$ and $\pi_2 - \pi_1$ are $Q$-independent. Since $Q\pi_1^{-1} = N(0, s)$ and $Q\pi_2^{-1} = N(0, t)$, it follows that $Q(\pi_2 - \pi_1)^{-1} = N(0, t - s)$. This proves (5.1.6).

The remainder of the proof needs only to check (5.1.3). From (5.1.6) it follows that

$$\limsup_{n\to\infty} P\{|W_n((k+1)\delta) - W_n(k\delta)| \geq \varepsilon/3\} \leq N(0,\delta)\{x : |x| \geq \varepsilon/3\}. \tag{5.1.7}$$

Furthermore

$$\begin{aligned}
&\frac{1}{\delta}N(0,\delta)\{x : |x| \geq \varepsilon/3\} \\
&= \frac{1}{\delta}\frac{1}{\sqrt{2\pi\delta}}\int_{|x|\geq\varepsilon/3} \exp\Big(-\frac{x^2}{2\delta}\Big)\,dx \\
&= \frac{2}{\delta}\Big(1-\Phi(\frac{\varepsilon}{3\sqrt{\delta}})\Big) \\
&\leq \frac{2}{\delta}\frac{3\sqrt{\delta}}{\varepsilon}\frac{1}{\sqrt{2\pi}}e^{-\varepsilon^2/18\delta} \to 0 \quad \text{as } \delta \to 0.
\end{aligned}$$

Now (5.1.5) and (5.1.7) imply (5.1.3). This completes the proof of Corollary 5.1.1.

**Remark 5.1.1.** The conditions (i) and (ii) of Theorem 5.1.1 are also necessary for the WIP.

Suppose that $W_n$ weakly converges to $W$. For $\delta > 0$ and a function $f$ let $w(f,\delta) = \sup\{|f(x) - f(y)| : 0 \leq x,\, y \leq 1,\, |x-y| < \delta\}$. For any $\varepsilon > 0$, the set $\{f : f \in D[0,1],\, w(f,\delta) \geq \varepsilon\}$ is closed with respect to the uniform topology. Hence

$$\begin{aligned}
\limsup_{n\to\infty} P\{\max_{1\leq i\leq n}|X_i| \geq \varepsilon\sigma_n\} &\leq \limsup_{n\to\infty} P\{w(W_n,\,\delta) \geq \varepsilon\} \\
&\leq P\{w(W,\,\delta) \geq \varepsilon\} \to 0 \quad \text{as}\, \delta \to 0.
\end{aligned}$$

This proves that (ii) holds true.

In order to prove (i), we show that $h(t) := \max(1, \sigma^2_{[t]})/t, t > 0$, is slowly varying. Since $\sigma_n^2 \to \infty$ as $n \to \infty, h(t) = \sigma^2_{[t]}/t$ for large $t$. For $t \in [0,1]$ we have

$$S_{[nt]}/\sigma_{[nt]} \xrightarrow{d} N(0,1), \quad S_{[nt]}/\sigma_n \xrightarrow{d} N(0,t) \quad \text{as}\, n \to \infty.$$

Therefore we obtain for every $t \in [0,1]$

$$\sigma^2_{[nt]}/\sigma^2_n \longrightarrow t \quad \text{as} \quad n \to \infty. \tag{5.1.8}$$

Moreover (ii) implies $X_n/\sigma_n \xrightarrow{P} 0$. Hence both $S_n/\sigma_n$ and $S_n/\sigma_{n+1}$ weakly converge to $N(0,1)$ and further $\sigma_n/\sigma_{n+1} \to 1$. Consequently for $t \in [0,1]$

$$\sigma_{[ts]}/\sigma_{[t[s]]} \longrightarrow 1 \quad \text{as} \quad s \to \infty. \tag{5.1.9}$$

From (5.1.8) and (5.1.9), we obtain $\lim_{s\to\infty} h(ts)/h(s) = 1$ for every $t \in [0,1]$. Hence $h$ is slowly varying.

Now we can write the following Corollary for the strictly stationary case immediately.

**Corollary 5.1.2.** *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\varphi$-mixing sequence with $EX_1 = 0, EX_1^2 < \infty$ and $\sigma_n^2 \to \infty$. Denote*

$$Y_n(t) = \frac{1}{\sigma_n}(S_{[nt]} + (nt - [nt])X_{[nt]+1}).$$

*Then the following assertions are equivalent*

*(a)* $W_n \Longrightarrow W$,

*(b)* $Y_n \Longrightarrow W$,

*(c)* $\{X_n\}$ *obeys the CLT and (ii) is fulfilled,*

*(d)* $\{S_n^2/\sigma_n^2, n \geq 1\}$ *is uniformly integrable and (ii) is fulfilled.*

**Proof.** Obviously, (a) and (b) are equivalent and (b) implies (c). From Theorem 5.4 of Billingsley (1968), it follows that (c) implies (d). Finally "(d) implies (a)" follows from Theorem 5.1.1 .

Peligrad (1985) weakened the conditions of Herrndorf (1983) and gave the following theorem.

**Theorem 5.1.2.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence with $EX_n = 0, EX_n^2 < \infty$. Suppose that the conditions (i), (iv) and the Lindeberg condition*

*(v)* $\lim_{n\to\infty} \frac{1}{\sigma_n^2}\sum_{i=1}^n EX_i^2 I(X_i^2 > \varepsilon\sigma_n^2) = 0$ for any $\varepsilon > 0$ *are satisfied. Then $W_n \Longrightarrow W$.*

**Remark 5.1.2.** If $\varphi^* = \lim_{n\to\infty}\varphi(n) < 1$, then (ii) is equivalent to

$$\sum_{i=1}^{n} P\{|X_i| > \varepsilon\sigma_n\} \longrightarrow 0 \qquad \text{for any}\, \varepsilon > 0 \tag{5.1.10}$$

and the Lindeberg condition (v) is equivalent to

$$E \max_{1\leq i\leq n} X_i^2/\sigma_n^2 \longrightarrow 0. \tag{5.1.11}$$

In fact, (5.1.10) implies (ii) obviously. On the other hand, by (ii) we can choose $n_0$ and $p_0$ such that

$$P\Big\{\max_{1\leq i\leq n} X_i^2 < \varepsilon\sigma_n^2\Big\} - \varphi(p_0) \geq a > 0 \qquad \text{for}\, n \geq n_0. \tag{5.1.12}$$

Therefore for every $x \geq \varepsilon > 0, n \geq n_0$ and $j = 0, 1, \cdots, p_0 - 1$ we have

$$\begin{aligned} & P\Big\{\max_{1\leq i\leq n} X_i^2 \geq x\sigma_n^2\Big\} \\ & \geq P\Big\{X_j^2 \geq x\sigma_n^2, \max_{p_0+j\leq i\leq n} X_i^2 < x\sigma_n^2\Big\} \\ & \quad + P\Big\{X_{p_0+j}^2 \geq x\sigma_n^2, \max_{2p_0+j\leq i\leq n} X_i^2 < x\sigma_n^2\Big\} \\ & \quad + \cdots + P\{X_{[(n-j)/p_0]p_0+j}^2 \geq x\sigma_n^2\} \\ & \geq \sum_{0\leq i\leq[(n-j)/p_0]} P\{X_{ip_0+j}^2 \geq x\sigma_n^2\} \\ & \quad \Big(P\Big\{\max_{1\leq i\leq n} X_i^2 < x\sigma_n^2\Big\} - \varphi(p_0)\Big) \\ & \geq a \sum_{0\leq i\leq[(n-j)/p_0]} P\{X_{ip_0+j}^2 \geq x\sigma_n^2\}, \end{aligned}$$

which implies

$$\sum_{j=1}^{n} P\{X_j^2 > x\sigma_n^2\} \leq \frac{p_0}{a} P\Big\{\max_{1\leq j\leq n} X_j^2 > x\sigma_n^2\Big\}. \tag{5.1.13}$$

Hence we can obtain (5.1.10) from (ii). The fact that (v) implies (5.1.11) follows from

$$\begin{aligned} & E\max_{1\leq i\leq n} X_i^2/\sigma_n^2 \\ & \qquad \leq \varepsilon + E\max_{1\leq i\leq n} X_i^2 I(X_i^2 > \varepsilon\sigma_n^2)/\sigma_n^2 \qquad \text{for any } \varepsilon > 0. \end{aligned}$$

Note that, (5.1.12) and further (5.1.13) hold under (5.1.11). Hence, (5.1.11) implies that (v) follows from the following well-known relation: for every positive integrable random variable X,

$$EXI(X > b) = bP(X > b) + \int_b^{\infty} P(X > x)dx. \tag{5.1.14}$$

**Remark 5.1.3.** Utev (1990) showed that for a $\varphi$-mixing sequence $\{X_n\}$, the Lindeberg condition (v) implies the CLT. Furthermore, Grin (1991) showed that for a stationary $\varphi$-mixing sequence $\{X_n\}$ with $\varphi(1) < 1$, there exists a sequence $\{A_n\}$ of numbers such that $B_n^{-1}S_n - A_n \to N(0,1)$, where $B_n = \sup\{z \geq 0 : Var(\sum_{i=1}^n X_i I(|X_i| \leq z)) \geq z^2\}$, if and only if $\{B_n\}$ is regularly varying with the exponent $1/2$.

The proof of theorem 5.1.2 needs the following lemmas. First we state an analogue of Lemma 2.2.7.

**Lemma 5.1.2.** *Let $\{Y_n, n \geq 1\}$ be a sequence of random variables. Denote $T_n = \sum_{i=1}^n Y_i$. If for some $b > 0, p \in \mathbb{N}$ and $a_0 > 0$*

$$\varphi(p) + \max_{1\leq i\leq n} P\{T_n - T_i| > ba_0/2\} \leq \eta < 1, \tag{5.1.15}$$

*then for every $a \geq a_0$ and $n > p$ the following relations hold:*

$$\begin{aligned} P\Big\{\max_{1\leq i\leq n} |T_i| > (1+b)a\Big\} \leq & \frac{1}{1-\eta} P(|T_n| > a) \\ & + \frac{1}{1-\eta} P\Big\{\max_{1\leq i\leq n} |Y_i| > \frac{ba}{2(p-1)}\Big\} \end{aligned} \tag{5.1.16}$$

*and*

$$\begin{aligned} P\{|T_n| > (1+2b)a\} \leq & \frac{\eta}{1-\eta} P\{|T_n| > a\} \\ & + \frac{1}{1-\eta} P\Big\{\max_{1\leq i\leq n} |Y_i| > \frac{ba}{2p}\Big\}. \end{aligned} \tag{5.1.17}$$

For simplicity, denote

$$E_a X = EXI(X > a).$$

**Lemma 5.1.3.** *Let $\{Y_n, n \geq 1\}$ be a sequence of random variables satisfying (5.1.15). Then for every $A \geq a_0^2$ we have*

$$\begin{aligned} E_{(1+2b)^2 A} T_n^2 \leq & (1+2b)^2 \frac{\eta}{1-\eta} E_A T_n^2 \\ & + \Big(\frac{2p(1+2b)}{b}\Big)^2 \frac{1}{1-\eta} E_{A(b/2p)^2} \max_{1\leq i\leq n} Y_i^2. \end{aligned}$$

**Proof.** By (5.1.14) and a change of variables one obtains

$$\begin{aligned} E_{(1+2b)^2 A} T_n^2 = & (1+2b)^2 A P\{T_n^2 > (1+2b)^2 A\} \\ & + (1+2b)^2 \int_A^\infty P\{T_n^2 > (1+2b)^2 y\} dy. \end{aligned}$$

The lemma follows by (5.1.17) and then (5.1.14) again.

**Lemma 5.1.4.** *Let $\{X_n, n \geq 1\}$ be a centered sequence such that $\varphi^* < 1/4$ and $\{\max_{1\leq i\leq n} EX_i^2/\sigma_n^2, n \geq 1\}$ is bounded. Then*

$$\Big\{\max_{1\leq i\leq n} E(S_n - S_i)^2/\sigma_n^2, n \geq 1\Big\}$$

*is bounded as well.*

**Proof.** Let $p$ be an integer such that $\varphi(p) < 1/4$. We have

$$\max_{1\leq i\leq n} E(S_n - S_i)^2 \leq \max_{1\leq i<n-p} E(S_n - S_i)^2 + p^2 \max_{1\leq i\leq n} EX_i^2. \tag{5.1.18}$$

For every $i < n - p$ we also have

$$\big|\|S_n\|_2 - \|S_i + (S_n - S_{i+p})\|_2\big| \leq p \max_{1\leq i\leq n} \|X_i\|_2.$$

By Lemma 1.2.8, we have

$$\begin{aligned}\|S_i + (S_n - S_{i+p})\|_2 \\ &\geq (1 - 2\varphi^{1/2}(p))^{1/2}(\sigma_i^2 + E(S_n - S_{i+p})^2)^{1/2}.\end{aligned}$$

Therefore

$$\sigma_i < (1 - 2\varphi^{1/2}(p))^{-1/2}\Big(\sigma_n + p \max_{1\leq i\leq n} \|X_i\|_2\Big)$$

for every $i < n - p$. Whence, from (5.1.18)

$$\begin{aligned}&\max_{1\leq i\leq n} E(S_n - S_i)^2/\sigma_n^2 \\ &\qquad \leq 2 + 4(1 - 2\varphi^{1/2}(p))^{-1} \\ &\qquad\quad + p^2(1 + 4(1 - 2\varphi^{1/2}(p))^{-1}) \max_{1\leq i\leq n} EX_i^2/\sigma_n^2\end{aligned}$$

as desired.

**Lemma 5.1.5.** *Let $\{X_n, n \geq 1\}$ be a centered sequence with $\phi^* < 1/4$. Then $\{\max_{1\leq i\leq n} S_i^2/\sigma_n^2, n \geq 1\}$ is uniformly integrable if and only if $\{\max_{1\leq i\leq n} X_i^2/\sigma_n^2, n \geq 1\}$ is uniformly integrable.*

**proof.** First, because

$$P\Big\{\max_{1\leq i\leq n} |X_i| > 2x\sigma_n\Big\} \leq P\Big\{\max_{1\leq i\leq n} |S_i| > x\sigma_n\Big\} \tag{5.1.19}$$

for any $x > 0$, one of the implications follows by the relation (5.1.14).

We prove the part of "*if*", and hence, assume that $\{\max_{1\le i\le n} X_i^2/\sigma_n^2, n \ge 1\}$ is uniformly integrable. By the Chebyshev inequality and by Lemma 5.1.4 for any $b > 0$

$$\lim_{t\to\infty} \sup_n \max_{1\le i\le n} P\{(S_n - S_i)^2 > (b/2)^2 t\sigma_n^2\} = 0. \tag{5.1.20}$$

By $\phi^* < 1/4$ and (5.1.20), we can find some constants $b > 0, \eta < 1/2, p \in \mathbb{N}$ and $a_0 \in R$ such that

$$(1 + 2b)^2\eta/(1 - \eta) < 1 \tag{5.1.21}$$

and

$$\varphi(p) + \max_{1\le i\le n} P\{(S_n - S_i)^2 > (b/2)^2 a_0^2\sigma_n^2\} \le \eta \quad \text{for every } n \ge 1. \tag{5.1.22}$$

From Lemma 5.1.3 and (5.1.21), (5.1.22) it follows that

$$E_{(1+2b)^2A}\Big(\frac{S_n^2}{\sigma_n^2}\Big) \le (1+2b)^2\frac{\eta}{1-\eta}E_A\Big(\frac{S_n^2}{\sigma_n^2}\Big) + \Big(\frac{2p(1+2b)}{b}\Big)^2\frac{1}{1-\eta}E_{A(b/2p)^2}\Big(\max_{1\le i\le n}\frac{X_i^2}{\sigma_n^2}\Big)$$

for any $A > a_0^2$ and every $n \ge 1$. Taking the supremum on $n$ in this relation and noting that $\sup_n E_A(S_n^2/\sigma_n^2)$ is decreasing in $A$ and that $\{\max_{1\le i\le n}(X_i^2/\sigma_n^2), n \ge 1\}$ is uniformly integrable, we obtain

$$\lim_{A\to\infty} \sup_n E_A\Big(\frac{S_n^2}{\sigma_n^2}\Big) \le (1+2b)^2\frac{\eta}{1-\eta}\lim_{A\to\infty}\sup_n E_A\Big(\frac{S_n^2}{\sigma_n^2}\Big).$$

Whence it follows by (5.1.21), (5.1.22) that $\{S_n^2/\sigma_n^2, n \ge 1\}$ is uniformly integrable. This implies by (5.1.16) and (5.1.14) that $\{\max_{1\le i\le n} S_i^2/\sigma_n^2, n \ge 1\}$ is uniformly integrable.

**Proof of Theorem 5.1.2.** By the proof of Theorem 5.1.1, $\{W_n, n \ge 1\}$ also has asymptotically independent increments. By Remark 5.1.2 it follows that $\{\max_{1\le i\le n}(X_i^2/\sigma_n^2), n \ge 1\}$ is uniformly integrable under the Lindeberg condition. Whence $\{W_n^2(t), n \ge 1\}$ is uniformly integrable for each $t$ by (i) and Lemma 5.1.5. Moreover, $EW_n(t) = 0$ and $EW_n^2(t) \to t$ as $n \to \infty$ by (i) again.

For the tightness condition of $\{W_n, n \ge 1\}$, from the proof of Theorem 8.3 of Billingsley (1968), it suffices to show

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sum_{i=0}^{[1/\delta]-1} P\Big\{\max_{i\delta\le s\le(i+1)\delta}|W_n(s) - W_n(i\delta)| > \varepsilon\Big\} = 0. \tag{5.1.23}$$

For every $0 \le i \le 1/\delta - 1$ denote

$$f_i = f_i(n, \delta, a) := \max_{in\delta \le j \le (i+1)n\delta} P\Big\{\Big| \sum_{k=j}^{(i+1)n\delta} X_k \Big| > \frac{b}{2} a^{1/2} \sigma_n \Big\}.$$

By the Chebyshev inequality we have

$$f_i \le \Big(\frac{2}{b}\Big)^2 \frac{1}{a} \max_{in\delta \le j \le (i+1)n\delta} E\Big( \sum_{k=j}^{(i+1)n\delta} X_k \Big)^2 / \sigma_n^2.$$

By (i), (iv) and the properties of a slowly varying function that follows from the Karamata representation (see Appendix) we obtain

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \max_{0 \le i \le 1/\delta - 1} f_i = 0. \tag{5.1.24}$$

Choose $p$ and $b$ such that

$$\varphi(p)(1+2b)^2/(1-\varphi(p)) < 1$$

and choose $\delta_0$ and $n_0$ such that for any $\delta < \delta_0$ and $n > n_0$

$$\varphi(p) + \max_{1 \le i \le 1/\delta} f_i = \eta'(n, \delta, a) := \eta' < 1. \tag{5.1.25}$$

From (5.1.25) and Lemma 5.1.3 we obtain for every $0 \le i \le 1/\delta - 1$

$$\begin{aligned} E_{(1+2b)^2 a}\Big(\Big( \sum_{ni\delta \le j \le n(i+1)\delta} X_j \Big)^2 / \sigma_n^2 \Big) \\ \le (1+2b)^2 \frac{\eta'}{1-\eta'} E_a \Big( \frac{\sum_{ni\delta \le j \le n(i+1)\delta} X_j}{\sigma_n} \Big)^2 \\ + \Big( \frac{2p(1+2b)}{b} \Big)^2 \frac{1}{1-\eta'} \sum_{ni\delta \le j \le n(i+1)\delta} E_{a(b/2p)^2} \Big( \frac{X_j^2}{\sigma_n^2} \Big). \end{aligned} \tag{5.1.26}$$

Noting conditions (i) and (iv), for fixed $\delta > 0$ we have

$$\begin{aligned} \limsup_{n\to\infty} \sum_{i=0}^{[1/\delta]-1} E\Big( \sum_{ni\delta \le j \le n(i+1)\delta} X_j \Big)^2 / \sigma_n^2 \\ = O\Big( \limsup_{n\to\infty} \sum_{i \le 1/\delta} \sigma_{[n\delta]}^2 / \sigma_n^2 \Big) = O(1). \end{aligned}$$

Denote

$$l(a) = \limsup_{\delta\to 0} \limsup_{n\to\infty} \sum_{i=0}^{[1/\delta]-1} E_a \Big( \sum_{ni\delta \le j \le n(i+1)\delta} X_j \Big)^2 / \sigma_n^2.$$

From (5.1.26), (5.1.24) and condition (v) we obtain

$$l((1+2b)^2a) \le \frac{(1+2b)^2\varphi(p)}{1-\varphi(p)} l(a) \quad \text{for every} a > 0.$$

Since $l(a)$ is a decreasing function in $a$, and $[(1+2b)^2\varphi(p)]/(1-\varphi(p)) < 1$, we obtain $\lim_{a\to 0} l(a) = 0$. Hence $l(a) = 0$ for every $a > 0$, which implies

$$\limsup_{\delta\to 0}\limsup_{n\to\infty} \sum_{i=0}^{[1/\delta]-1} P\Big(\Big|\sum_{ni\delta\le j\le n(i+1)\delta} X_j\Big| > \varepsilon\sigma_n\Big) = 0$$

for any $\varepsilon > 0$. The relation (5.1.23) follows now by (5.1.16), (v) and (5.1.24). The proof of Theorem 5.1.2 is completed.

Particularly, by Theorem 5.1.2 we have the following corollaries.

**Corollary 5.1.3.** *Let $\{X_n, n \ge 1\}$ be a stationary $\varphi$-mixing sequence with $EX_1 = 0, EX_1^2 < \infty, \sigma_n^2 \to \infty$ and the Lindeberg condition (v) is satisfied. Then $W_n \Longrightarrow W$.*

**Corollary 5.1.4.** *Let $\{X_n, n \ge 1\}$ be a strictly stationary $\varphi$-mixing sequence with $EX_1 = 0, EX_1^2 < \infty$, $\sigma_n^2 \to \infty$ and for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{n}{\sigma_n^2} EX_1^2 I(X_1^2 > \varepsilon\sigma_n) = 0. \tag{5.1.27}$$

*Then $W_n \Longrightarrow W$.*

A special case of Corollary 5.1.4 is

**Corollary 5.1.5.** *Let $\{X_n, n \ge 1\}$ be a strictly stationary $\varphi$-mixing sequence with $EX_1 = 0, EX_1^2 < \infty$ and $\liminf_n \sigma_n^2/n > 0$. Then $W_n \Longrightarrow W$.*

**Remark 5.1.4.** Peligrad (1985) pointed out that in some cases the Lindeberg condition (v) also is necessary: If $\{X_n, n \ge 1\}$ is a $\varphi$-mixing sequence such that $W_n \Longrightarrow W$, $\sigma_n^2 \to \infty$ and $\varphi(1) < 1$, then the Lindeberg condition (v) is satisfied.

In fact, by Remark 5.1.1 we have $\sigma_i^2 = ih(i)$, where $h$ is a slowly varying function on $R^+$, whence,

$$\Big\{\max_{1\le i\le n} E(S_n - S_i)^2/\sigma_n^2, n \ge 1\Big\}$$

is bounded. So there exists a $t_0 > 0$ such that for every $n$

$$\varphi(1) + \max_{1\le i\le n} P\{|S_n - S_i| > t_0\sigma_n\} \le c < 1.$$

Then by the proof of Lemma 2.2.7 for any $x > t_0^2$ and for each $n \in \mathbb{N}$, we have

$$P\Big\{\max_{1\le i\le n} S_i^2 > 4x\sigma_n^2\Big\} \le \frac{1}{1-c} P\{S_n^2 > x\sigma_n^2\}. \tag{5.1.28}$$

On the other hand, the weak convergence to $W$ implies the uniform integrability of $\{S_n^2/\sigma_n^2, n \ge 1\}$. This fact together with (5.1.28) and (5.1.14) implies $\{\max_{1\le i\le n} S_i^2/\sigma_n^2, n \ge 1\}$ is uniformly integrable. By Lemma 5.1.5 $\{\max_{1\le i\le n} X_i^2/\sigma_n^2, n \ge 1\}$ is also uniformly integrable. From Remark 5.1.1 we have

$$\lim_{n\to\infty} P\Big\{\max_{1\le i\le n} |X_i| > \varepsilon\sigma_n\Big\} = 0 \qquad \text{for any } \varepsilon > 0.$$

Hence we obtain (5.1.11) since $\{\max_{1\le i\le n} X_i^2/\sigma_n^2, n \ge 1\}$ is uniformly integrable. By Remark 5.1.2, (v) is a necessary condition for the weak convergence to $W$.

## 5.2 The Ibragimov-Linnik-Iosifescu conjecture

We have showed in Corollary 5.1.5 that the Iosifescu conjecture is true under the assumption $\liminf \sigma_n^2/n > 0$. But Herrndorf (1983b) showed by an example that the Iosifescu conjecture does not hold true if $\liminf \sigma_n^2/n = 0$.

**Example 5.2.1.** Assume that $\{\eta_n, n \ge 1\}$ be a strictly stationary $\varphi$-mixing sequence with $E\eta_n = 0$, $E\eta_n^2 < \infty$, $\tau_n^2 = E\Big(\sum_{i=1}^n \eta_i\Big)^2 \to \infty$, $\liminf \tau_n^2/n = 0$. If $\{\eta_n, n \ge 1\}$ does not satisfies the WIP, then we can take $X_n = \eta_n$ for $n \in \mathbb{N}$. Suppose that $\{\eta_n, n \ge 1\}$ satisfies the WIP. Let $\{\alpha_n, n \ge 0\}$ be a sequence of independent identically distributed random variables, which are independent of $\{\eta_n\}$ and satisfy

$$\begin{aligned} &P\{\alpha_0 = a_k\tau_{n_k}\} = b_k n_k^{-1} \quad \text{for} k \in \mathbb{N}, \\ &P\{\alpha_0 = 0\} = 1 - \sum_k b_k n_k^{-1}, \end{aligned} \tag{5.2.1}$$

where the positive integers $n_1 < n_2 < n_3 < \cdots$, and the sequences of real numbers $\{a_k\}, \{b_k\}$ with $a_k \to \infty, b_k \to \infty$ such that

$$\sum_k b_k n_k^{-1} \le 1/2, \qquad \sum_k a_k^2 \tau_{n_k}^2 b_k n_k^{-1} < \infty. \tag{5.2.2}$$

( For instance, from $\tau_n^2/n \to 0$, we can choose $n_k$ such that $\tau_{n_k}^2 n_k^{-1} < 2^{-2k-1}$ and $n_k$ to be increasing, $a_k = k$, $b_k = 2^k$.) (5.2.1) and (5.2.2) imply $E\alpha_0^2 < \infty, P(\alpha_0 = 0) \geq 1/2$. Now, define

$$X_n = \eta_n + \alpha_n - \alpha_{n-1}. \tag{5.2.3}$$

Denote $S_n = \sum_{j=1}^n X_j$, $T_n = \sum_{j=1}^n \eta_j$. We have

$$S_n = T_n + \alpha_n - \alpha_0. \tag{5.2.4}$$

It is clear that $\{X_n, n \geq 1\}$ is a strictly stationary $\varphi$-mixing sequence with $EX_1 = 0, EX_1^2 < \infty$, and

$$\sigma_n^2/\tau_n^2 \to 1. \tag{5.2.5}$$

For $n,\ m \geq 1$ we can write

$$\begin{aligned}&\Big(\frac{S_{n+m} - S_m}{\sigma_n}\Big)^2\\ &\leq 2\Big(\frac{T_{n+m} - T_m}{\tau_n}\Big)^2\Big(\frac{\tau_n}{\sigma_n}\Big)^2 + 2\Big(\frac{\alpha_{n+m} - \alpha_m}{\sigma_n}\Big)^2.\end{aligned} \tag{5.2.6}$$

It is well-known that if $\{\eta_n\}$ satisfies the WIP, then $\{(T_{n+m}-T_m)^2/\tau_n^2, m \geq 0, n \geq 1\}$ is uniformly integrable. Thus, from (5.2.6), (5.2.5) and

$$\|(\alpha_{n+m} - \alpha_m)/\sigma_n\|_2 \leq 2\|\alpha_0\|_2/\sigma_n \to 0, \tag{5.2.7}$$

it follows that $\{(S_{n+m} - S_m)^2/\sigma_n^2,\ m \geq 0, n \geq 1\}$ is also uniformly integrable. Since $\{\eta_n\}$ satisfies the WIP and the random vector

$$\sigma_n^{-1}(S_{[nt_1]}, \cdots, S_{[nt_k]}) - \tau_n^{-1}(T_{[nt_1]}, \cdots, T_{[nt_k]}) \xrightarrow{P} 0$$

for any $0 \leq t_1 \leq \cdots \leq t_k \leq 1$, $(W_{[nt_1]}, \cdots, W_{[nt_k]})$ converges weakly to $W\pi_{t_1\cdots t_k}^{-1}$, where $W_n(t) = S_{[nt]}/\sigma_n$. We shall show that $W_n$ does not weakly converge to $W$. For $t > 0$ we have

$$\begin{aligned}&P\Big\{\max_{1\leq i\leq n} |S_i| \geq t\sigma_n\Big\}\\ &\geq P\Big\{\max_{1\leq i\leq n} |\alpha_i - \alpha_0| \geq 2t\sigma_n\Big\} - P\Big\{\max_{1\leq i\leq n} |T_i| \geq t\sigma_n\Big\},\end{aligned} \tag{5.2.8}$$

$$\begin{aligned}&P\Big\{\max_{1\leq i\leq n} |\alpha_i - \alpha_0| \geq 2t\sigma_n\Big\}\\ &\geq P\{\alpha_0 = 0, \max_{1\leq i\leq n} |\alpha_i| \geq 2t\sigma_n\Big\}\\ &\geq \frac{1}{2}P\{\max_{1\leq i\leq n} |\alpha_i| \geq 2t\sigma_n\Big\}\\ &= \frac{1}{2}(1 - (P\{|\alpha_0| < 2t\sigma_n\})^n)\\ &\geq \frac{1}{2} - \frac{1}{2}\exp(-nP\{|\alpha_0| \geq 2t\sigma_n\}).\end{aligned} \tag{5.2.9}$$

Using (5.2.9), (5.2.5), $a_k \to \infty$, (5.2.1) and $b_k \to \infty$, we obtain

$$\limsup_{n\to\infty} P\Big\{\max_{1\le i\le n} |\alpha_i - \alpha_0| \ge 2t\sigma_n\Big\}$$
$$\ge \frac{1}{2} - \frac{1}{2}\lim_{k\to\infty} \exp(-n_k b_k/n_k) = \frac{1}{2}.$$

Since $\{\eta_n\}$ fulfills the WIP and (5.2.5), we can choose $t_0 > 0$ with

$$\limsup_{n\to\infty} P\{\max_{1\le i\le n} |T_i| > t_0\sigma_n\} \le 1/4.$$

Then (5.2.8) and (5.2.9) imply:

$$\limsup_{n\to\infty} P\Big\{\max_{1\le i\le n} |S_i| \ge t\sigma_n\Big\} \ge \frac{1}{4} \quad \text{for every} \quad t \ge t_0,$$

whence $\{X_n, n \ge 1\}$ does not satisfy the WIP.

Furthermore, Peligrad (1990) pointed out that conjectures 1 and 2 are not the most general results which one can expect for a $\varphi$-mixing sequence. After comprehensive survey of the relevant works, she showed that the following conjecture might be true.

**Conjecture 3** (Peligrad 1990). Let $\{X_n, n \ge 1\}$ be a strictly stationary centered $\varphi$-mixing sequence satisfying:

$$H(x) := EX_1^2 I(|X_1| \le x) \quad \text{is slowly varying as} \quad x \to \infty \tag{5.2.10}$$

and $\varphi(1) < 1$. Then $\tilde{W}_n$ weakly converges to $W$, where

$$\tilde{W}_n(t) = S_{[nt]}/((\pi/2)^{1/2} b_n) \qquad 0 \le t \le 1,$$

$b_n = E|S_n|$.

**Remark 5.2.1.** There are at least two situations of interest when it is easy to verify (5.2.10). One is $EX_1^2 < \infty$, and the other is that $P(|X_1| > x)$ is regularly varying with the exponent $-2$, i.e.,

$$P(|X_1| > x) = 1/(x^2 h(x)), \tag{5.2.11}$$

where $h(x)$ is slowly varying as $x \to \infty$.

In fact, we can rewrite $P(|X_1| > x) = h(x)/x^2$. By the partial integration we have

$$H(x) = -\int_0^x y^2 dP(|X_1| > y)$$
$$= -h(x) + \int_0^x h(y) y^{-1} dy.$$

By the Karamata representation (see Theorem A1), we can choose $z_1 = z_1(x) < x$ such that

$$\limsup_{x \to \infty} \sup_{z_1 \le y \le x} h(y)/h(x) = 1,$$
$$\lim_{x \to \infty} x/z_1(x) = \infty.$$

So that

$$\int_0^x h(y)y^{-1}dy \ge \int_{z_1}^x h(y)y^{-1}dy > \frac{1}{2}h(x)\log(xz_1^{-1}).$$

Then

$$H(x) = (1 + o(1)) \int_0^x h(y)y^{-1}dy.$$

For any given $k > 0$ we have

$$\Big|\int_x^{kx} h(y)y^{-1}dy\Big| \le 2(\log k)h(x) = o\Big(\int_0^x h(y)y^{-1}dy\Big).$$

It follows that

$$\lim_{x \to \infty} H(kx)/H(x) = 1.$$

Peligrad (1990) proved that Conjecture 3 is true under condition (5.2.11).

**Theorem 5.2.1.** *Let $\{X_n, n \ge 1\}$ be a centered, strictly stationary $\varphi$-mixing sequence satisfying (5.2.11) and $\varphi(1) < 1$. Then*

$$\tilde{W}_n \Longrightarrow W.$$

The proof of Theorem 5.2.1 will not be presented here.

# Chapter 6 Weak Convergence for Mixing Random Fields

There are two kinds of definitions of mixing dependence for a random field. One is a natural generalization from the classical case, the sequence of dependent random variables, to the dependent random field, which has been discussed by Bulinskii-Zurbenko (1981), Gorodezkii (1982, 1984), Bolthausen (1982), Nahapetian (1987), Bradley (1992), Donkhan and Guyon (1991), Guyon(1992) and Donkhan (1994), etc. Another is appeared in the study of set-index partial sum process for weakly dependent random fields, which has been discussed by Goldie and Greenwood (1986 a,b), Chen (1991) and Lu (1995), etc. We shall introduce the first case in Section 6.1 and the second case in Sections 6.2 and 6.3 respectively.

## 6.1 The CLT for mixing random fields

A natural generalization from an $\alpha$-mixing sequence to an $\alpha$-mixing random field has been discussed by some mathematicians.

A random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}, d \geq 1$, is said to be $\alpha_*$-mixing, if

$$\begin{aligned}\alpha_{m,n}(r) = \sup\{&|P(AB) - P(A)P(B)| : A \in \sigma_U, B \in \sigma_V,\\ &U, V \subset \mathbb{Z}^d, |U| \leq m, |V| \leq n, d(U,V) \geq r\}\\ \to 0 \qquad & as \quad r \to \infty. \end{aligned} \tag{6.1.1}$$

where $d(U,V) = \inf\{d(\mathbf{t},\mathbf{s}) : \mathbf{t} \in U, \ \mathbf{s} \in V\}$, $d(\mathbf{t},\mathbf{s}) = \max_{1\leq i\leq d}|t_i - s_i|$, $m, n \in \mathbb{N} \cup \{\infty\}$, $\sigma_A = \sigma\{\xi_{\mathbf{t}}, \mathbf{t} \in A\}, |A|$ is the number of elements of $A$.

For some subsets $\Delta_j^{(n)}, j = 1, \cdots, k$, of d-dimensional cube $\mathbf{J}_n = [-n, n]^d$, denote

$$S(n,j) = \sigma_n^{-1} S_{\Delta_j^{(n)}}, \qquad \sigma_n^2 = \mathrm{Var} S_{\mathbf{J}_n},$$
$$S_I = \sum_{\mathbf{t} \in I} \xi_{\mathbf{t}}, \qquad I \subset \mathbb{Z}^d, \quad |I| < \infty,$$

$$M_k = \left| E \prod_{j=1}^{k} e^{itS(n,j)} - \prod_{j=1}^{k} E e^{itS(n,j)} \right|,$$
$$L_k(\varepsilon, \delta) = \sum_{j=1}^{k} \int_{|S(n,j)| \ge \varepsilon} |S(n,j)|^{2+\delta} P(d\omega), \qquad \varepsilon,\ \delta \ge 0. \quad (6.1.2)$$

Nahapetian (1987) proved the following conclusion, which is a generalization of Theorem 3.2.1.

**Theorem 6.1.1.** *Let $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ be a strictly stationary $\alpha_*$-mixing random field with $EX_{\mathbf{t}} = 0, E|X_{\mathbf{t}}|^{2+\delta} < \infty$ for some $\delta > 0$. If for some $\tau > 0$,*

*(i) $\alpha_{m,n}(r) \le f(m) n^{\tau} \alpha(r)$, where $f(m)$ is a non-negative function, $m \in \mathbb{N} \cup \{\infty\}$;*

*(ii) $\sum_{r=1}^{\infty} r^{d-1} \alpha^{\delta/(2+\delta)}(r) < \infty, \quad \alpha(r) = o(r^{-(2\tau+1)d}), \qquad r \to \infty,$ then*

$$\sigma^2 = \sum_{\mathbf{t} \in \mathbb{Z}^d} EX_{\mathbf{0}} X_{\mathbf{t}} < \infty,$$

*and if $\sigma \ne 0$, we have*

$$S_{\mathbf{J}_n} / \sigma_n \xrightarrow{d} N(0,1). \quad (6.1.3)$$

The proof of Theorem 6.1.1 will need some lemmas. It is clear from the proof of Lemma 1.2.3 that we have

**Lemma 6.1.1.** *Let random variables $X$ and $Y$ be measurable with respect to $\sigma$-fields $\sigma_U$ and $\sigma_V$ respectively, $|U| \le m,\ |V| \le n,\ d(U,V) \ge r,\ E|X|^p < \infty,\ E|Y|^q < \infty,\ p, q > 1,\ p^{-1} + q^{-1} \le 1$. Then*

$$|EXY - EXEY| \le c(E|X|^p)^{1/p} (E|Y|^q)^{1/q} \alpha_{m,n}^{1-p^{-1}-q^{-1}}(r).$$

*Particularly, if $|X| \le C_1$ a.s., $|Y| \le C_2$ a.s., we have*

$$|EXY - EXEY| \le c C_1 C_2 \alpha_{m,n}(r).$$

**Lemma 6.1.2.** *Let $\xi_1, \cdots, \xi_n$ be a sequence of random vectors, $|E\prod_{j=i}^n \times\xi_j| < \infty$, $i = 1, \cdots, n-1$, $|E\xi_i| \le 1$, $i = 1, \cdots, n$. Then*

$$\begin{aligned} &\left|E\prod_{s=1}^n \xi_s - \prod_{s=1}^n E\xi_s\right| \\ &\quad \le \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Big|E(\xi_i - 1)(\xi_j - 1) \\ &\qquad \times \prod_{s=j+1}^{n} \xi_s - E(\xi_i - 1)E(\xi_j - 1)\prod_{s=j+1}^{n} \xi_s\Big|. \end{aligned} \tag{6.1.4}$$

**Proof.** Obviously, we have

$$\left|E\prod_{s=1}^n \xi_s - \prod_{s=1}^n E\xi_s\right| \le \sum_{i=1}^{n-1}\left|E\xi_i\prod_{j=i+1}^{n}\xi_j - E\xi_i E\prod_{j=i+1}^{n}\xi_j\right|. \tag{6.1.5}$$

Moreover

$$\begin{aligned} &\left|E\xi_i\prod_{j=i+1}^{n}\xi_j - E\xi_i E\prod_{j=i+1}^{n}\xi_j\right| \\ &\quad \le \Big|E(\xi_i - 1)(\xi_{i+1} - 1)\prod_{j=i+2}^{n}\xi_j \\ &\qquad - E(\xi_i - 1)E(\xi_{i+1} - 1)\prod_{j=i+2}^{n}\xi_j\Big| \\ &\qquad + \left|E(\xi_i - 1)\xi_{i+2}\prod_{j=i+3}^{n}\xi_j - E(\xi_i - 1)E\xi_{i+2}\prod_{j=i+3}^{n}\xi_j\right|. \end{aligned}$$

By this recursive process, we obtain

$$\begin{aligned} &\left|E\xi_i\prod_{j=i+1}^{n}\xi_j - E\xi_i E\prod_{j=i+1}^{n}\xi_j\right| \\ &\quad \le \sum_{j=i+1}^{n}\Big|E(\xi_i - 1)(\xi_j - 1) \\ &\qquad \times \prod_{s=j+1}^{n}\xi_s - E(\xi_i - 1)E(\xi_j - 1)\prod_{s=j+1}^{n}\xi_s\Big|, \end{aligned} \tag{6.1.6}$$

inserting (6.1.6) into (6.1.5) yields (6.1.4).

**Proof of Theorem 6.1.1.**

Let $p = p(n)$, $q = q(n)$ be the positive integers such that

$$p, q \to \infty, \quad p = o(n), \ q = o(p) \text{as } n \to \infty.$$

Denote

$$\hat{k} = \hat{k}(n) = [2n/(p+q)],$$
$$I_n(i) = [-n + ip + iq, -n + (i+1)p + iq] \quad i = 0, 1, \cdots, \hat{k} - 1,$$
$$I_n = \bigcup_{j=0}^{\hat{k}-1} I_n(j), \quad I_n^d = \overbrace{I_n \times I_n \times \cdots \times I_n}^{d\text{times}}.$$

Then $I_n^d$ consists of $\hat{k}^d$ d-dimensional cubes with side $p$. For given $n$, we denote these $k = \hat{k}^d$ $d$-dimensional cubes by $\Delta_j^{(n)}, j = 1, \cdots, k$, i.e.

$$I_n^d = \bigcup_{j=1}^{k} \Delta_j^{(n)}.$$

Put $A_n = \mathbf{J}_n \backslash I_n^d$. By Bernstein's blocking technique, we need only to prove

1) $\sigma_n^{-2} \mathrm{Var} S_{A_n} \to 0$,

2) $M_k \to 0$, $\sum_{j=1}^{k} \mathrm{Var} S(n, j) \to 1$ and for any given $\varepsilon > 0$

$$L_k(\varepsilon, \delta) \to 0, \quad \text{as } n \to \infty.$$

From $E|X_{\mathbf{t}}|^{2+\delta} < \infty$, $\sum_{r=1}^{\infty} r^{d-1} \alpha^{\delta/(2+\delta)}(r) < \infty$ and Lemma 6.1.1, it follows that $\sigma^2$ is finite and $\mathrm{Var} S_I \sim \sigma^2 |I|$ for any $d$-dimensional cube $I$ with $|I| < \infty$. Therefore we have $\sigma_n^{-2} \mathrm{Var} S_{A_n} \to 0$ and

$$\begin{aligned}\sum_{j=1}^{k} \mathrm{Var} S(n, j) &= k \sigma_n^{-2} \mathrm{Var} S_{\Delta_1^{(n)}} \\ &= \frac{k}{(2n)^d \sigma^2 (1 + o(1))} p^d \sigma^2 (1 + o(1)) \to 1, \quad n \to \infty.\end{aligned}$$

It also permit us to prove 2) only for the bounded random field (cf. Ibragimov and Linnik 1971, the proofs of Theorems 18.5.3 and 18.5.4).

Now we prove $M_k \to 0$. From (6.1.5) it follows that

$$
\begin{aligned}
M_k &\le \sum_{j=1}^{k-1} \Big| E e^{itS(n,j)} \prod_{m=j+1}^{k} e^{itS(n,m)} - E e^{itS(n,j)} E \prod_{m=j+1}^{k} e^{itS(n,m)} \Big| \\
&\le \sum_{j=1}^{k-1} \Big| E(e^{itS(n,j)} - 1 - itS(n,j) \\
&\quad + \frac{t^2}{2} S(n,j)^2) \prod_{m=j+1}^{k} e^{itS(n,m)} \\
&\quad - E(e^{itS(n,j)} - 1 - itS(n,j) + \frac{t^2}{2} S(n,j)^2) E \prod_{m=j+1}^{k} e^{itS(n,m)} \Big| \\
&\quad + |t| \sum_{j=1}^{k-1} \Big| ES(n,j) \prod_{m=j+1}^{k} e^{itS(n,m)} \\
&\quad - ES(n,j) E \prod_{m=j+1}^{k} e^{itS(n,m)} \Big| \\
&\quad + \frac{t^2}{2} \sum_{j=1}^{k-1} \Big| ES(n,j)^2 \prod_{m=j+1}^{k} e^{itS(n,m)} \\
&\quad - ES(n,j)^2 E \prod_{m=j+1}^{k} e^{itS(n,m)} \Big| \\
&=: T_1 + |t| T_2 + \frac{t^2}{2} T_3. \qquad (6.1.7)
\end{aligned}
$$

Take $p = o(n^{1/2})$, we have

$$
\begin{aligned}
T_1 &\le 2 \frac{|t|^3}{3!} \sum_{j=1}^{k-1} E|S(n,j)|^3 \\
&\le c|t|^3 \Big(\frac{n}{p}\Big)^d \sigma_n^{-3} E|S_{\Delta_1^{(n)}}|^3 \\
&\le c|t|^3 \Big(\frac{n}{p}\Big)^d (n^d)^{-3/2} p^d E S^2_{\Delta_1^{(n)}} \\
&\le c|t|^3 \Big(\frac{p}{n^{1/2}}\Big)^d \to 0, \quad n \to \infty. \qquad (6.1.8)
\end{aligned}
$$

Using the proof of inequality (6.1.4) we have

$$T_2 \le \sigma_n^{-1} \sum_{j=1}^{k-1} \sum_{\mathbf{s}\in\Delta_j^{(n)}} \Big| EX_{\mathbf{s}} \prod_{m=j+1}^{k} e^{itS(n,m)} - EX_{\mathbf{s}} E \prod_{m=j+1}^{k} e^{itS(n,m)} \Big|$$

$$\le \sigma_n^{-1} \sum_{j=1}^{k-1} \sum_{\mathbf{s}\in\Delta_j^{(n)}} \sum_{m=j+1}^{k} \Big| EX_{\mathbf{s}} (e^{itS(n,m)} - 1) \prod_{r=m+1}^{k} e^{itS(n,r)}$$

$$- EX_{\mathbf{s}} E (e^{itS(n,m)} - 1) \prod_{r=m+1}^{k} e^{itS(n,r)} \Big|. \tag{6.1.9}$$

Noting $|X_{\mathbf{s}}| \le C_0$, $|e^{itS(n,m)} - 1| \le cp^d/\sigma_n$ a.s. and applying Lemma 6.1.1 in (6.1.9) we obtain

$$T_2 \le cp^d \sigma_n^{-2} \sum_{j=1}^{k-1} \sum_{\mathbf{s}\in\Delta_j^{(n)}} \sum_{m=j+1}^{k} \alpha_{1,n^d}(d(\Delta_j^{(n)}, \Delta_m^{(n)}))$$

$$\le cp^{2d} f(1) n^{\tau d} \sigma_n^{-2} \sum_{j=1}^{k-1} \sum_{m=j+1}^{k} \alpha(d(\Delta_j^{(n)}, \Delta_m^{(n)}))$$

$$\le cp^{2d} n^{\tau d} p^{-d} \sum_{j=1}^{k} \alpha(d(\Delta_1^{(n)}, \Delta_j^{(n)}))$$

$$\le cp^d n^{\tau d} \sum_{l=1}^{\infty} \sum_{j:(l-1)q \le d(\Delta_1^{(n)}, \Delta_j^{(n)}) < lq} \alpha(d(\Delta_1^{(n)}, \Delta_j^{(n)}))$$

$$\le cp^d n^{\tau d} \sum_{l=1}^{\infty} l^{d-1} \alpha(lq).$$

From condition (ii) of Theorem 6.1.1 it follows that

$$T_2 \le cp^d n^{\tau d} q^{-(2\tau+1)d} \sum_{l=1}^{\infty} \frac{\beta(lq)}{l^{1+2\tau d}}, \tag{6.1.10}$$

where $\beta(n) = \alpha(n) n^{(2\tau+1)d}$. Since $\beta(n) \to 0$ as $n \to 0$ we can take $p = p(n) = o(n^{1/2})$, $p(n) \to \infty$ as $n \to \infty$ and $q = q(n) \to \infty$, $q = o(p)$ such that the right hand side of (6.1.10) approaches zero.

Now we estimate $T_3$.

$$\begin{aligned}
T_3 \le \sigma_n^{-2}\sum_{j=1}^{k-1}\sum_{\mathbf{s},\mathbf{u}\in\Delta_j^{(n)}}\Big|EX_{\mathbf{s}}X_{\mathbf{u}}\prod_{m=j+1}^{k} e^{itS(n,m)} \\
- EX_{\mathbf{s}}X_{\mathbf{u}}E\prod_{m=j+1}^{k} e^{itS(n,m)}\Big| \\
\le \sigma_n^{-2}\sum_{j=1}^{k-1}\sum_{m=j+1}^{k}\sum_{\mathbf{s},\mathbf{u}\in\Delta_j^{(n)}}\Big|EX_{\mathbf{s}}X_{\mathbf{u}}(e^{itS(n,j+1)}-1)\prod_{m=j+2}^{k} e^{itS(n,m)} \\
- EX_{\mathbf{s}}X_{\mathbf{u}}E(e^{itS(n,j+1)}-1)\prod_{m=j+2}^{k} e^{itS(n,m)}\Big| \\
\le c\frac{p^{2d}}{n^d}\frac{p^d}{n^{d/2}}\sum_{j=1}^{k-1}\sum_{m=j+1}^{k}\alpha_{2,n^d}(d(\Delta_j^{(n)},\Delta_m^{(n)})) \\
\le c\frac{p^{3d}}{n^{3d/2}}\Big(\frac{n}{p}\Big)^d f(2)n^{\tau d}\sum_{s=1}^{\infty}s^{d-1}\alpha(sq) \\
\le c\frac{p^{2d}n^{\tau d}\beta(q)}{n^{d/2}q^{(2\tau+1)d}} \to 0, \quad \text{as} \quad n\to\infty. \qquad (6.1.11)
\end{aligned}$$

Combining (6.1.7)-(6.1.11) yields that $M_k \to 0$ as $n\to\infty$.

Finally, we prove $L_k(\varepsilon,\delta)\to 0$. Noting that the stationarity of $\{\xi_{\mathbf{t}}\}$, we have

$$\begin{aligned}
L_k(\varepsilon,\delta) &\le c\Big(\frac{n}{p}\Big)^d n^{-d(1+\delta/2)}\int_{|S_{\Delta_1^{(n)}}|\ge\varepsilon\sigma_n}|S_{\Delta_1^{(n)}}|^{2+\delta}P(d\omega) \\
&\le c\Big(n^{\delta/2}p\Big)^{-d}\int_{|S_{\Delta_1^{(n)}}|\ge\varepsilon\sigma_n}|S_{\Delta_1^{(n)}}|^{2+\delta}P(d\omega).
\end{aligned}$$

There exist $p=p(n)\to\infty$, $p(n)=o(n)$, and $q=q(n)\to\infty$, $q=o(p)$ as $n\to\infty$ such that the right hand of the above inequality approaches zero. This completes the proof of Theorem 6.1.1.

**Remark 6.1.1.** In the above definition of $\alpha_*$-mixing, the positions of set $U$ and $V$ are symmetric, so the assumption:

$$\alpha_{m,n}(r)\le f(m)n^{\tau}\alpha(r)$$

in the condition (i) is not very reasonable. The random field $\{X_{\mathbf{t}},\mathbf{t}\in\mathbb{Z}^d\}$

is said to be $\alpha$-mixing, if

$$\begin{aligned}\alpha(r) = \sup\{|P(AB) - P(A)P(B)| : A \in \sigma_U, B \in \sigma_V,\\ U, V \subset \mathbb{Z}^d, d(U,V) \geq r\}\\ \to 0 \qquad as \quad r \to \infty. \end{aligned} \tag{6.1.12}$$

A more general result was given by Lu (1995), which is a generalization of Theorem 3.2.3 to $\alpha$-mixing random fields.

**Theorem 6.1.2.** *Let $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ be an $\alpha$-mixing random field with $EX_{\mathbf{t}} = 0$, $EX_{\mathbf{t}}^2 < \infty$. If there exists a $g \in \mathcal{G}$ such that*

$$\sup_{\mathbf{t}} Eg(|X_{\mathbf{t}}|) < \infty, \qquad \sum_{r=1}^{\infty} r^{d-1} f_g(\alpha(r)) < \infty, \tag{6.1.13}$$

$$\lim_{n\to\infty} \mathrm{Var} S_{J_n}/n^d = \sigma^2 > 0. \tag{6.1.14}$$

*Then*

$$W_n \Longrightarrow W$$

*where $W_n(\mathbf{t}) = S_{J[n\mathbf{t}]}/\sigma_n$, $\mathbf{0} \leq \mathbf{t} \leq \mathbf{1}$.*

The proof of Theorem 6.1.2 is similar to that of Theorem 3.2.3.

**Remark 6.1.2.** A similar theorem for non-stationary random fields has been discussed by Guyon (1992). The central limit theorem for $\alpha$-mixing random fields with continuous parameters has been discussed by Gorodezkii (1984), Zhurbenko (1984), etc, and the weak invariance principle for this case has also been given there.

Bradley (1992) proved the CLT of strictly stationary random fields under the "unrestricted $\rho$-mixing" condition and just finite or "barely infinite" second moments. No mixing rate is assumed. Let $\{\xi_{\mathbf{t}}, \ \mathbf{t} \in \mathbb{Z}^d\}$ be a strictly stationary random field. For any nonempty disjoint sets $S, D \subset \mathbb{Z}^d$, put

$$\rho(S,D) = \rho(\sigma(\xi_{\mathbf{k}}, \mathbf{k} \in S), \ \sigma(\xi_{\mathbf{k}}, \mathbf{k} \in D)).$$

For each $r \geq 1$, define

$$\rho^*(r) = \sup \ \rho(S,D),$$

where the sup is taken over all pairs of nonempty disjoint subsets $S, D \subset \mathbb{Z}^d$ such that $d(S,D) \geq r$. Denote

$$L := L^{(n)} = (l_1^{(n)}, \cdots, l_d^{(n)}) \in \mathbb{N}^d, \ S(\xi, L) = \sum_{\mathbf{1} \leq \mathbf{t} \leq \mathbf{L}} \xi_{\mathbf{t}}.$$

Bradley (1992) proved the following theorem.

**Theorem 6.1.3.** *Let $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ be a centered strictly stationary random field with $0 < EX_{\mathbf{0}}^2 < \infty$, $\rho^*(r) \to 0$ as $r \to \infty$ and the continuous positive spectral density $f(\cdot)$ on $T^d$, satisfying $f(1, \cdots, 1) > 0$, where $T$ denotes the unit circle in the complex plane. Then as $\|L^{(n)}\| = l_1^{(n)} \cdots l_d^{(n)} \to \infty$ one has that $\|S(\xi, L)\|_2 \to \infty$ and $S(\xi, L)/\|S(\xi, L)\|_2 \xrightarrow{d} N(0, 1)$.*

The proof of Theorem 6.1.3 will not be presented here.

## 6.2 Convergence of finite dimensional distributions

Denote

$$\begin{aligned}
\mathcal{J}_n &= \{\mathbf{j} = (j_1/n, j_2/n, \cdots, j_d/n) : \ j_1, j_2, \cdots, j_d \in \{1, 2, \cdots, n\}\}, \\
C_{n,\mathbf{j}} &= (\mathbf{j} - n^{-1}\mathbf{1}, \mathbf{j}), \\
(\mathbf{a}, \mathbf{b}] &= \{(x_1, x_2, \cdots, x_d) : a_i < x_i \leq b_i, i = 1, 2, \cdots, d\}, \\
\mathcal{I}^d &= \{(\mathbf{a}, \mathbf{b}] : \ \mathbf{a}, \mathbf{b} \in [0, 1]^d\}.
\end{aligned}$$

Let $\{\xi_{n,\mathbf{j}}, \ \mathbf{j} \in \mathcal{J}_n, \ n \geq 1\}$ be a triangular array field. It is easy to see that a random field $\{\xi_{\mathbf{t}}, \ \mathbf{t} \in \mathbb{Z}^d\}$ on a $d$-dimensional integer lattice is a special case of a triangular array random field $\{\xi_{n,\mathbf{j}}, \ \mathbf{j} \in \mathcal{J}_n, n \geq 1\}$, if we write $\xi_{n,\mathbf{j}} = \xi_{n\mathbf{j}}$, $n\mathbf{j} = (j_1, j_2, \cdots, j_d)$. Now from a random field $\{\xi_{n,\mathbf{j}}, \ \mathbf{j} \in \mathcal{J}_n, n \geq 1\}$ we form the set-indexed partial-sum process of the $n$-th level as

$$Z_n(A) = n^{-d/2} \sum_{\mathbf{j} \in \mathcal{J}_n} \frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}(\xi_{n,\mathbf{j}} - E\xi_{\mathrm{n},\mathbf{j}}) \quad A \in \mathcal{B}^d, \ n \in \mathbb{N}, \tag{6.2.1}$$

where $|\cdot|$ is Lebesgue measure and $\mathcal{B}^d$ is the class of Borel sets of $[0, 1]^d$. For the random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}\}$, correspondingly, define

$$Z_n(A) = n^{-d/2} \sum_{\mathbf{t} \in \mathbb{Z}^d} |A \cap C_{\mathbf{t}}|(\xi_{\mathbf{t}} - E\xi_{\mathbf{t}}) \tag{6.2.1'}$$

where $A/n^d \in \mathcal{B}^d$, $\quad C_{\mathbf{t}} = (\mathbf{t} - \mathbf{1}, \mathbf{t}), \mathbf{t} = (t_1, t_2, \cdots, t_d), t_i \in \mathbb{Z}$.

In Sections 6.2 and 6.3, we prove the weak convergence of $Z_n$ to a Wiener process $W$, restricting its domain of definition to a subset of $\mathcal{B}^d$

satisfying a metric-entropy bound. We also impose moment and mixing conditions on the $\{\xi_{n,\mathbf{j}}\}$. Let $x > 0$.

**Definition 6.2.1.** The random field $\{\xi_{n,\mathbf{j}},\ \mathbf{j} \in \mathcal{J}_n\}$ is said to be *$\alpha$-mixing* if $\alpha(nx) \to 0$ as $n \to \infty$, where

$$\alpha(nx) = \sup_{\substack{I,J\subset\mathcal{J}_n \\ d(I,J)\geq x}} \sup_{\substack{A\in\sigma(\xi_{n,\mathbf{j}},\mathbf{j}\in I) \\ B\in\sigma(\xi_{n,\mathbf{j}},\mathbf{j}\in J)}} |P(AB) - P(A)P(B)|.$$

**Definition 6.2.2.** The random field $\{\xi_{n,\mathbf{j}},\ \mathbf{j} \in \mathcal{J}_n,\}$ is said to be *$\rho$-mixing* if $\rho(nx) \to 0$ as $n \to \infty$, where

$$\rho(nx) = \sup_{\substack{I,J\subset\mathcal{J}_n \\ d(I,J)\geq x}} \sup_{\substack{X\in L_2(\sigma(\xi_{n,\mathbf{j}},\mathbf{j}\in I)) \\ Y\in L_2(\sigma(\xi_{n,\mathbf{j}},\mathbf{j}\in J))}} \frac{|\mathrm{Cov}(X,Y)|}{\sqrt{\mathrm{Var}X\mathrm{Var}Y}}$$

and $L_2(\mathcal{F})$ is the set of $L_2$ random variables measurable with respect to $\mathcal{F}$.

**Definition 6.2.3.** The random field $\{\xi_{n,\mathbf{j}},\ \mathbf{j} \in \mathcal{J}_n\}$ is said to be *symmetric $\varphi$-mixing* if $\varphi(nx) \to 0$ as $n \to \infty$, where

$$\varphi(nx) = \sup_{\substack{I,J\subset\mathcal{J}_n \\ d(I,J)\geq x}} \sup_{\substack{A\in\sigma(\xi_{n,\mathbf{j}},\mathbf{j}\in I) \\ B\in\sigma(\xi_{n,\mathbf{j}},\mathbf{j}\in J) \\ P(A)P(B)>0}} \max(|P(A|B) - P(A)|, |P(B|A) - P(B)|).$$

**Definition 6.2.4.** The random field $\{\xi_{n\mathbf{j}},\ \mathbf{j} \in \mathcal{J}_n\}$ is said to be *absolutely regular* if $\beta(nx) \to 0$ as $n \to \infty$, where

$$\beta(nx) = \sup_{I,J\subset\mathcal{J}_n, d(I,J)\geq x} \|\mathcal{L}(\xi_{n,\mathbf{j}}, \mathbf{j} \in I \cup J) - \mathcal{L}(\xi_{n,\mathbf{j}}, \mathbf{j} \in I)\mathcal{L}(\xi_{n,\mathbf{j}}, \mathbf{j} \in J)\|_{\mathrm{Var}}$$

$\mathcal{L}(\xi(\cdot))$ is the distribution law of $\{\xi(\cdot)\}$ and $\|\cdot\|_{\mathrm{Var}}$ is variation norm.

It is clear that

$$\alpha(nx) \leq \rho(nx) \leq 2\varphi(nx), \qquad \alpha(nx) \leq \beta(nx) \leq \varphi(nx). \tag{6.2.2}$$

Next, we introduce the metric entropy condition. We say that Borel sets $A, B$ in $\mathcal{B}^d$ are equivalent if $|A\triangle B| = 0$, and denote the set of equivalence classes by $\mathcal{E}$. Define $d_L(A, B) = |A\triangle B|$, it can be proved that $d_L(\cdot,\cdot)$ is a metric on $\mathcal{E}$. The set $\mathcal{E}$ forms a complete metric space under $d_L$.

**Definition 6.2.5.** A subset $\mathcal{A}$ of $\mathcal{E}$ is called *totally bounded* with inclusion, if for every $\delta > 0$ there is a finite set $\mathcal{A}_\delta \subset \mathcal{E}$, such that for every $A \in \mathcal{A}$ there exist $A^+, A^- \in \mathcal{A}_\delta$ with $A^- \subset A \subset A^+$ and $|A^+ \setminus A^-| \leq \delta$.

Note that $\mathcal{A}_\delta$ is a $\delta$-net with respect to $d_L$ for $\mathcal{A}$.

Let $\mathcal{A}$ be a totally bounded subset of $\mathcal{E}$. Its closure $\overline{\mathcal{A}}$ is complete and totally bounded, hence compact. Let $C(\overline{\mathcal{A}})$ be the space of continous functions on $\overline{\mathcal{A}}$ with the sup norm $\|\cdot\|$. Because $\overline{\mathcal{A}}$ is compact, $C(\overline{\mathcal{A}})$ is separable. Thus $C(\overline{\mathcal{A}})$ is a complete, separable metric space. Let $CA(\overline{\mathcal{A}})$ be the set of everywhere additive elements of $C(\overline{\mathcal{A}})$, namely, elements $f$ such that $f(A\cup B) = f(A)+f(B)-f(A\cap B)$ whenever $A, B, A\cup B, A\cap B \in \overline{\mathcal{A}}$. It can be shown that for fixed $\omega, Z_n(\cdot) \in CA(\overline{\mathcal{A}})$, i.e. $Z_n$ are random elements of $CA(\overline{\mathcal{A}})$. A standard Wiener process on $\overline{\mathcal{A}}$ is a random element $W$ of $CA(\overline{\mathcal{A}})$ whose finite dimensional laws are Gaussian with $EW(A) = 0, EW(A)W(B) = |A\cap B|$. In order that $W$ should exist it is necessary (see Dudley 1973) that $\mathcal{A}$ satisfies a metric entropy condition.

**Definition 6.2.6.** Let $\mathcal{A}$ be a totally bounded subset of $\mathcal{E}$, $\mathcal{A}_\delta$ be the smallest $\delta$-net of $\mathcal{A}$. Denote

$$N(\delta, \mathcal{A}) = \text{Card } \mathcal{A}_\delta, \qquad H(\delta) = \log N(\delta, \mathcal{A}).$$

$\mathcal{A}$ is said to satisfy a *metric entropy condition*, or a *convergent entropy integral*, if

$$\int_0^1 \left(\frac{H(\delta)}{\delta}\right)^{1/2} d\delta < \infty. \tag{6.2.3}$$

Define the *exponent of metric entropy* of $\mathcal{A}$, denoted by $r := \inf\{s,\ s > 0, H(\delta) = O(\delta^{-s}) \text{ as } \delta \to 0\}$. If $r < 1$, then (6.2.3) holds.

**Remark 6.2.1.** Some examples of classes of sets which satisfy the metric entropy condition are as follows:

If $\mathcal{C}^d$ denotes the convex subsets of $[0,1]^d$, then $r = (d-1)/2$ (see Dudley 1974).

If $\mathcal{I}^d = \{(\mathbf{a}, \mathbf{b}] : \mathbf{a}, \mathbf{b} \in [0,1]^d\}$ as above, then $r = 0$.

If $\mathcal{P}^{d,m}$ denotes the family of all polygonal regions of $[0,1]^d$ with no more than $m$ vertices, then $r = 0$ (see Erickson 1981).

If $\mathcal{E}^d$ denotes the sets of all ellipsoidal regions in $[0,1]^d$, then $r = 0$ (see Gaeussler 1983).

For the Vapnik-Červonenkis class $\mathcal{V}$ that includes the above three examples, it is known that $N(\delta, \mathcal{V}, d_\lambda) = \text{Card } \mathcal{V}_\delta \leq c\delta^{-v}$ for some $c$ and $v > 0$ (Dudley 1978).

When $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ is independent, the weak convergence of $Z_n$ to $W$ has been studied by Bass and Pyke (1984, 1985), Alexander and Pyke

(1986), Lu (1992), etc. For the mixing random field $\{\xi_{n,\,\mathbf{j}},\ \mathbf{j} \in \mathcal{J}_n,\ n \geq 1\}$, the weak convergence of $Z_n$ to $W$ was first discussed by Goldie and Greenwood (1986a, b). They proved the following theorem.

**Theorem 6.2.1.** *Assume that* $E\xi_{n,\,\mathbf{j}} = 0$, *and*

*(i) for some* $s > 2$, $\{|n^{d/2}\xi_{n\mathbf{j}}|^s, \mathbf{j} \in \mathcal{J}_n, n \geq 1\}$ *is uniformly integrable;*

*(ii) the exponent* $r$ *of metric entropy (with inclusion) of* $\mathcal{A}$ *satisfies* $r < 1$;

*(iii)* $\beta(nx) = O((nx)^b)$ *as* $nx \to \infty$, *the exponent* $b$ *of absolute regularity satisfies* $b \geq ds/(s-2)$ *and* $b > d(1+r)/(1-r)$;

*(iv) the symmetric* $\varphi$*-mixing coefficients satisfy*

$$\sup_{n\geq 1}\sum_{j=1}^{\infty}\varphi^{1/2}(2^j n^{-1}) < \infty;$$

*(v) for any null family* $\{D_h, 0 < h < h_0\}$ *in* $\mathcal{I}^d$ *(a null family is a collection such that* $D_h \subseteq D_{h'}$ *for* $h \leq h'$ *and* $|D_h| = h$ *for each* $h$*),*

$$\lim_{h\downarrow 0}\limsup_{n\to\infty}\left|\frac{EZ_n^2(D_h)}{|D_h|} - 1\right| = 0.$$

*Then* $Z_n$ *converges weakly in* $CA(\overline{\mathcal{A}})$ *to* $W$.

For a random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$, the versions of Definitions 6.2.1, 6.2.2, 6.2.3 and 6.2.4 are as follows:

$$\alpha(x) = \sup_{\substack{I,J\subset\mathbb{Z}^d\\ d(I,J)\geq x}}\ \sup_{\substack{A\in\sigma(\xi_{\mathbf{i}},\mathbf{i}\in I)\\ B\in\sigma(\xi_{\mathbf{j}},\mathbf{j}\in J)}} |P(AB) - P(A)P(B)|,$$

$$\rho(x) = \sup_{\substack{I,J\subset\mathbb{Z}^d\\ d(I,J)\geq x}}\ \sup_{\substack{X\in L_2(\sigma(\xi_{\mathbf{i}},\mathbf{i}\in I))\\ Y\in L_2(\sigma(\xi_{\mathbf{j}},\mathbf{j}\in J))}} \frac{|\mathrm{Cov}(X,Y)|}{\sqrt{\mathrm{Var}X\,\mathrm{Var}Y}},$$

$$\beta(x) = \sup_{I,J\subset\mathbb{Z}^d,\ d(I,J)\geq x} \|\mathcal{L}(\xi_{\mathbf{j}}, \mathbf{j} \in I\cup J) - \mathcal{L}(\xi_{\mathbf{j}}, \mathbf{j} \in I)\mathcal{L}(\xi_{\mathbf{j}}, \mathbf{j} \in J)\|_{\mathrm{Var}},$$

$$\varphi(x) = \sup_{\substack{I,J\subset\mathbb{Z}^d\\ d(I,J)\geq x}}\ \sup_{\substack{A\in\sigma(\xi_{\mathbf{i}},\mathbf{i}\in I)\\ B\in\sigma(\xi_{\mathbf{j}},\mathbf{j}\in J)P(A)P(B)>0}} \times \max(|P(A|B) - P(A)|, |P(B|A) - P(B)|).$$

**Corollary 6.2.1.** *Let* $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ *be a strictly stationary real random field with* $E\xi(\mathbf{0}) = 0$. *Assume that*

*(i)* $E|\xi_{\mathbf{0}}|^s < \infty$ *for some* $s > 2$;

*(ii)* $\mathcal{A}$ *has exponent of metric entropy (with inclusion)* $r < 1$;

*(iii)* $\beta(x) = O(x^{-b})$ $(x \to \infty)$ *for some* $b > \max(ds/(s-2), d(1+r)/(1-r))$;

*(iv)* $\sum_{j=1}^{\infty} \rho^{1/2}(2^j) < \infty$;

*(v)* $\sum_{\mathbf{i}\in\mathbb{Z}^d} E\xi_{\mathbf{0}}\xi_{\mathbf{i}} = 1$.

*Then* $Z_n$ *converges weakly in* $CA(\overline{\mathcal{A}})$ *to* $W$.

Dobrushin (1968) showed that the $\varphi$-mixing condition is not satisfied even for some simple examples of Gibbs random fields. Dobrushin and Nahapetian (1974) introduced the nonuniform $\varphi$-mixing condition.

**Definition 6.2.7.** The random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ is said to be *nonuniform $\varphi$-mixing*, if for $\Lambda_i \subset \mathbb{Z}^d$, $|\Lambda_i| < \infty, i = 1, 2$, there exists a nonnegative function $\varphi_{|\Lambda_1|}(\cdot)$ depending only on $|\Lambda_1|$, such that

$$\sup_{E\in\sigma(\Lambda_1),F\in\sigma(\Lambda_2),P(F)>0} |P(E|F) - P(E)| \le \varphi_{|\Lambda_1|}(d(\Lambda_1, \Lambda_2))$$

and $\varphi_{|\Lambda_1|}(x) \to 0$ as $x \to \infty$, where $|\Lambda|$ is the cardinality of $\Lambda$.

Chen (1991) gave a sufficient condition under which a sequence of partial-sum set-indexed processes with the nonuniform $\varphi$-mixing condition converges to a Brownian motion when the indexed set $\mathcal{A} = \mathcal{I}^d = \{(\mathbf{a}, \mathbf{b}], \mathbf{a}, \mathbf{b} \in [0,1]^d\}$.

**Theorem 6.2.2.** *Let* $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ *be a strictly stationary nonuniform $\varphi$-mixing random field and satisfy*

*(i) there exists a non-negative function* $\varphi(\cdot)$ *on* $R^1$*, such that for any* $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, $\varphi_{|\Lambda|}(\cdot) \le |\Lambda|\varphi(\cdot)$, *and for some* $\delta > 0$

$$\limsup_{r\to\infty}(\varphi(r))^{1/2} r^{3d+4d/\delta} < \infty, \tag{6.2.4}$$

*(ii)* $E\xi_{\mathbf{0}} = 0$, $E|\xi_{\mathbf{0}}|^{2+\delta} < \infty$,

*(iii)*

$$0 < \sigma^2 := \sum_{\mathbf{t}\in\mathbb{Z}^d} \operatorname{Cov}(\xi_{\mathbf{0}}, \xi_{\mathbf{t}}) < \infty. \tag{6.2.5}$$

*Then* $Z_n/\sigma$ *converges weakly in* $CA(\overline{\mathcal{A}})$ *to a Brownian motion as* $n \to \infty$.

By a direct calculation, Lu (1995) proved that the conclusion of Theorem 6.2.2 holds for a more general indexed set $\mathcal{A}$ (with the metric entropy exponent $r, 0 < r < 1$) and the condition for the rate of nonuniform $\varphi$-mixing was weakened.

**Theorem 6.2.3.** *Let $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ be a strictly stationary nonuniform $\varphi$-mixing random field, and satisfy*

*(i)* $\varphi_{|\Lambda|}(\cdot) \le |\Lambda|\varphi(\cdot)$ *and* $\varphi(x) = O(x^{-2d-1-2d/\delta})$ *for some* $\delta > 0$,

*(ii)* $E\xi_{\mathbf{0}} = 0, E|\xi_{\mathbf{0}}|^{2+\delta} < \infty$,

*(iii)* $\mathcal{A}$ *has a exponent of metric entropy (with inclusion)* $r < 1$. *Then* $Z_n/\sigma$ *converges weakly in* $CA(\overline{\mathcal{A}})$ *to a Brownian motion, as* $n \to \infty$, *where* $\sigma^2$ *is defined as in (6.2.5).*

**Remark 6.2.1.** From Theorem 6.2.2 a uniform central limit theorem for certain Gibbs fields has been given in Chen (1991) which is also true for indexed set $\mathcal{A}$, if $\mathcal{A}$ has the exponent of metric entropy $r < 1$.

The proofs of Theorems 6.2.1 and 6.2.2 are omitted here.

The proof of Theorem 6.2.3 will need the following lemmas. Some lemmas for moment estimations are of independent interest.

A *slice* in $R^d$ is a set

$$S(\mathbf{c}, a, \eta) = \{\mathbf{x} \in R^d : a \le \mathbf{c}'\mathbf{x} \le a + \eta\},$$

where $\mathbf{x} \in R^d$, $|\mathbf{c}| = 1$, $a \in R$, and $\eta > 0$. The *thickness* is $\eta$, the *direction* (of the normal to the two bounding hyperplanes) $\mathbf{c}$, and the *displacement* $a$. The slice splits a set $A \subseteq R^d$ into three parts, namely $A \cap S(\mathbf{c}, a, \eta)$ and two sets

$$A_+ = A \cap \{\mathbf{x} \in R^d : \mathbf{c}'\mathbf{x} > a + \eta\}, \quad A_- = A \cap \{\mathbf{x} \in R^d : \mathbf{c}'\mathbf{x} < a\}.$$

If $A$ is measurable and $|A_+| = |A_-|$ we say the slice bisects $A$.

**Lemma 6.2.1.** *There exist $C_0, q$, depending only on $d$, such that for all $p$ satisfying $0 < p < 1/d$, and for every measurable $A \subset R^d$ of finite measure, we can find a slice $S$ that bisects $A$, has thickness $(|A|/2)^p$, and is such that*

$$|A \cap S| \le C_0(|A|/2)^{(q+pd)/(q+1)}. \tag{6.2.6}$$

The proof of Lemma 6.2.1 was given in Goldie and Greenwood (1986b).

Let $\{X_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^d\}$ be a $\rho$-mixing random field with $EX_{\mathbf{i}} = 0$. Denote

$$\rho = \sum_{i=0}^{\infty} \rho(2^i), \qquad \sigma = \sup_{\mathbf{i}} \|X_{\mathbf{i}}\|_2,$$

$$Z(A) = \sum_{\mathbf{i} \in \mathbb{Z}^d} |A \cap C_{\mathbf{i}}| X_{\mathbf{i}}, \qquad A \in \mathcal{B}^d.$$

**Lemma 6.2.2.** *There exist constants $a, b$, depending only on $d$, such that*

$$\|Z(A)\|_2 \le a\, e^{b\rho}\sigma|A|^{1/2}, \qquad A \in \mathcal{B}^d. \tag{6.2.7}$$

**Proof.** Without loss of generality we assume $\sigma = 1$. Set

$$\sigma(h) = \sup_{A\in\mathcal{B}^d, |A|=h} \|Z(A)\|_2, \qquad \overline{\sigma}(h) = \sup_{h'\le h} \sigma(h').$$

Observe

$$\|Z(A)\|_2 \le \sum_{\mathbf{j}\in\mathbb{Z}^d} |A\cap C_{\mathbf{j}}|\|X_{\mathbf{j}}\|_2 \le \sum_{\mathbf{j}\in\mathbb{Z}^d} |A\cap C_{\mathbf{j}}| = |A|,$$

so

$$\sigma(h) \le h. \tag{6.2.8}$$

Take $p$ in Lemma 6.2.1 to be such that the exponent $r = (q+pd)/(q+1)$ does not have any positive integer power equal to $1/2$. This is to avoid a minor technically later. Pick $A$ of positive finite measure $2m$, say. The slice $S$ in Lemma 6.2.1 is $S(\mathbf{c}, a, m^p)$ for some $\mathbf{c}, a$. We refer to $\{\mathbf{x} \in R^d : \mathbf{c}'\mathbf{x} > a + m^p\}$ and $\{\mathbf{x} \in R^d : \mathbf{c}'\mathbf{x} < a\}$ as the "sides" of $S$ containing $A_+$ and $A_-$, respectively. Since $|A_+| = |A_-|$ we know

$$m \ge |A_+| = |A_-| \ge m - |A\cap S| \ge m - C_0 m^r.$$

We may find $A'_+$, such that $A'_+$ is in the side of $S$ containing $A_+$, is disjoint from $A_-$, and has measure $|A'_+| = m - |A_+|$. Thus $|A'_+| \le C_0 m^r$. Let $A''_+ = A_+ \cup A'_+$; then $|A''_+| = m$. Similarly we construct $A'_-$ and $A''_-$ on the side of $S$ containing $A_-$. Now if $\mathbf{x} \in A''_+$, $\mathbf{y} \in A''_-$ then

$$\|\mathbf{x}-\mathbf{y}\| \ge d^{-1/2}|\mathbf{x}-\mathbf{y}| \ge d^{-1/2}m^p.$$

Hence if $C_{\mathbf{i}}$ and $C_{\mathbf{j}}$ intersect $A''_+$ and $A''_-$, respectively, then

$$\|\mathbf{i}-\mathbf{j}\| \ge d^{-1/2}m^p - 2,$$

whence

$$\begin{aligned} &E(Z(A''_+) + Z(A''_-))^2 \\ &\quad\le (1+\rho(d^{-1/2}m^p - 2))(EZ^2(A''_+) + EZ^2(A''_-)) \\ &\quad\le (1+\rho(d^{-1/2}m^p - 2))2\sigma(m)^2. \end{aligned}$$

Since

$$Z(A) = Z(A''_+) + Z(A''_-) - Z(A'_+) - Z(A'_-) + Z(A\cap S) \tag{6.2.9}$$

the triangle inequality gives

$$\sigma(2m) \leq 2^{1/2}(1 + \rho(d^{-1/2}m^p - 2))^{1/2}\sigma(m) + 3\overline{\sigma}(C_0 m^r).$$

Choose $h > 1$, then $h = 2^k m$ where $k \in \mathbb{N}$ and $1/2 < m \leq 1$, and we have

$$\sigma(m) \leq 1, \qquad \sigma(2^{j+1}m) \leq \alpha_j \sigma(2^j m) + \beta_j,$$

where

$$\alpha_j := 2^{1/2}(1 + \rho(d^{-1/2}2^{jp}m^p - 2))^{1/2}, \qquad \beta_j := 3\overline{\sigma}(C_0 2^{jr}).$$

Iterating,

$$\sigma(h) \leq \prod_{j=0}^{k-1} \alpha_j + \sum_{j=0}^{k-1} \beta_j \prod_{i=j+1}^{k-1} \alpha_i. \tag{6.2.10}$$

Let $l \in \mathbb{N}$ satisfy $p > 1/l$. Now $d^{1/2}2^{jp}2^{-p} - 2 > 2^{j/l}$ for $j \geq j_0 \geq 1$, so

$$\begin{aligned}\sum_{j=j_0}^{\infty} \rho(d^{-1/2}2^{jp}m^p - 2) &\leq \sum_{j=1}^{\infty} \rho(2^{j/l}) \\ &= \sum_{j=0}^{\infty}\sum_{i=1}^{l} \rho(2^j 2^{i/l}) \\ &\leq \sum_{j=0}^{\infty}\sum_{i=1}^{l} \rho(2^j) = l\rho.\end{aligned}$$

Thus

$$\prod_{j=0}^{\infty}(1 + \rho(d^{-1/2}2^{jp}m^p - 2))^{1/2} \leq 2^{j_0/2}e^{l\rho/2} =: a_1 e^{b_1\rho}, \tag{6.2.11}$$

and (6.2.10) yields

$$\sigma(h) \leq a_1 e^{b_1\rho}\Big(2^{k/2} + 3\sum_{j=0}^{k-1} 2^{(k-1-j)/2}\overline{\sigma}(C_0 2^{jr})\Big). \tag{6.2.12}$$

Let $\nu$ be the integer such that $r^\nu < 1/2 < r^{\nu-1}$ (recall that $r^\nu = 1/2$ was excluded by choice of $r$). We shall use (6.2.12) iteratively, $\nu$ times, to obtain the result of the lemma,

$$\sigma(h) \leq a e^{b\rho} h^{1/2}. \tag{6.2.13}$$

Applying (6.2.8) to the right-hand side of (6.2.12), if $\nu = 1$ we find

$$\sigma(h) \leq a_1 e^{b_1 \rho} 2^{k/2} \Big\{ 1 + 3C_0 (1 - 2^{-(1/2-r)})^{-1} \Big\},$$

and since $2^{k/2} \leq 2^{1/2} h^{1/2}$ the proof is concluded. In the other case, i.e. when $r > 1/2$, we obtain

$$\sigma(h) \leq a_1 e^{b_1 \rho} 2^{k/2} \Big\{ 1 + \frac{3}{2} C_0 (2^{r-1/2} - 1)^{-1} 2^{kr} \Big\},$$

whence $\overline{\sigma}(h) \leq a_1' e^{b_1 \rho} h^r$ for some $a_1'$. Substituting this in (6.2.12), if $r^2 < 1/2$ we obtain (6.2.13) and otherwise $\sigma(h) \leq a_2 e^{b_2 \rho} h^{r^2}$. After a total of $\nu$ uses of (6.2.12) we obtain (6.2.13).

By the same way, we can prove the following lemma.

**Lemma 6.2.3.** *Suppose that* $\tau := \sup_{\mathbf{i}} \|X_{\mathbf{i}}\|_s < \infty$ *for* $s = 2 + \delta, 0 \leq \delta \leq 1$ *and*

$$\rho" = \sum_{i=1}^{\infty} \rho^{2/s}(2^i) < \infty.$$

*Then for any* $A \in \mathcal{B}^d$ *we have*

$$\|Z(A)\|_s \leq c e^{c\rho''} \tau |A|^{1/2}. \tag{6.2.14}$$

**Proof.** First it follows from Lemma 6.2.2 that Lemma 6.2.3 holds true for $\delta = 0$. For the case of $0 < \delta \leq 1$, let

$$\tau(m) = \sup_{A \in \mathcal{B}^d, |A| = m} \|Z(A)\|_s, \quad \overline{\tau}(m) = \sup_{m' \leq m} \tau(m').$$

From (6.2.9), we obtain

$$\tau(2m) \leq \|Z(A''_+) + Z(A''_-)\|_{2+\delta} + 3\overline{\tau}(C_0 m^r). \tag{6.2.15}$$

By using

$$|1 + x|^{2+\delta} \leq 1 + 9|x| + 9|x|^{1+\delta} + |x|^{2+\delta}, \tag{6.2.16}$$

we have

$$\begin{aligned} &E|Z(A''_+) + Z(A''_-)|^{2+\delta} \\ &\quad \leq 2\tau^{2+\delta}(m) + 9(E|Z(A''_+)||Z(A''_-)|^{1+\delta} \\ &\qquad + E|Z(A''_+)|^{1+\delta}|Z(A''_-)|). \end{aligned} \tag{6.2.17}$$

By Lemma 1.2.7, we have

$$\begin{aligned} &E|Z(A''_+)||Z(A''_-)|^{1+\delta} \\ &\quad \leq E|Z(A''_+)|E|Z(A''_-)|^{1+\delta} + 4\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\tau^{2+\delta}(m) \\ &\quad \leq \Big(1 + 4\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\Big)\tau^{2+\delta}(m). \qquad (6.2.18) \end{aligned}$$

Similarly we have

$$\begin{aligned} &E|Z(A''_+)|^{1+\delta}|Z(A''_-)| \\ &\quad \leq \Big(1 + 4\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\Big)\tau^{2+\delta}(m). \qquad (6.2.19) \end{aligned}$$

Inserting (6.2.18), (6.2.19) into (6.2.17) yields

$$\|Z(A''_+) + Z(A''_-)\|_{2+\delta} \leq 20^{\frac{1}{2+\delta}}\Big(1 + 2\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}m^p - 2)\Big)^{\frac{1}{2+\delta}}\tau(m).$$

Write $\rho_0 = \rho^{2/(2+\delta)}(d^{-1/2}m^p - 2)$, whence

$$\tau(2m) \leq 20^{\frac{1}{2+\delta}}\Big(1 + 2\rho_0\Big)^{\frac{1}{2+\delta}}\tau(m) + 3\overline{\tau}(C_0 m^r).$$

Iterating, for $h = 2^k m, k \in \mathbb{N}, 1/2 < m \leq 1$ we have

$$\tau(h) \leq \prod_{j=0}^{k-1}\alpha_j + \sum_{j=0}^{k-1}\beta_j \prod_{i=j+1}^{k-1}\alpha_i,$$

where

$$\alpha_j \leq 20^{\frac{1}{2+\delta}}\Big\{1 + 2\rho^{\frac{2}{2+\delta}}(d^{-\frac{1}{2}}2^{jp}m^p - 2)\Big\}^{\frac{1}{2+\delta}}, \quad \beta_j = 3\overline{\tau}(C_0 2^{jr}m^r).$$

The conditon $\rho'' < \infty$ implies that

$$\sum_{j=j_0}^{\infty}\rho^{\frac{2}{2+\delta}}\Big(d^{-\frac{1}{2}}2^{jp}m^p - 2\Big) < \infty,$$

where $j_0$ is defined as in Lemma 6.2.2. The remainder of the proof is the same as in the case of $\delta = 0$. The proof of Lemma 6.2.3 is completed.

Denote

$$\begin{aligned} \rho' &= \sum_{i=0}^{\infty}\rho^{1/2}(2^i), \\ g(y) &= \sup_{\mathbf{i}} EX_{\mathbf{i}}^2 I(|X_{\mathbf{i}}| > y). \end{aligned}$$

**Lemma 6.2.4.** *Let $\{X_{\mathbf{i}},\ \mathbf{i} \in \mathbb{Z}^d\}$ be a $\rho$-mixing random field. If $\{X_{\mathbf{i}}^2,\ \mathbf{i} \in \mathbb{Z}^d\}$ is uniformly integrable and $\rho' < \infty$, then the set of random variables $\{Z^2(A)/|A|, A \in \mathcal{B}^d, |A| < \infty\}$ is uniformly integrable, i.e. there exists a constant $C$ depending only on d, such that*

$$E\Big(\frac{Z^2(A)}{|A|}I\Big(\frac{Z^2(A)}{|A|} > y\Big)\Big) < \Big\{C(1 \wedge y^{-1}) + Cg(y^{1/4})\Big\}e^{C\rho'}. \qquad (6.2.20)$$

**Lemma 6.2.5.** *Let $\{W(A), A \in \mathcal{B}^d\}$ be an additive processs that satisfying*

*(i)* $EW(C) = 0$ *for any* $C \in \mathcal{I}^d$,

*(ii)* $EW^2(C) = |C|$ *for any* $C \in \mathcal{I}^d$,

*(iii)* $W(C_1), \cdots, W(C_k)$ *are independent whenever* $C_1, \cdots, C_k \in \mathcal{I}^d$ *and* $d(C_i, C_j) > 0$ *for* $i \neq j$,

*(iv)* $\lim_{m\to\infty} \sum_{\mathbf{j}\in J_m} P\{|W(C_{m,\mathbf{j}})| \geq \varepsilon\} = 0$ *for any* $\varepsilon > 0$.

*Then $W$ is a standard Wiener process on $\mathcal{R} = \mathcal{R}(\mathcal{I}^d)$ (the ring of all finite unions of elements of $\mathcal{I}^d$).*

**Lemma 6.2.6.** *Let $\{Z_n(A),\ A \in \mathcal{R} = \mathcal{R}(\mathcal{I}^d),\ n \geq 1\}$ be a sequence of additive processes such that*

*(i)* $EZ_n(C) \to 0$ $(n \to \infty)$ *for any* $C \in \mathcal{I}^d$,

*(ii)* $EZ_n^2(C) \to |C|$ $(n \to \infty)$ *for any* $C \in \mathcal{I}^d$,

*(iii) whenever* $C_1, \cdots, C_k \in \mathcal{I}^d$ *are such that* $\rho(C_i, C_j) > 0$ *for* $i \neq j$ *we have for all real* $z_1, \cdots, z_k$ *that*

$$P\{\cap_{i=1}^k (Z_n(C_i) \leq z_i)\} - \prod_{i=1}^k P\{Z_n(C_i) \leq z_i\} \to 0, \quad as\, n \to \infty, \quad (6.2.21)$$

*(iv)* $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{\mathbf{j}\in J_m} P\{|Z_n(C_{m,\mathbf{j}})| \geq \varepsilon\} = 0$ *for any* $\varepsilon > 0$,

*(v) for each* $C \in \mathcal{I}^d$, *the set* $\{Z_n^2(C), n \geq 1\}$ *is uniformly integrable.*

*Then the finite dimensional distributions of $Z_n$ converge weakly to the corresponding finite dimensional distributions of a Wiener process $W$ on $\mathcal{R}$.*

**Lemma 6.2.7.** *Let $\{Z_n\}$ be a sequence of additive processes on $\mathcal{B}^d$ such that the set $\{Z_n^2(A)/|A|, A \in \mathcal{B}^d, n \geq 1\}$ is uniformly integrable, and satisfying Lemma 6.2.6 (i), (ii), (iii). Then the finite dimensional distributions of $Z_n$ converge weakly to the corresponding finite dimensional distributions of a Wiener process $W$ on $\mathcal{B}^d$.*

The proofs of Lemmas 6.2.4–6.2.7 are given in Goldie and Greenwood (1986 a, b).

**Theorem 6.2.4.** *Let $\{\xi_{n\mathbf{j}}\ \mathbf{j} \in \mathcal{J}_n\}$, be a $\rho$-mixing triangular array. The smoothed partial-sum processes $Z_n$ are defined by (6.2.1). Suppose that*

*(i)* $E\xi_{n\mathbf{j}} = 0$ *for any* $n \geq 1, \mathbf{j} \in \mathcal{J}_n$*;*

*(ii) the set* $\{n^d \xi_{n\mathbf{j}}^2, n \geq 1, \mathbf{j} \in \mathcal{J}_n\}$ *is uniformly integrable;*

*(iii)* $\sup_n \sum_{j=1}^{\infty} \rho_n^{1/2}(n^{-1}2^j) < \infty$, *and Lemma 6.2.6 (ii).*

*Then the finite dimensional distributions of $Z_n$ on $\mathcal{B}^d$ converges weakly to the corresponding finite dimensional distributions of a Wiener process.*

**Proof.** By Lemma 6.2.4 we have uniform integrability of the set $\{Z_n^2(A)/|A|,\ n \geq 1, A \in \mathcal{B}^d\}$. Because $\rho_n(x)$ is non-increasing in $x$, condition (iii) implies $\rho_n(x) \to 0 (n \to \infty)$ for each fixed $x$. Thus $\alpha_n(x) \to 0 (n \to \infty)$. Clearly the left hand side of (6.2.21) does not exceed $(k-1)\alpha_n(\theta)$, where $\theta > 0$ is the least separation distance between the sets $C_1, \cdots, C_k$. Hence Lemma 6.2.6 (iii) is satisfied. By Lemma 6.2.7, the proof of Theorem 6.2.4 is completed.

## 6.3 Tightness

Fisrt we present a lemma for nonuniform $\varphi$-mixing sequence, which is similar to Lemma 6.2.3.

**Lemma 6.3.1.** *Let $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}, \{Z_n(A), A \in \mathcal{B}^d\}$ be as in Theorem 6.2.3 with $\varphi(x) = O(x^{-(d+1+\theta)})$, for some $\theta > 0$, instead of $\varphi(x) = O(x^{-(2d+1+2d/\delta)})$. Then for any $A \in \mathcal{B}^d$*

$$\|Z_n(A)\|_{2+\delta} \leq c\sigma_0 |nA|^{1/2} \tag{6.3.1}$$

*where* $\sigma_0^2 = E\xi_{\mathbf{0}}^2$.

**Proof.** First, we prove that for $\delta = 0$

$$\sigma(2m) \leq 2^{1/2}(1 + 2m^{1/2}\varphi^{1/2}(d^{-1/2}m^p - 2))^{1/2}\sigma(m) + 3\overline{\sigma}(cm^r),$$

where $\sigma(m) = \sup_{A \in \mathcal{B}^d, |A|=m} \|Z_n(A)\|_2$, $\overline{\sigma}(m) = \sup_{m' \leq m} \sigma(m')$. Take $0 < p < 1/d$ in the bisection lemma to be such that the exponent $r = (q+pd)/(q+1)$ does not have any positive power equal to $1/2$. For any given $h > 1$, we write $h = 2^k m, k \in \mathbb{N}, 1/2 < m \leq 1$. Note that $\sigma(m) \leq 1$,

and for nonuniform $\varphi$-mixing, by the same discussion as in the proof of Lemma 6.2.3 we have

$$\sigma(2^{j+1}m) \le \alpha_j \sigma(2^j m) + \beta_j,$$

where

$$\alpha_j = 2^{1/2}(1 + 2 \cdot 2^{j/2} m^{j/2} \varphi(d^{-1/2} 2^{jp} m^p - 2))^{1/2},$$
$$\beta_j = 3\, \overline{\sigma}(c2^{js}).$$

Iterating,

$$\sigma(h) \le \prod_{j=0}^{k-1} \alpha_j + \sum_{j=0}^{k-1} \beta_j \prod_{i=j+1}^{k-1} \alpha_i. \tag{6.3.2}$$

Furthermore, we take $p$, $\frac{1}{d} > p > \frac{1}{d+1}$, such that $r = \frac{q+pd}{q+1}$ does not also have any positive power equal to $1/2$. There exists a $j_0 \ge 1$ such that $d^{1/2} 2^{jp} 2^{-p} > 2^{j/(d+1)}$ for $j \ge j_0$, so

$$\begin{aligned}
\varphi &:= \sum_{j=j_0}^{\infty} 2^{j/2} \varphi^{1/2}(d^{-1/2} 2^{jp} m^p - 2) \\
&\le \sum_{j=1}^{\infty} 2^{j/2} \varphi^{1/2}(2^{j/(d+1)}) \\
&\le c \sum_{j=1}^{\infty} 2^{\frac{j}{2}} 2^{-\frac{1}{2} \cdot \frac{j}{d+1}(d+1+\theta)} \\
&= c \sum_{j=1}^{\infty} \left(2^{\theta/(2(d+1))}\right)^{-j} < \infty.
\end{aligned}$$

Thus we obtain

$$\sigma(h) \le c e^{c\varphi} \Big( 2^{\frac{k}{2}} + 3 \sum_{j=0}^{k-1} 2^{\frac{k-j-1}{2}} \overline{\sigma}(c2^{jr}) \Big). \tag{6.3.3}$$

The remainder of the proof is the same as in the proof of Lemma 6.2.2. This proves that (6.3.1) holds for $\delta = 0$.

Consider the case of $0 < \delta \le 1$. Recall the notation of Lemma 6.2.2 and write

$$Z_n(A) = Z_n(A''_+) + Z_n(A''_-) - Z_n(A'_+) - Z_n(A'_-) + Z_n(A \cap S), \tag{6.3.4}$$

where $|A| = 2m, S$ is a slice of $A$, $|A''_+| = |A''_-| = m, A''_+$ and $A''_-$ is situated on the different side of $S$, the separation distance $d(A''_+, A''_-) \ge$

$d^{-1/2}m^p, |A'_+|, |A'_-|$, and $A \cap S$ all do not exceed $cm^r$. Denote $\tau(h) = \sup_{|A|=h, A\in\mathcal{B}^d} \|Z_n(A)\|_{2+\delta}$, $\overline{\tau}(h) = \sup_{h'\le h}\tau(h')$. From (6.3.4),

$$\tau(2m) \le \|Z_n(A''_+) + Z_n(A''_-)\|_{2+\delta} + 3\overline{\tau}(cm^r). \tag{6.3.5}$$

By (6.2.16) we have

$$\begin{aligned} &E|Z_n(A''_+) + Z_n(A''_-)|^{2+\delta} \\ &\le 2\tau^{2+\delta}(m) + 9(E|Z_n(A''_+)||Z_n(A''_-)|^{1+\delta} \\ &\quad + E|Z_n(A''_+)|^{1+\delta}|Z_n(A''_-)|). \end{aligned} \tag{6.3.6}$$

By the property of nonuniform $\varphi$-mixing, we have

$$\begin{aligned} &E|Z_n(A''_+)||Z_n(A''_-)|^{1+\delta} \\ &\le E|Z_n(A''_+)|E|Z_n(A''_-)|^{1+\delta} + 2\varphi^{\frac{1}{2+\delta}}_{|A''_+|}(d^{-\frac12}m^p - 2)\tau^{2+\delta}(m) \\ &\le \Big(1 + 2m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac12}m^p - 2)\Big)\tau^{2+\delta}(m). \end{aligned} \tag{6.3.7}$$

Similarly we have

$$\begin{aligned} &E|Z_n(A''_+)|^{1+\delta}|Z_n(A''_-)| \\ &\le \Big(1 + 2m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac12}m^p - 2)\Big)\tau^{2+\delta}(m). \end{aligned} \tag{6.3.8}$$

Inserting (6.3.7), (6.3.8) into (6.3.6) yields

$$\|Z_n(A''_+) + Z_n(A''_-)\|_{2+\delta} \le c\Big(1 + m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac12}m^p - 2)\Big)^{\frac{1}{2+\delta}}\tau(m).$$

Whence

$$\tau(2m) \le c\Big(1 + m^{\frac{1}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac12}m^p - 2)\Big)^{\frac{1}{2+\delta}}\tau(m) + 3c\overline{\tau}(cm^r).$$

Iterating, for $h = 2^k m, k \in \mathbb{N}, 1/2 < m \le 1$ we have

$$\tau(h) \le \prod_{j=0}^{k-1}\alpha_j + \sum_{j=0}^{k-1}\beta_j\prod_{i=j+1}^{k-1}\alpha_i,$$

where

$$\alpha_j \le c\Big\{1 + 2^{\frac{j}{2+\delta}}m^{\frac{j}{2+\delta}}\varphi^{\frac{1}{2+\delta}}(d^{-\frac12}2^{jp}m^p - 2)\Big\}^{\frac{1}{2+\delta}}.$$

Note

$$\begin{aligned} &\sum_{j=j_0}^{\infty} 2^{\frac{j}{2+\delta}}\varphi^{\frac{1}{2+\delta}}\Big(d^{-\frac12}2^{jp}m^p - 2\Big) \\ &\le c\sum_{j=1}^{\infty}\Big(2^{\varepsilon/(2(2+\delta)(d+1))}\Big)^{-j} < \infty. \end{aligned}$$

The remainder of the proof is the same as in the case of $\delta = 0$. The proof of Lemma 6.3.1 is completed.

**Proof of tightness in Theorem 6.2.3.** For $f \in C(\overline{\mathcal{A}})$ write the modulus of continuity

$$\omega_\delta(f) = \sup_{A,B \in \mathcal{A}, |A \triangle B| \le \delta} |f(A) - f(B)|. \tag{6.3.9}$$

Then, since $\overline{\mathcal{A}}$ is compact, we can use a version of the Arzela-Ascoli theorem: a subset $U$ of $C(\overline{\mathcal{A}})$ has a compact closure iff it is equibounded ($\sup_{f\in U} \sup_{A \in \overline{\mathcal{A}}} |f(A)| < \infty$) and equicontinuous ($\lim_{\delta\downarrow 0} \sup_{f \in U} \omega_\delta(f) = 0$). Using this, from Theorem 8.2 of Billingsley (1968) it follows that a sequence $\{Z_n\}$ of random elements of $C(\overline{\mathcal{A}})$ is relatively compact, i.e. every subsequence of $\{Z_n\}$ contains a weakly convergent subsequence iff

(a) for each element $A$ of some countable dence set in $\mathcal{A}$, the family $\{Z_n(A), n \ge 1\}$ is tight, and

(b) for every $\lambda > 0$, $\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\{\omega(Z_n) > \lambda\} = 0$.

(a) follows from Theorem 6.2.4. We need only prove (b). Without loss of generality, we assume that $\sigma^2 = 1, E\xi_{\mathbf{t}} = 0$. For $0 \le u \le v < \infty$, define

$$\eta_{n\mathbf{j}}(u,v) = n^{-(d/2)}\xi_{n\mathbf{j}} I(u \le n^{d\delta/(2(1+\delta))} n^{-d/2} |\xi_{n\mathbf{j}}| < v) \qquad \mathbf{j} \in \mathcal{J}_n,$$
$$Z_n(A,u,v) = \sum_{\mathbf{j}\in\mathcal{J}_n} \frac{|A \cap C_{n\mathbf{j}}|}{|C_{n\mathbf{j}}|}(\eta_{n\mathbf{j}}(u,v) - E\eta_{n\mathbf{j}}(u,v)).$$

**Lemma 6.3.2.** *Suppose $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ satisfies the conditions of Theorem 6.2.3. Then as $n \to \infty$, $U_n(I^d, a, \infty) \to 0$, a.s., $EU_n(I^d, a, \infty) \to 0$, where $I^d = [0,1]^d, a > 0$ and*

$$U_n(A,a,\infty) = \sum_{\mathbf{j}\in\mathcal{J}_n} \frac{|A \cap C_{n\mathbf{j}}|}{|C_{n\mathbf{j}}|} |\eta_{n\mathbf{j}}(a,\infty)|.$$

**Proof.** We use Bass's technique (1985). For $\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{Z}_+^d$, let $\mu(\mathbf{k}) = \max\{k_1, \cdots, k_d\}$ and $\Psi(i,a) = \sup\{k \in \mathbb{Z}_+ : ak^{d/(2(1+\delta))} < i+1\}$. It is well known that $\text{Card}\{\mathbf{k} \in \mathbb{Z}_+^d : \mu(\mathbf{k}) = r\} \le Cr^{d-1}$, where $C$

is a positive constant depending on $d$.

$$\begin{aligned}
&\sum_{\mathbf{k}\in\mathbb{Z}_+^d} I\Big(i+1 > a\mu(\mathbf{k})^{d/(2(1+\delta))}\Big)\mu(\mathbf{k})^{-d/2} \\
&\quad = \sum_{r=1}^{\infty} \sum_{\mathbf{j}:\mu(\mathbf{j})=r} I\Big(i+1 > ar^{d/(2(1+\delta))}\Big) r^{-d/2} \\
&\quad = c\sum_{r=1}^{\infty} I\Big(i+1 > ar^{d/(2(1+\delta))}\Big) r^{(d-2)/2} \\
&\quad \le c(\Psi(i,a))^{d/2} \le ca^{-1}(i+1)^{1+\delta}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{\mathbf{k}\in\mathbb{Z}_+^d} E|\xi_{\mathbf{k}}| I\Big(a(\mu(\mathbf{k}))^{d/(2(1+\delta))} \le |\xi_{\mathbf{k}}| < \infty\Big)(\mu(\mathbf{k}))^{-d/2} \\
&\le \sum_{\mathbf{k}} \sum_{i+1 > a(\mu(\mathbf{k}))^{d/(2(1+\delta))}} (i+1)P\{i < |\xi_{\mathbf{k}}| \le i+1\}(\mu(\mathbf{k}))^{-d/2} \\
&\le \sum_{i=0}^{\infty}\Big[\sum_{\mathbf{k}} I\Big(i+1 > a(\mu(\mathbf{k}))^{d/(2(1+\delta))}\Big)(\mu(\mathbf{k}))^{-d/2}\Big] \\
&\quad \cdot (i+1)P\{i < |\xi_{\mathbf{0}}| \le i+1\} \\
&\le ca^{-1}\sum_{i=0}^{\infty}(i+1)^{2+\delta}P\{i < |\xi_{\mathbf{0}}| \le i+1\} \\
&\le ca^{-1}E(|\xi_{\mathbf{0}}|+1)^{2+\delta} < \infty.
\end{aligned}$$

Then for any $\varepsilon > 0$ and almost sure $\omega$, there exists an $n_1(\omega)$, such that

$$\sum_{\mathbf{k}:\mu(\mathbf{k})>n_1} \Big|\xi_{\mathbf{k}} I\Big(a\mu(\mathbf{k})^{d/(2(1+\delta))} \le |\xi_{\mathbf{k}}| < \infty\Big)\Big| / (\mu(\mathbf{k}))^{d/2} < \varepsilon.$$

From this,

$$U_n(I^d, a, \infty) \le \sum_{\mu(\mathbf{k})\le n_1} |\xi_{\mathbf{k}}| I\Big(an^{d/(2(1+\delta))} \le |\xi_{\mathbf{k}}| < \infty\Big) n^{-d/2}.$$

Thus, as $n \to \infty$, $U_n(I^d, a, \infty) \to 0$, a.s. Analogously, we can obtain

$$EU_n(I^d, a, \infty) \to 0.$$

Lemma 6.3.2 is proved.

In order to prove (b) we need only prove

$$\lim_{\nu\downarrow 0}\limsup_{n\to\infty} P\{\|Z_n\|_{\mathcal{A}_\nu} > \lambda\} = 0, \tag{6.3.10}$$

where $\mathcal{A}_\nu = \{A\backslash B : A, B \in \mathcal{A},\ |A\backslash B| \le \nu\}$. Since $Z_n(A) = Z_n(A, 0, a) + Z_n(A, a, \infty)$, and

$$|Z_n(A, a, \infty)| \le U_n(I^d, a, \infty) + EU_n(I^d, a, \infty),$$

by Lemma 6.3.2, in order to prove (6.3.10), we need only prove

$$\lim_{\nu\downarrow 0} \limsup_{n\to\infty} P\{\|Z_n(A, 0, a)\|_{\mathcal{A}_\nu} > \lambda\} = 0. \tag{6.3.11}$$

Let $p_n = [n^{(2+\delta)/(2(1+\delta))}]$ and $m_n = n/(2p_n)$. We divide $I^d$ in the following two ways: $C_{p_n,\mathbf{l}}, \mathbf{l} \in J_{p_n}$ and $C_{2p_n,\mathbf{l}}, \mathbf{l} \in J_{2p_n}$. There are $2^d$ $C_{2p_n,\mathbf{l}}$ in each $C_{p_n,\mathbf{l}}$. Denote by $I_{n,\mathbf{l},i}$ the $i$th $C_{2p_n\mathbf{j}}$ in $C_{p_n,\mathbf{l}}$, $\mathbf{l} \in J_{p_n}, \mathbf{j} \in J_{2p_n}$. Let

$$I_{n,i} = \bigcup_{\mathbf{l}\in J_{p_n}} I_{n,\mathbf{l},i}, \qquad i = 1, \cdots, 2^d.$$

Then

$$Z_n(\cdot, 0, a) = \sum_{i=1}^{2^d} Z_n(\cdot \cap I_{n,i}, 0, a).$$

Now in order to prove (6.3.11), we need only prove

$$\lim_{\nu\downarrow 0} \limsup_{n\to\infty} P\Big\{\|Z_n(A \cap I_{n,i}, 0, a)\|_{\mathcal{A}_\nu} > \lambda\Big\} = 0. \tag{6.3.12}$$

Write

$$\begin{aligned} Z_n(A \cap I_{n,i}, 0, a) &= \sum_{\mathbf{l}\in J_{p_n}} \sum_{\mathbf{j}\in S(n,\mathbf{l},i)} \frac{|A \cap I_{n,\mathbf{l},i} \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} (\eta_{n\mathbf{j}}(0, a) - E\eta_{n,\mathbf{j}}(0, a)) \\ &=: \sum_{\mathbf{l}\in J_{p_n}} V_{n\mathbf{l}}(A \cap I_{n,i}, 0, a), \\ \eta_{n,\mathbf{j}}(0, a) &= n^{-(d/2)} \xi_{n\mathbf{j}} I(n^{d\delta/(2(1+\delta))} n^{-d/2} |\xi_{n\mathbf{j}}| < a), \quad \mathbf{j} \in \mathcal{J}_n, \end{aligned}$$

where $S(n, \mathbf{l}, i) = \{\mathbf{j} \in J_n : I_{n,\mathbf{l},i} \cap C_{n,\mathbf{j}} \ne \emptyset\}$. Denote $V_{n\mathbf{l}} = V_{n\mathbf{l}}(A \cap I_{n,i}, 0, a)$. By the property of nonuniform $\varphi$-mixing, we have

$$\begin{aligned} Ee^{\alpha \sum_{\mathbf{k}\in J_{p_n}} V_{n\mathbf{k}}} \le& Ee^{\alpha V_{n\mathbf{l}}} Ee^{\alpha \sum_{\mathbf{k}\ne\mathbf{l}, \mathbf{k}\in J_{p_n}} V_{n\mathbf{k}}} \\ &+ 2\varphi_{|S(n,\mathbf{l},i)|}\Big(\frac{n}{2p_n}\Big) Ee^{\alpha \sum_{\mathbf{k}\ne\mathbf{l}} V_{n\mathbf{k}}} \|e^{V_{n\mathbf{l}}}\|_\infty. \end{aligned} \tag{6.3.13}$$

Note that $|V_{n\mathbf{l}}| \le 2a$ and $e^{\alpha V_{n\mathbf{l}}} \le 1 + \alpha V_{n\mathbf{l}} + \alpha^2 V_{n\mathbf{l}}^2$ when $\alpha a \le 1/4$. From Lemma 6.3.1 it follows that

$$EV_{n\mathbf{l}}^2 \le C|A \cap I_{n,\mathbf{l},i}| \qquad \text{for} \quad i = 1, 2, \cdots, 2^d.$$

Therefore for $\alpha a \le 1/4$

$$Ee^{\alpha V_{n\mathbf{l}}} \le e^{E\alpha^2 V_{n\mathbf{l}}^2} \le e^{C\alpha^2 |A\cap I_{n,\mathbf{l},i}|}, \tag{6.3.14}$$

Inserting (6.3.14) into (6.3.13) yields

$$Ee^{\alpha \sum_{\mathbf{l}\in J_{p_n}} V_{n\mathbf{l}}} \le wEe^{\alpha \sum_{\mathbf{k}\ne\mathbf{l},\, \mathbf{k}\in J_{p_n}} V_{n\mathbf{k}}},$$

where

$$\begin{aligned} w &\le e^{C\alpha^2|A\cap I_{n,\mathbf{l},i}|}\Big(1+2e^{1/2}|S(n,\mathbf{l},i)|\varphi\Big(\frac{n}{2p_n}\Big)\Big) \\ &\le e^{C\alpha^2|A\cap I_{n,\mathbf{l},i}|}\Big(1+2e^{1/2}\frac{n^d}{(2p_n)^d}\varphi\Big(\frac{n}{2p_n}\Big)\Big). \end{aligned}$$

Iterating this procedure, we have

$$\begin{aligned} Ee^{\alpha \sum_{\mathbf{l}\in J_{p_n}} V_{n\mathbf{l}}} &\le e^{C\alpha^2|A\cap I_{n,i}|}\Big(1+2e^{1/2}\frac{n^d}{(2p_n)^d}\varphi\Big(\frac{n}{2p_n}\Big)\Big)^{p_n^d} \\ &\le \exp\Big(c\alpha^2|A| + cn^d\Big(\frac{n}{p_n}\Big)^{-2d-2d/\delta-1}\Big) \\ &= \exp\Big(c\alpha^2|A| + cn^{-\frac{\delta}{2(1+\delta)}}\Big) \\ &\le c\, \exp(\alpha^2|A|) \end{aligned} \tag{6.3.15}$$

for large $n$.

Now we return to estimate the left hand side of (6.3.12). Since $0 \le r < 1$, we can take $s > 0$ such that $r < 1/(1+s)$. Set

$$\begin{aligned} \delta_j &= \nu/2^j, \qquad j = 0,1,\cdots, \\ \lambda_j &= \lambda_0 e^{-j(1+r-r(2+s))/(2+s)} \qquad j = 1,2,\cdots, \\ \lambda_0 &= \lambda(1-2^{-(1+r-r(2+s))/(2+s)}), \\ a_j &= e^{-j(1+r)/(2+s)}a, \qquad j = 1,2,\cdots, \\ a &= c\nu^{1/(1+s)}, \qquad c_0 = (6E|\xi_0|^{2+s}/\lambda_0)^{1/(1+s)}. \end{aligned}$$

For any $A \in \mathcal{A}_0$ there exist $A_j, A_j^+ \in \mathcal{A}_0(\delta_j)$ such that $A_j \subseteq A \subseteq A_j^+$ and $|A_j^+ \backslash A_j| \le \delta_j$. Then

$$\begin{aligned} &Z_n(A\cap I_{n,i},0,a) \\ &\quad = Z_n(A_0\cap I_{n,i},0,a) + \sum_{j=0}^{\infty}\{Z_n(A_{j+1}\cap I_{n,i},0,a_j) - Z_n(A_j\cap I_{n,i},0,a_j)\} \\ &\qquad + \sum_{j=0}^{\infty}\{Z_n(A\cap I_{n,i},a_j,a_{j-1}) - Z_n(A_j\cap I_{n,i},a_j,a_{j-1})\}. \end{aligned}$$

So if $\|Z_n(\cdot \cap I_{n,i}, 0, a)\|_{\mathcal{A}_0}$ is to exceed $\lambda$, at least one of the following must hold:

(a) for some $A_0 \in \mathcal{A}_0(\delta_0)$, $\quad |Z_n(A_0 \cap I_{n,i}, 0, a)| > \lambda_0$;

(b) for some $j$, for some

$$A_j \in \mathcal{A}_0(\delta_j), A_{j+1} \in \mathcal{A}_0(\delta_{j+1}), |A_j \triangle A_{j+1}| \le 2\delta_j,$$

$$|Z_n(A_{j+1} \cap I_{n,i}, 0, a_j) - Z_n(A_j \cap I_{n,i}, 0, a_j)| > 2\lambda_j;$$

(c) for some $j$, for some

$$A_j, A_j^+ \in \mathcal{A}_0(\delta_j),\ A_j \subseteq A \subseteq A_j^+,\ |A_j^+ \backslash A_j| \le \delta_j,$$

$$|Z_n(A \cap I_{n,i}, a_j, a_{j-1}) - Z_n(A_j \cap I_{n,i}, a_j, a_{j-1})| > \lambda_j.$$

The number of pairs $A_j, A_j^+$ in $\mathcal{A}_0(\delta_j)$ is $\le \exp(4H(\delta_j/2))$, while the number of pairs $A_j \in \mathcal{A}_0(\delta_j)$, $A_{j+1} \in \mathcal{A}_0(\delta_{j+1})$ is $\le \exp(4H(\delta_{j+1}/2))$.

We have

$$P\{\|Z_n(\cdot \cap I_{n,i}, 0, a)\|_{\mathcal{A}_0} > \lambda\} \le p_0 + \sum_{j=0}^{\infty} r_j + \sum_{j=1}^{\infty} s_j,$$

where

$$p_0 \le 2\exp\{2H(\delta_0/2)\} \max_{|A_0| \le 2\delta_0} P\{|Z_n(A_0 \cap I_{n,i}, 0, a)| > \lambda_0\},$$

$$r_j \le 4\exp\{4H(\delta_{j+1}/2)\} \max_{|A_{j+1} \triangle A_j| \le 2\delta_j} (P\{|Z_n((A_{j+1} \backslash A_j) \cap I_{n,i}, 0, a)| > \lambda_j\}$$
$$+ P\{|Z_n((A_j \backslash A_{j+1}) \cap I_{n,i}, 0, a)| > \lambda_j\}),$$

$$s_j \le \exp\{4H(\delta_j/2)\} \max_{A_j \subseteq A_j^+, |A_j^+ \backslash A_j| \le 2\delta_j} P\{ \sup_{A_j \subseteq A \subseteq A_j^+} |Z_n(A \cap I_{n,i}, a_j, a_{j-1})$$
$$- Z_n(A_j \cap I_{n,i}, a_j, a_{j-1})| > \lambda_j\}.$$

By (6.3.15), taking $\alpha = 1/4a_0$, we have

$$p_0 \le 2\exp\{2H(\delta_0/2)\} \exp\Big(-\frac{\lambda_0}{4a_0} + c_0\frac{\delta_0}{a_0^2}\Big)$$
$$\le 2\exp\Big\{c2^{r+1}\delta_0^{-r} - \frac{\lambda_0}{4a_0} + c_0\frac{\delta_0}{a_0}\Big\}.$$

Similarly

$$r_j \le 4\exp\Big\{c2^{r+1}\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c_0\frac{\delta_j}{a_j}\Big\},$$

$$s_j \le \exp\Big\{c2^{r+1}\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c_0\frac{\delta_j}{a_j}\Big\}.$$

Thus

$$\begin{aligned}p_0+&\sum_{j=0}^{\infty} r_j + \sum_{j=1}^{\infty} s_j \\ &\leq 6\sum_{j=0}^{\infty}\exp\Big\{c\delta_j^{-r} - \frac{\lambda_j}{4a_j} + c_0\frac{\delta_j}{a_j}\Big\} \\ &\leq 6\sum_{j=0}^{\infty}\exp\Big\{\Big(c\nu^{-r} - c'\nu^{-\frac{1}{1+s}} + c\nu^{-\frac{1-s}{1+s}}2^{-\frac{jsr}{2+s}}\Big)2^{jr}\Big\}.\end{aligned}$$

Because of $r < 1/(1+s)$, the coefficient of $2^{ir}$ may be negative by choosing $\nu$ small enough, and (6.3.12) follows. Theorem 6.2.3 is proved.

# Chapter 7 The Berry-Esseen Inequality and the Rate of Weak Convergence

It is well-known that the uniform estimation of the difference between the distribution function $F_n(x)$ of the normalized sum of the first $n$ terms of the sequence of independent random variables $\{X_n, n \geq 1\}$ and the normal distribution function $\Phi(x)$ is given by the Esseen and the Berry-Esseen inequalities. Furthermore, there exists a succinct non-uniformly estimation

$$|F_n(x) - \Phi(x)| \leq A\rho_0/(\sqrt{n}(1+|x|^3))$$

for the case of i.i.d. random variables $\{X_n\}$ with $E|X_1|^3 < \infty$, $EX_1 = 0$, where $\rho_0 = E|X_1|^3/\sigma^3, \sigma^2 = EX_1^2, F_n(x) = P\{S_n/\sigma\sqrt{n} < x\}$.

In this chapter, we shall give the uniform estimations for $\alpha$-mixing and $\rho$-mixing sequences in Section 7.1

In Section 7.2 we shall discuss the Prohorov distance $L(P \circ W_n^{-1}, W)$ between the measure $W_n$ generated by the partial sums processes $\{W_n(t),\ 0 \leq t \leq 1,\ n \geq 1\}$ and the Wiener measure $W$. We shall give the estimation of $L(P \circ W_n^{-1}, W)$ for a $\varphi$-mixing sequence.

## 7.1 Rate of convergence in distribution for $\alpha$-mixing and $\rho$-mixing sequences

The proofs of the Esseen and the Berry-Esseen inequalities for the independent random variables are based on the following proposition (Petrov 1975):

**Proposition 7.1.1.** Let $F_n(x)$ be a distribution, $f_n(t)$ be the characteristic function of $F_n(x)$. Then for any given $T > 0$ and $b > 1/(2\pi)$ we

have

$$\sup_x |F_n(x) - \Phi(x)| \leq b \int_{-T}^{T} \Big|\frac{f_n(t) - e^{-t^2/2}}{t}\Big| dt + \gamma(b)\frac{1}{\sqrt{2\pi}T} \tag{7.1.1}$$

for some $\gamma(b) > 0$.

To obtain the estimation of $|f_n(t) - e^{-t^2/2}|$ is the key to the proof. It is simpler in the case of independent random variables. How to get the estimation of $|f_n(t) - e^{-t^2/2}|$ for the case of mixing dependent random variables ? A method was given by Tikhomirov (1980), a modification of which was given by Sunklodas (1984). The sketch of this method is as follows:

Let $\{X_n, n \geq 1\}$ be an $\alpha$-mixing sequence with $EX_n = 0$. Denote

$$\sigma_n^2 = ES_n^2, \qquad Z_n = S_n/\sigma_n, \qquad F_n(x) = P(Z_n < x).$$

Assume that $1 \leq h \leq n-1$, $2 \leq k \leq n-1$ such that $2kh + 1 < n$. Put

$$\begin{aligned}
Y_j &= X_j/\sigma_n, \\
Z_j^{(0)} &= X_j/\sigma_n = Y_j, \\
Z_j^{(l)} &= \sum_{|p-j|\leq lh} Y_p, \\
z_j^{(0)} &= Z_n, \\
z_j^{(l)} &= Z_n - Z_j^{(l)}, \\
\xi_j^{(l)} &= \exp\{it(z_j^{(l-1)} - z_j^{(l)})\} - 1, \\
a_j^{(r-1)} &= E\Big(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)}\Big), \\
\eta_j^{(r)} &= e^{-itZ_j^{(r)}} - 1, \\
f_n(t) &= Ee^{itZ_n},
\end{aligned} \tag{7.1.2}$$

for $j = 1, 2, \cdots, n$; $l = 1, 2, \cdots, r-1$; $r = 2, \cdots, k$. Since

$$f_n'(t) = i\sum_{j=1}^{n} EY_j e^{itZ_n} = i\sum_{j=1}^{n} EY_j e^{itz_j^{(0)}}$$

and

$$Ee^{itz_j^{(r)}} = E(\eta_j^{(r)} + 1)f_n(t) + E[(\eta_j^{(r)} - E\eta_j^{(r)})e^{itZ_n}]$$

for $j = 1, 2, \cdots, n$; $r = 2, \cdots, k$, the derivative of the characteristic function $f_n(t)$ has the representation :

$$
\begin{aligned}
f_n'(t) = & i\Big\{\sum_{j=1}^{n} a_j^{(1)} E(\eta_j^{(2)} + 1) + \sum_{r=3}^{k}\sum_{j=1}^{n} a_j^{(r-1)} E(\eta_j^{(r)} + 1)\Big\} f_n(t) \\
& + i\sum_{r=2}^{k}\sum_{j=1}^{n}\Big\{E(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itz_j^{(r)}}) - E(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)}) E e^{itz_j^{(r)}}\Big\} \\
& + i\sum_{r=2}^{k}\sum_{j=1}^{n} a_j^{(r-1)} E\{(\eta_j^{(r)} - E\eta_j^{(r)}) e^{itZ_n}\} \\
& + i\sum_{j=1}^{n} EY_j e^{itz_j^{(1)}} + i\sum_{j=1}^{n} EY_j \prod_{l=1}^{k} \xi_j^{(l)} e^{itz_j^{(k)}}.
\end{aligned} \tag{7.1.3}
$$

If we can give the estimation for each term of the right hand side of (7.1.3) under some suitable conditions, then we get the differential equation

$$f_n'(t) = (-t + a(t)) f_n(t) + b(t). \tag{7.1.4}$$

By solving this differential equation, one obtains the estimation of $|f_n(t) - e^{-t^2/2}|$. Finally, from (7.1.1) the estamation of $\Delta_n = \sup_x |F_n(x) - \Phi(x)|$ follows.

Denote $s = 2 + \delta$. Assume that

$$d := \max_{1\le j\le n} E|X_j|^{2+\delta} < \infty, \qquad 0 < \delta \le 1, \tag{7.1.5}$$

$$\sigma_n^2 := ES_n^2 \ge c_0 n \qquad for \quad 0 < c_0 < \infty. \tag{7.1.6}$$

Sunklodas (1984) gave the following theorems.

**Theorem 7.1.1.** *Let $\{X_n, n \ge 1\}$ be an $\alpha$-mixing sequence with $EX_n = 0$, and satisfy conditions (7.1.5), (7.1.6) and*

$$\alpha(n) \le K e^{-\lambda n} \qquad \lambda > 0,\ K > 0.$$

*Then there exist $c_1 = c_1(K, \delta)$, $c_2 = c_2(K, \delta)$ such that for $\lambda$, $\lambda_1 \le \lambda \le \lambda_2$,*

$$\Delta_n \le c_1 \frac{d}{c_0 \sigma_n^\delta}\Big(\frac{\log(\sigma_n / c_0^{1/2})}{\lambda}\Big)^{1+\delta}, \qquad n \ge 1, \tag{7.1.7}$$

*where*

$$\lambda_1 = c_2 (\log(\sigma_n / c_0^{1/2}))^b / n, \qquad b > 2(1+\delta)/\delta;$$
$$\lambda_2 = 4(2+\delta)\delta^{-1} \log(\sigma_n / c_0^{1/2}).$$

**Theorem 7.1.2.** *Let $\{X_n, n \geq 1\}$ be as in Theorem 7.1.1, but*

$$\alpha(n) \leq Kn^{-\mu}, \qquad \mu = 2\beta(1+\delta)(2+\delta)/\delta^2, \quad \beta > 1,\ K > 0.$$

*Then there exists a $C = C(K, \beta, \delta)$ such that for every $n \geq 1$*

$$\Delta_n \leq Cdc_0^{-(1+\beta+\delta)/(1+\beta)}\sigma_n^{-(\beta-1)\delta/(1+\beta)}. \tag{7.1.8}$$

**Remark 7.1.1.** Theorem 7.1.1 implies the Theorem 2 of Tikhomirov (1980). Theorem 4 of Tikhomirov (1980) also points out that if $E|X_1|^{4+\gamma} < \infty$, $\gamma > 0$, then

$$\Delta_n \leq cn^{-1/2}\log n. \tag{7.1.9}$$

The proofs of Theorems will use the following lemmas.

**Lemma 7.1.1.** *We have*

$$i\sum_{j=1}^{n} a_j^{(1)} = -t + \Theta 20(1+12\alpha)^{1/2}d^{2/s}\sigma_n(\alpha(h+1))^{(s-2)/2s}|t|/c_0^{3/2}$$
$$+\Theta(2^{3-s}/(s-1))d(2h+1)^{s-1}|t|^{s-1}/(c_0\sigma_n^{s-2}), \tag{7.1.10}$$

*where $\alpha = \sum_{j=1}^{n}(\alpha(j))^{(s-2)/s}$, $s = 2+\delta$, $0 < \delta \leq 1$ and $\Theta$ is a constant with $|\Theta| \leq 1$.*

**Proof.** Let $\tau$ be a random variable uniformly distributed on the set $\{1, 2, \cdots, n\}$, and independent of $\{X_1, X_2, \cdots, X_n\}$. It is easy to see that for $i = 0, 1, \cdots, k$ and any $b \geq 1$

$$E|Z_\tau^{(l)}|^b \leq (2hl+1)^b \sum_{j=1}^{n} E|Y_j|^b/n. \tag{7.1.11}$$

By the Hölder inequality and (7.1.11), we obtain

$$\sum_{j=1}^{n} E(|Y_j||Z_j^{(1)}|^{s-1}) \leq n\|Y_\tau\|_s\|Z_\tau^{(1)}\|_s^{s-1}$$
$$\leq (2h+1)^{s-1}\sum_{j=1}^{n} E|Y_j|^s. \tag{7.1.12}$$

Denote by $\hat{z}_j{}^{(l)}$ the sum of those $Y_p$ from $z_j^{(l)}$, for which $p < j - lh$, and by $\tilde{z}_j{}^{(l)}$ the sum of those $Y_p$ from $z_j^{(l)}$, for which $p > j + lh$. Thus $z_j^{(l)} = \hat{z}_j{}^{(l)} +$

$\tilde{z}_j^{(l)}$. From Lemma 1.2.4 it follows that

$$|EY_j\hat{z}_j^{(l)}| \le 10(\alpha(h+1))^{1-\frac{1}{2}-\frac{1}{s}}\|Y_j\|_s\|\hat{z}_j^{(l)}\|_2$$
$$\le 10(1+12\alpha)^{1/2}d^{2/s}(\alpha(h+1))^{(s-2)/2s}/(c_0^{1/2}\sigma_n).$$

The same quantity also bounds $|EY_j\tilde{z}_j^{(l)}|$ for $j = 1,2,\cdots,n$. Since $\sum_{j=1}^n EY_jZ_n = 1$, we have

$$\sum_{j=1}^n EY_jZ_j^{(1)} = 1 - \sum_{j=1}^n(EY_j\hat{z}_j^{(1)} + EY_j\tilde{z}_j^{(1)})$$
$$= 1 + \Theta 20(1+12\alpha)^{1/2}d^{2/s}$$
$$\times (\alpha(h+1))^{(s-2)/2s}\sigma_n/c_0^{3/2}, \tag{7.1.13}$$

where we have used (7.1.6). According to Taylor's formula

$$i\sum_{j=1}^n a_j^{(1)} = -\sum_{j=1}^n EY_jZ_j^{(1)}t + \Theta(2^{3-s}/(s-1))\sum_{j=1}^n E(|Y_j||Z_j^{(1)}|^{s-1})|t|^{s-1}.$$

Inserting (7.1.12) and (7.1.13) into this equation and using (7.1.6), we obtain (7.1.10).

**Lemma 7.1.2.** *For* $|t| \le (\sigma_n/32d^{1/s}(h+1)) =: T_1$, $j = 1,2,\cdots,n$ *and* $r = 3,4,\cdots,k$, *we have*

$$|a_j^{(r-1)}| \le 4^r d^{3/s}(h+1)^2t^2(1/2)^{4r}/\sigma_n^3$$
$$+ c\frac{d^{1/s}}{\sigma_n}(\alpha(h+1))^{(s-2)/s}\Big[r\Big(\frac{1}{2}\Big)^r + r4^r(\alpha(h+1))^{1/s}\Big]$$
$$=: a^{(r-1)} \tag{7.1.14}$$

*and*

$$|a_j^{(1)}| \le 2d^{2/s}(h+1)|t|/\sigma_n^2. \tag{7.1.15}$$

**Proof.** Note that $a_j^{(r-1)} = E\Big\{Y_j\prod_{l=1}^{r-1}\xi_j^{(l)}\Big\}$. From

$$|a_j^{(1)}| = |EY_j\xi_j^{(1)}| \le |t|E|X_j|\sum_{|p-j|\le h}|X_p|/\sigma_n^2$$
$$\le 2d^{2/s}(h+1)|t|/\sigma_n^2,$$

(7.1.15) follows.

Now, to prove (7.1.14). Denote

$$\xi_{j1}^{(l)} = \exp\Big\{it \sum_{p=j-lh+1}^{j-(l-1)h} Y_p\Big\} - 1,$$

$$\xi_{j2}^{(l)} = \exp\Big\{it \sum_{p=j+(l-1)h+1}^{j+lh} Y_p\Big\} - 1.$$

Obviously

$$|\xi_j^{(l)}| \le |\xi_{j1}^{(l)}| + |\xi_{j2}^{(l)}|.$$

It follows that

$$\begin{aligned} |a_j^{(r-1)}| &\le E\Big|Y_j \prod_{l=1}^{r-1} \xi_j^{(l)}\Big| \\ &\le \sum_{k=0}^{r-1} \sum{}^{*} E\Big|Y_j \prod_{\nu=1}^{k} \xi_{j1}^{(l_\nu)} \prod_{\mu=k+1}^{r-1} \xi_{j2}^{(l_\mu)}\Big| \end{aligned} \tag{7.1.16}$$

where the summation $\sum^*$ sums up for $1 \le l_1 < \cdots < l_k \le r-1, \quad 1 \le l_{k+1} < \cdots < l_{r-1} \le r-1, \quad l_\nu \ne l_\mu \ (\nu \ne \mu)$. For each term of the right hand side of (7.1.16) we have

$$\begin{aligned} E\Big|Y_j &\prod_{\nu=1}^{k} \xi_{j1}^{(l_\nu)} \prod_{\mu=k+1}^{r-1} \xi_{j2}^{(l_\mu)}\Big| \\ &\le \Big(E\Big|Y_j \prod_\nu{}' \xi_{j1}^{(l_\nu)} \prod_\mu{}' \xi_{j2}^{(l_\mu)}\Big|^{\frac{2+\delta}{1+\delta}}\Big)^{\frac{1+\delta}{2+\delta}} \cdot \\ &\quad \cdot \Big(E\Big|\prod_\nu{}'' \xi_{j1}^{(l_\nu)} \prod_\mu{}'' \xi_{j2}^{(l_\mu)}\Big|^{2+\delta}\Big)^{\frac{1}{2+\delta}} \end{aligned} \tag{7.1.17}$$

where $\prod'$ is the product for all even $l$, $\prod''$ is the product for all odd $l$. It is easy to see that

$$\begin{aligned} E\Big|Y_j &\prod_\nu{}' \xi_{j1}^{(l_\nu)} \prod_\mu{}' \xi_{j2}^{(l_\mu)}\Big|^{\frac{2+\delta}{1+\delta}} \\ &\le (E|Y_j|^{2+\delta})^{\frac{1}{1+\delta}} \prod_\nu{}' E|\xi_{j1}^{(l_\nu)}|^{\frac{2+\delta}{1+\delta}} \prod_\mu{}' E|\xi_{j2}^{(l_\mu)}|^{\frac{2+\delta}{1+\delta}} \\ &\quad + cr2^{r-1}(\alpha(h+1))^{\frac{\delta}{1+\delta}} (E|Y_j|^{2+\delta})^{\frac{1}{1+\delta}}, \end{aligned} \tag{7.1.18}$$

$$\begin{aligned} E\Big|&\prod_\nu{}'' \xi_{j1}^{(l_\nu)} \prod_\mu{}'' \xi_{j2}^{(l_\mu)}\Big|^{2+\delta} \\ &\le cr2^{r-1}\alpha(h+1) + \prod_\nu{}'' E|\xi_{j1}^{(l_\nu)}|^{2+\delta} \prod_\mu{}'' E|\xi_{j2}^{(l_\mu)}|^{2+\delta} \end{aligned} \tag{7.1.19}$$

and

$$\max\Big(E|\xi_{j1}^{(l_\nu)}|^{2+\delta}, \quad E|\xi_{j2}^{(l_\mu)}|^{2+\delta}\Big) \le \Big(\frac{|t|b_h}{\sigma_n}\Big)^{2+\delta} \wedge 2^{2+\delta},$$

where $b_h = \max_{1\le i\le h+1}\Big\|\sum_{j=1}^{i} X_j\Big\|_{2+\delta}$. It follows from (7.1.17)–(7.1.19) that we have

$$\begin{aligned} &E\Big|Y_j\prod_\nu \xi_{j1}^{(l_\nu)}\prod_\mu \xi_{j2}^{(l_\mu)}\Big| \\ &\le 2^r d^{3/s}(h+1)^2t^2(1/2)^{4r}/\sigma_n^3 \\ &\quad + c\frac{d^{1/s}}{\sigma_n}(\alpha(h+1))^{\frac{s-2}{s}}\Big[r(\frac{1}{2})^{2r} + r2^r(\alpha(h+1))^{1/s}\Big]. \qquad (7.1.20)\end{aligned}$$

Note that the number of terms of the right hand side of (7.1.16) does not exceed $2^r$. (7.1.14) is proved.

We shall use the following results: for any finite $p \ge 1$

$$\sum_{r=1}^{\infty} r^p/e^r < \infty. \qquad (7.1.21)$$

And for $j = 1, \cdots, n$; $r = 2, \cdots, k$

$$(E|\eta_j^{(r)}|^s)^{1/s} \le |t|d^{1/s}(2rh+1)/\sigma_n. \qquad (7.1.22)$$

Assume that $1 \le h \le n-1$, $2 \le k \le n-1$ and

$$k^{3/2}4^k(\alpha(h+1))^{1/s} \le 1. \qquad (7.1.23)$$

**Lemma 7.1.3.** *If (7.1.23) is satisfied and $|t| \le T_1$. Then*

$$\begin{aligned}&\Big|\sum_{j=1}^{n} a_j^{(1)}E\eta_j^{(2)} + \sum_{r=3}^{k}\sum_{j=1}^{n} a_k^{(r-1)}E(\eta_j^{(r)}+1)\Big| \\ &\quad \le \frac{c}{c_0}\Big[d^{3/s}(h+1)^2t^2/\sigma_n + d^{1/s}\sigma_n(\alpha(h+1))^{(s-2)/s}\Big], \qquad (7.1.24)\end{aligned}$$

$$\begin{aligned}&\sum_{r=2}^{k}\Big|\sum_{j=1}^{n} a_j^{(r-1)}E[(\eta_j^{(r)} - E\eta_j^{(r)})e^{itZ_n}]\Big| \\ &\quad \le \frac{c}{c_0^{1/2}}[(h+1)^{1/2} + \alpha^{1/2}][d^{3/s}(h+1)^2t^2/\sigma_n^2 \\ &\qquad + d^{1/s}(\alpha(h+1))^{(s-2)/s}]. \qquad (7.1.25)\end{aligned}$$

*In addition, if* $k \geq (\log n)/8\log 2$, *then*

$$\sum_{j=1}^{n}\Big|E\Big(Y_j\prod_{l=1}^{k}\xi_j^{(l)}e^{itz_j^{(k)}}\Big)\Big|$$
$$\leq cc_0^{-1/2}d^{3/s}(h+1)^2t^2/\sigma_n^2$$
$$+ cc_0^{-1}d^{1/s}\sigma_n(\alpha(h+1))^{(s-2)/s}. \tag{7.1.26}$$

**Proof.**

$$\Big|\sum_{j=1}^{n}a_j^{(1)}E\eta_j^{(2)} + \sum_{r=3}^{k}\sum_{j=1}^{n}a_j^{(r-1)}E(\eta_j^{(r)}+1)\Big|$$
$$\leq \sum_{j=1}^{n}|a_j^{(1)}|E|\eta_j^{(2)}| + \sum_{r=3}^{k}\sum_{j=1}^{n}|a_j^{(r-1)}|,$$

which implies (7.1.24) by using Lemma 7.1.2, (7.1.22), (7.1.23) and (7.1.6).

Further, $\mathrm{Cov}(\xi,\eta) = E(\xi - E\xi)\overline{(\eta - E\eta)}$. Applying the Hölder inequality and Lemma 1.2.4, we obtain that for $r = 2, 3, \cdots, k$

$$\Big|\sum_{j=1}^{n}a_j^{(r-1)}E\Big[(\eta_j^{(r)} - E\eta_j^{(r)})e^{itZ_n}\Big]\Big|$$
$$\leq\Big\{\Big(\sum_{j=1}^{n}\sum_{|p-j|\leq 2rh} + \sum_{j=1}^{n}\sum_{|p-j|>2rh}\Big)a_j^{(r-1)}\overline{a_p^{(r-1)}}\mathrm{Cov}(\eta_j^{(r)},\eta_p^{(r)})\Big\}^{1/2}$$
$$\leq\Big\{\sum_{j=1}^{n}\sum_{|p-j|\leq 2rh}|a_j^{(r-1)}||a_p^{(r-1)}|||\eta_j^{(r)}||_2||\eta_p^{(r)})||_2\Big\}^{1/2}$$
$$+\Big\{24\sum_{j=1}^{n}\sum_{|p-j|>2rh}|a_j^{(r-1)}||a_p^{(r-1)}|$$
$$\times(\alpha(|p-j|-2rh))^{(s-2)/s}||\eta_j^{(r)}||_s||\eta_p^{(r)})||_s\Big\}^{1/2}. \tag{7.1.27}$$

Let $r = 3, 4, \cdots, k$. Since for $j = 1, \cdots, n$, $|\eta_j^{(r)}| \leq 2$ and for $|t| \leq T_1$, $|a_j^{(r-1)}| \leq a^{(r-1)}$, and $a^{(r-1)}$ is independent of j, considering (7.1.6) we get that right hand side of (7.1.27) does not exceed

$$2a^{(r-1)}\Big\{\sum_{j=1}^{n}\sum_{|p-j|\leq 2rh}1\Big\}^{1/2}$$
$$+ca^{(r-1)}\Big\{\sum_{j=1}^{n}\sum_{|p-j|>2rh}(\alpha(|p-j|-2rh))^{(s-2)/s}\Big\}^{1/2}$$
$$\leq cc_0^{-1/2}\sigma_n[r^{1/2}(h+1)^{1/2} + \alpha^{1/2}]a^{(r-1)}. \tag{7.1.28}$$

For $r = 2$ we proceed in the same way but we estimate $\eta_j^{(2)}$ for $j = 1, 2 \cdots, n$ according to (7.1.22). One gets that for $r = 2$ the right hand side of (7.1.27) does not exceed

$$cc_0^{-1/2} d^{3/s}(h+1)^{5/2} t^2/\sigma_n^2 + cc_0^{-1/2} \alpha^{1/2} d^{3/s}(h+1)^2 t^2/\sigma_n^2. \qquad (7.1.29)$$

Adding over $r$ from 2 to $k$ in (7.1.27) and considering (7.1.28), (7.1.29) and (7.1.23), we get the proof of (7.1.25).

Finally, since for $k \geq (\log n)/(8 \log 2)$ we have that $(1/2)^{4k} \leq n^{-1/2}$, (7.1.26) follows from the proof of Lemma 7.1.2, (7.1.6) and (7.1.23). Lemma 7.1.3 is proved.

**Lemma 7.1.4.** *We have*

$$\sum_{j=1}^n |EY_j e^{itz_j^{(1)}}| \leq 32 c_0^{-1} d^{1/s} \sigma_n (\alpha(h+1))^{(s-1)/s}. \qquad (7.1.30)$$

*If (7.1.23) is satisfied, then*

$$\sum_{r=2}^k \sum_{j=1}^n \Big| EY_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itz_j^{(r)}} - E\Big(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)}\Big) Ee^{itz_j^{(r)}} \Big|$$
$$\leq 48 c_0^{-1} d^{1/s} \sigma_n (\alpha(h+1))^{(s-1)/s}.$$

**Proof.** We only prove the second inequality since the first can be proved analogously. From the definition of $z_j^{(r)}$, we have that for all $r = 2, 3, \cdots, k$, $\widehat{z}_j^{(r)}$ and $\widetilde{z}_j^{(r)}$ cannot be equal to zero simultaneously. Without loss of generality we assume that $\widehat{z}_j^{(r)} \neq 0$ and $\widetilde{z}_j^{(r)} \neq 0$. Then, according to Lemma 1.2.4

$$\Big| E\Big(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itz_j^{(r)}}\Big) - E\Big(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)}\Big) Ee^{itz_j^{(r)}} \Big|$$
$$\leq \Big| E\Big(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itz_j^{(r)}}\Big) - E\Big(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{it\widehat{z}_j^{(r)}}\Big) Ee^{it\widetilde{z}_j^{(r)}} \Big|$$
$$+ \Big| E\Big(Y_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{it\widehat{z}_j^{(r)}}\Big) - EY_j \prod_{l=1}^{r-1} \xi_j^{(l)} Ee^{it\widehat{z}_j^{(r)}} \Big|$$
$$+ \Big| Ee^{it\widehat{z}_j^{(r)}} Ee^{it\widetilde{z}_j^{(r)}} - Ee^{itz_j^{(r)}} \Big| \cdot \Big| EY_j \prod_{l=1}^{r-1} \xi_j^{(l)} \Big|$$
$$\leq 48 d^{1/s} (\alpha(h+1))^{(s-1)/s} 2^{r-1}/\sigma_n.$$

Adding the inequalities obtained over all $j = 1, 2, \cdots, n$ and $r = 2, 3, \cdots, k$, we use (7.1.6) and (7.1.23). Lemma 7.1.4 is proved.

**Proof of Theorem 7.1.1.**

According to the basic method of the proof , we first establish the differential equation (7.1.4). Denote $\alpha = \sum_{k=1}^{n} \alpha(k)^{\delta/(2+\delta)}$, let

$$\begin{aligned}
a_0 &= cd^{1/s}\sigma_n(\alpha(h+1))^{(s-2)/s}/c_0,\\
a_1 &= cd^{2/s}(1+\alpha)^{1/2}\sigma_n(\alpha(h+1))^{(s-2)/2s}/c_0^{3/2},\\
a_2 &= cd^{3/s}(h+1)^2/c_0\sigma_n, \qquad a_3 = cd(h+1)^{s-1}/c_0\sigma_n^{s-2},\\
b_0 &= cd^{1/s}((h+1)^{1/2}+\alpha^{1/2})(\alpha(h+1))^{(s-2)/s}/c_0^{1/2} + a_0,\\
b_2 &= cd^{3/s}((h+1)^{1/2}+\alpha^{1/2})(h+1)^2/c_0^{1/2}\sigma_n^2,\\
T_2 &= \min\{1/a_0, 1/(6a_2), (1/6a_3)^{1/(s-2)}\}.
\end{aligned}$$

Suppose that $1 \le h \le n-1$, $\quad 2 \le k \le n-1$ such that

$$k \ge \frac{\log n}{8\log 2}, \qquad k^{3/2}4^k(\alpha(h+1))^{1/s} \le 1, \quad 2kh+1 < n. \tag{7.1.31}$$

From (7.1.3), Lemmas 7.1.1, 7.1.3 and 7.1.4, it follows that

$$f_n'(t) = (-t + \Theta a(t))f_n(t) + \Theta b(t) \qquad \text{as} \quad |t| \le T_1 \tag{7.1.32}$$

where $|\Theta| \le 1$,

$$a(t) = a_0 + a_1|t| + a_2t^2 + a_3|t|^{s-1}, \qquad b(t) = b_0 + b_2t^2.$$

Next, we solve the linear differential equation (7.1.32) and get

$$\begin{aligned}
|f_n(t) - e^{-t^2/2}|
&\le |x_0|e^{-t^2/2+|x_0|}\\
&\quad + e^{-t^2/2}\int_0^{|t|} b(u)\exp\Big\{\frac{u^2}{2} + \int_{|u|}^{|t|} a(v)dv\Big\}du,
\end{aligned} \tag{7.1.33}$$

where $x_0 = \int_0^t \Theta a(u)du$. Let $0 \le u \le t$. Then for $a_1 \le 1/6$ and $|t| \le T_2$ we have

$$\int_u^t a(v)dv \le 1 + \frac{t^2-u^2}{4}, \tag{7.1.34}$$

and it is easy to see that

$$\int_0^{|t|} u^2\exp(u^2/4)du \le 2|t|\exp(t^2/4), \tag{7.1.35}$$

$$\int_0^{|t|} \exp(u^2/4)du \le \min(4/|t|, |t|)\exp(t^2/4). \tag{7.1.36}$$

From (7.1.33)–(7.1.36) it follows that for $|t| \le \min(T_1, T_2)$ and $a_1 \le 1/6$

$$|f_n(t) - e^{-t^2/2}| \le (a_0|t| + \frac{a_1}{2}t^2 + \frac{a_2}{3}|t|^3 + \frac{a_3}{s}|t|^s)e^{-t^2/4+1} + eb_0\Big\{\frac{4}{|t|} \wedge t\Big\} + 2eb_2|t|. \tag{7.1.37}$$

By Lemma 1.2.4 we have

$$\sigma_n^2 \le (1 + 12\alpha)d^{2/s}n,$$

and hence $1 \le (1+12\alpha)^{1/2}\mathcal{K}$, $\mathcal{K} = d^{1/s}/c_0^{1/2}$. Thus from (7.1.37) and (7.1.1) we get

$$\begin{aligned}\Delta_n \le & c\Big\{\mathcal{K}^s \frac{(h+1)^{s-1}}{\widetilde{\sigma}_n^{s-2}} + \mathcal{K}^3 \frac{(h+1)^2}{\widetilde{\sigma}_n} \\ & + \mathcal{K}^2((h+1)^{1/2} + \alpha^{1/2})\frac{h+1}{\widetilde{\sigma}_n} \\ & + \mathcal{K}^2(1+\alpha)^{1/2}\widetilde{\sigma}_n(\alpha(h+1))^{(s-2)/2s} \\ & + \mathcal{K}((h+1)^{1/2} + \alpha^{1/2})(\alpha(h+1))^{(s-2)/s}\Big\}\end{aligned} \tag{7.1.38}$$

where $\widetilde{\sigma}_n^2 = \sigma_n^2/c_0$.

Finally, we prove (7.1.7). Let $0 < \lambda \le \lambda_2 = \frac{4(2+\delta)}{\delta}\log\widetilde{\sigma}_n$ and $h = \left[\frac{4(2+\delta)}{\lambda\delta}\log\widetilde{\sigma}_n\right]$, $n \ge 1$. Then $\frac{4(2+\delta)}{\lambda\delta} \le \frac{h+1}{\log\widetilde{\sigma}_n}$. Hence, by the condition $\alpha(n) \le Ke^{-\lambda n}$, we obtain

$$\begin{aligned}\alpha^{1/2} &= \Big(\sum_{\tau=1}^{n}(\alpha(\tau))^{\delta/(2+\delta)}\Big)^{1/2} \\ &\le \frac{1}{2}K^{\delta/2(2+\delta)}(h+1)^{1/2}/(\log\widetilde{\sigma}_n)^{1/2} \\ &\le C(K,\delta)(h+1)^{1/2}.\end{aligned} \tag{7.1.39}$$

This means

$$1 \le C(K,\delta)(h+1)^{1/2}\mathcal{K}. \tag{7.1.40}$$

For the chosen $h$,

$$\begin{aligned}(\alpha(h+1))^{\delta/2(2+\delta)} &\le K^{\delta/2(2+\delta)}e^{-\lambda\delta(h+1)/2(2+\delta)} \\ &\le K^{\delta/2(2+\delta)}\widetilde{\sigma}_n^{-2},\end{aligned} \tag{7.1.41}$$

$$(\alpha(h+1))^{\delta/(2+\delta)} \le K^{\delta/(2+\delta)}\widetilde{\sigma}_n^{-4}. \tag{7.1.42}$$

We get from (7.1.39)-(7.1.42) and (7.1.37) that

$$\Delta_n \le C(K,\delta)\Big[\mathcal{K}^{2+\delta}\frac{(h+1)^{1+\delta}}{\widetilde{\sigma}_n^\delta} + \mathcal{K}^3\frac{(h+1)^2}{\widetilde{\sigma}_n}\Big]$$
$$\le C(k,\delta)\mathcal{K}^{2+\delta}\frac{(h+1)^{1+\delta}}{\widetilde{\sigma}_n^\delta}. \tag{7.1.43}$$

It remains to check whether there exists a $k, 2 \le k \le n-1$, such that for $h = \left[\frac{4(2+\delta)}{\lambda\delta}\log\widetilde{\sigma}_n\right]$, where $\lambda_1 \le \lambda \le \lambda_2$ we have

$$k \ge \frac{\log n}{8\log 2}, \qquad k^{3/2}4^k(\alpha(h+1))^{1/(2+\delta)} \le 1, \qquad 2kh+1 < n. \tag{7.1.44}$$

It is easy to verify that for all large $n (n > n_0)$ and for

$$k = \Big[\frac{(4/\delta)\log\widetilde{\sigma}_n - (\log K)/(2+\delta)}{(3/2)+\log 4}\Big]$$

the first two inequalities hold and the third inequality does not constrict the interval of change of the parameter $\lambda$.

Considering (7.1.40), we have

$$n^{-1} \le C(K,\delta)\mathcal{K}^{2+\delta}(h+1)^{1+\delta}/\widetilde{\sigma}_n^\delta.$$

This also proves Theorem 7.1.1 for all $n \le n_0$. The proof of Theorem 7.1.1 is completed.

The proof of Theorem 7.1.2 is analogous to that of Theorem 7.1.1. We only indicate that for this it is sufficient to set $h = [\widetilde{\sigma}_n^\alpha]$, where $\alpha = 2\delta/(\beta+1)(\delta+1)$, and

$$k = \Big[\frac{(\alpha\mu/(2+\delta))\log\widetilde{\sigma}_n - (\log K)/(2+\delta)}{(3/2)+\log 4}\Big]$$

in (7.1.31)-(7.1.33).

**Remark 7.1.2.** Tikhomirov (1980) proved the Berry-Esseen inequality for the $\rho$-mixing sequence with the exponential rate of decay of $\rho(n)$. Zuparov (1991) improved his method to prove the following theorem in which there is a better order of $\Delta_n$ when $E|X_1|^s < \infty$, $2 < s < s_0 (< 1+\sqrt{3})$, and the rate of $\rho(\cdot)$ was weakened.

**Theorem 7.1.3.** *Let $\{X_n, n \ge 1\}$ be a strictly stationary $\rho$-mixing sequence with $EX_1 = 0$ and*

$$\rho(n) \le Kn^{-\theta}, \qquad \theta > 0, \quad K > 0, E|X_1|^s < \infty,$$

$$2 < s < s_0(\theta) = \frac{\theta-1}{\theta} + \sqrt{\left(\frac{\theta-1}{\theta}\right)^2 + \frac{4+2\theta}{\theta}}.$$

*If $\sigma_n^2 = ES_n^2 \geq \tau n E X_1^2$. Then there exists a constant $C = C(s,\theta,K,\tau)$ depending only on $s,\theta,K$ and $\tau$ such that*

$$\Delta_n \leq C(s,\theta,K,\tau)\beta_s/n^{(s-2)/2},$$

*where $\beta_s = E|X_1|^s/(EX_1^2)^{s/2}$,* $s_0 < 1+\sqrt{3}$.

The proof of Theorem 7.1.3 will not be presented here.

## 7.2 The rate of weak convergence for a $\varphi$-mixing sequence

Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $EX_n = 0$, $\sigma_n^2 = ES_n^2$. Define the partial sum process as follows:

$$W_n(t) = \frac{1}{\sigma_n}\Big(S_{[nt]} + \frac{t\sigma_n^2 - \sigma_{[nt]}^2}{\sigma_{[nt]+1}^2 - \sigma_{[nt]}^2} X_{[nt]+1}\Big), \tag{7.2.1}$$

if $\sigma_{[nt]}^2 \leq t\sigma_n^2 \leq \sigma_{[nt]+1}^2$.

Denote by $P_n$ the distribution of $W_n$ in $C[0,1]$, i.e. $P_n = P \circ W_n^{-1}$. The weak invariance principle holds, i.e. $P_n \Longrightarrow W$, that is equivalent to

$$L(P_n, W) \longrightarrow 0, \tag{7.2.2}$$

where $L(P,Q)$ is the *Lévy-Prohorov distance*:

$$L(P,Q) = \inf\{\varepsilon : \forall B \in \mathcal{F}, P(B) \leq Q(B^\varepsilon)+\varepsilon, Q(B) \leq P(B^\varepsilon)+\varepsilon\}, \tag{7.2.3}$$

where $\mathcal{F}$ is the Borel $\sigma$-field of $C[0,1]$, $B^\varepsilon$ is the $\varepsilon$-neighborhood of Borel set $B$,

$$B^\varepsilon = \{y : y \in C[0,1], \exists z \in B, \|y-z\| < \varepsilon\}, \qquad \|y-z\| = \sup_{o \leq t \leq 1} |y(t)-z(t)|.$$

Denote

$$\Lambda_n(\varepsilon) = \sup_{B\in\mathcal{F}} |P(W_n \in B) - P(W \in B^\varepsilon)|,$$

It is obvious that

$$L(P_n, W) = \inf_\varepsilon(\varepsilon \vee \Lambda_n(\varepsilon)). \tag{7.2.4}$$

Then without changing the distribution of $\{W_n, n \geq 1\}$, we can redefine process $\{\widetilde{W}_n, n \geq 1\}$ on a richer probability space together with the standard Wiener process $\{W(t), t \geq 0\}$ such that

$$\Lambda_n(\varepsilon) \leq P\{\|\widetilde{W}_n - W\| \geq \varepsilon\}.$$

Therefore, in order to estimate the rate of weak convergence, we need only consider the following inequality:

$$L(P_n, W) \le \inf_{\varepsilon}(\varepsilon \vee P\{\|W_n - W\| \ge \varepsilon\}).$$

For the independent random variables $\{X_n, n \ge 1\}$, Prohorov (1956) gave a precise estimation :

$$L(P_n, W) = O(L_3^{1/4} \log^2 L_3),$$

where $L_3 = \sum_{k=1}^n E|X_k|^3/\sigma_n^3$. For i.i.d.r.v.'s, Borovkov (1973) proved: if $E|X_1|^{2+\delta}$
$< \infty,\ 0 < \delta \le 1$, then

$$L(P_n, W) = O(L_{2+\delta}^{1/(3+\delta)}) = O(n^{-\frac{\delta}{2(3+\delta)}}). \tag{7.2.5}$$

Borovkov (1973) also pointed out that the weak convergence rate (7.2.5) can not be improved if we use only the Skorohod method. Utev (1981) generalized this estimation to the case $0 < \delta < 3$ and gave

$$L(P_n, W) = O(n^{-\frac{\delta}{2(3+\delta)}}), \text{when} E|X_1|^{2+\delta} < \infty,\ 0 < \delta < 3. \tag{7.2.6}$$

Utev (1984) gave the same estimation as (7.2.6) for $\varphi$-mixing sequence under a stronger condition on the rate of decay of $\varphi(n)$.

**Theorem 7.2.1.** *Let $\{X_n, n \ge 1\}$ be a strictly stationary $\varphi$-mixing sequence of random variables with $EX_1 = 0, EX_1^2 = 1$ and $E|X_1|^{2+\delta} < \infty$ for some $0 < \delta < 3$. Suppose*

$$\varphi(n) \le cn^{-g}$$

*where $g > j(u)(j(u) - 1)$, $u = (12 + 5\delta)/(2(3 - \delta))$, $j(u) = \min\{2k : 2k \ge u, k \in \mathbb{N}\}$. Then*

$$L(P_n, W) \le cn^{-\frac{\delta}{2(3+\delta)}}.$$

By using Lemma 2.2.10, Lu (1993) weakened the conditions and proved the following theorem.

**Theorem 7.2.2.** *Let $\{X_n, n \ge 1\}$ be a $\varphi$-mixing sequence of random variables with $EX_n = 0$, $A_0 = \sup_n EX_n^2 < \infty$, $A_\delta = \sup_n E|X_n|^{2+\delta} < \infty$, $0 < \delta < 3$. Suppose that*

*(i) if $\sigma_n^2 \to \infty$ as $n \to \infty$,*

*(ii)* $\varphi(n) \le cn^{-g}$, $g \ge (3\delta+\varepsilon)/(2(3-\delta)) \vee (2+\varepsilon)$ *for any* $\varepsilon > 0$. *Then we have*

$$L(P_n, W) = O(n^{-\frac{\delta}{2(3+\delta)}}). \tag{7.2.7}$$

**Proof.** By using (i), $A_\delta < \infty$, Lemma 2.2.2 and Lemma 2.2.10, we have

$$\begin{aligned} E\max_{1\le i\le n}|S_k(i)|^{2+\delta} &\le c\Big((n)^{\frac{2+\delta}{2}} + E\max_{k\le i\le k+n}|X_i|^{2+\delta}\Big) \\ &\le cn^{1+\delta/2}. \end{aligned} \tag{7.2.8}$$

Without loss of generality, we can assume that $\sigma_n^2 = c_0 n$, $c_0 = 1$. Then $W_n$ is a random polygonal line with nodes at $(k/n, S_k/\sqrt{n})$, $k = 0, 1, \cdots, n$. Denote

$$X_i^{(1)} = X_i I(|X_i| \le y\sqrt{n}) - EX_i I(|X_i| \le y\sqrt{n}) \qquad i = 1, \cdots, n,$$

where $y = n^{-1/2}(\sum_{i=1}^n E|X_i|^{2+\delta})^{1/(2+\delta)} \le n^{-\frac{\delta}{2(2+\delta)}} A_\delta^{\frac{1}{2+\delta}}$. Note that there exist all finite moments of $X_i^{(1)}$ for any given $n$.

Let $S_k^{(1)} = \sum_{i=1}^k X_i^{(1)}$ and $W_n^{(1)}$ be a random polygonal line with nodes at $(k/n, S_k^{(1)}/\sqrt{n})$. By using Lemma 2.2.10 , we have

$$\begin{aligned} &P\Big\{\|W_n^{(1)} - W_n\| \ge cn^{-\delta/2(3+\delta)}\Big\} \\ &\le P\Big\{\max_{0\le k\le n}\Big|\frac{1}{\sqrt{n}}(S_k - S_k^{(1)})\Big| \ge cn^{-\delta/2(3+\delta)}\Big\} \\ &\le cn^{-3(2+\delta)/2(3+\delta)} E\max_{o\le k\le n}|S_k - S_k^{(1)}|^{2+\delta} \\ &\le cn^{-3(2+\delta)/2(3+\delta)}\Big[\Big(\max_{1\le k\le n} E|S_k - S_k^{(1)}|^2\Big)^{\frac{2+\delta}{2}} \\ &\quad + \sum_{i=1}^n E|X_i - X_i^{(1)}|^{2+\delta}\Big] \\ &\le cn^{-\delta/2(3+\delta)}. \end{aligned} \tag{7.2.9}$$

Next, put $l = [n^{3/(3+\delta)-\eta}]$, $m = [n^{\delta/(3+\delta)+\eta}]$, $\eta > 0$ (specified later on), write $n = lm + r$, $0 \le r < l$. Let $W_n^{(2)}$ be a random polygonal line

with nodes at $(kl/n, S_{kl}^{(1)}/\sqrt{n})$, $k = 0, 1, \cdots, m$ and $(1, S_n^{(1)}/\sqrt{n})$. We have

$$\begin{aligned}
&P\{\|W_n^{(2)} - W_n^{(1)}\| \geq b\} \\
&\leq \sum_{k=0}^{m} P\{\max_{1\leq i\leq l} |S_{kl+i}^{(1)} - S_{kl}^{(1)}| \geq b\sqrt{n}\} \\
&\leq c\sum_{k=0}^{m} b^{-u}n^{-u/2}E\max_{1\leq i\leq l} |S_{kl}^{(1)}(i)|^u \\
&\leq cmb^{-u}n^{-u/2}l^{u/2} \\
&= cb^{-u}(\frac{n}{l})^{1-u/2},
\end{aligned}$$

where we have applied Lemma 2.2.10. If we take $b = cn^{-\delta/2(3+\delta)}$, $u \geq 2 + 3\delta/[(3+\delta)\eta]$, then

$$\left(\frac{n}{l}\right)^{1-u/2} \leq n^{-\frac{\delta(u+1)}{2(3+\delta)}}.$$

Therefore we have

$$P\{\|W_n^{(2)} - W_n^{(1)}\| \geq cn^{-\frac{\delta}{2(3+\delta)}}\} \leq cn^{-\frac{\delta}{2(3+\delta)}}. \tag{7.2.10}$$

Put $h = [n^\theta]$, $\theta > 0$ (specified later on). Let $d = l - h$,

$$\xi_k^{(1)} = \sum_{i=1}^{d} X_{(k-1)l+i}^{(1)},$$

$W_n^{(3)}$ be a random polygonal line with nodes at $(kl/n, \sum_{j=1}^{k} \xi_j^{(1)}/\sqrt{n})$, we have

$$\begin{aligned}
&P\{\|W_n^{(3)} - W_n^{(2)}\| \geq b\} \\
&= P\Big\{\max_{1\leq k\leq m+1} |S_{kl}^{(1)} - \sum_{j=1}^{k}\sum_{i=1}^{d} X_{(j-1)l+i}^{(1)}| \geq b\sqrt{n}\Big\} \\
&= P\Big\{\max_{1\leq k\leq m+1} |\sum_{j=1}^{k}\sum_{i=1}^{h} X_{(j-1)l+d+i}^{(1)}| \geq b\sqrt{n}\Big\} \\
&\leq cb^{-u}n^{-u/2}\Big\{\Big[\max_{1\leq k\leq m+1} E\Big(\sum_{j=1}^{k}\sum_{i=1}^{h} |X_{(j-1)l+d+i}^{(1)}|\Big)^2\Big]^{u/2} \\
&\qquad + \sum_{j=1}^{m+1} E\Big[\sum_{i=1}^{h} |X_{(j-1)l+d+i}^{(1)}|\Big]^u\Big\}
\end{aligned}$$

where the first term does not exceed $c(mh)^{u/2}$ and the last sum does not exceed $c\,m\,h^{u/2} = o((mh)^{u/2})$ by Lemma 2.2.4, therefore

$$\begin{aligned} &P\{\|W_n^{(3)} - W_n^{(2)}\| \geq b\} \\ &\qquad \leq cb^{-u}n^{-u/2}(mh)^{u/2} \leq cb^{-u}(h/l)^{u/2}. \end{aligned}$$

If we take $b = cn^{-\delta/2(3+\delta)}$ and

$$0 < \theta \leq \frac{3-\delta}{3+\delta} - \frac{4\eta}{3}, \tag{7.2.11}$$

then $(h/l)^{u/2} \leq cn^{-\delta(u+1)/2(3+\delta)}$, so that

$$P\{\|W_n^{(3)} - W_n^{(2)}\| \geq cn^{-\delta/2(3+\delta)}\} \leq cn^{-\delta/2(3+\delta)}. \tag{7.2.12}$$

In the remain of the proof, imitating the proof of Utev (1984), i.e. by Berkes and Philipp (1979) Theorem 2, we have

$$P\{\|W_n^{(4)} - W_n^{(3)}\| \geq c\varphi(h)n/l\} \leq c\varphi(h)n/l, \tag{7.2.13}$$

where $W_n^{(4)}$ is the random polygonal line with nodes $(kl/n, \sum_{j=1}^k \xi_j^{(2)}/\sqrt{n})$, $k = 0, 1, \cdots, m+1$, the $\xi_j^{(2)}$ are independent and distributed same as $\xi_j^{(1)}$. And by Sakhanenko (1981), we have

$$P\{\|W_n^{(5)} - W_N^{(4)}\| \geq q_u^{\frac{1}{u+1}}\} \leq q_u^{\frac{1}{u+1}} \qquad u > 2, \tag{7.2.14}$$

where $W_n^{(5)}$ is the random polygonal line with nodes $(kl/n, \sum_{j=1}^k Y_j/\sqrt{n})$, $k = 0, 1, \cdots, m+1$, the $Y_j$ are independent normally distributed random variables with $EY_j = 0$, $\mathrm{Var}Y_j = \mathrm{Var}\xi_j^{(1)}, j = 1, 2, \cdots, m+1$ and

$$q_u = c\Big(\sum_{i=1}^{m+1} E|\xi_i^{(1)}|^u\Big)\Big/\Big(\sum_{i=1}^{m+1} \mathrm{Var}\xi_i^{(1)}\Big)^{u/2}.$$

It is easy to see that by (7.2.8) we have

$$\begin{aligned} q_u &= cn^{-u/2}\sum_{i=1}^{m} E|\xi_i^{(1)}|^u \leq cn^{-u/2}ml^{u/2} \\ &\leq c(n/l)^{1-u/2}, \end{aligned} \tag{7.2.15}$$

so that

$$q_u^{\frac{1}{u+1}} \leq cn^{-\delta/2(3+\delta)}$$

when $u \geq 2 + 3\delta/(3+\delta)\eta$.

At last, if we take

$$\theta \geq \frac{2(3-\delta)}{(3+\varepsilon)\delta}\Big(\frac{3\delta}{2(3+\delta)}+\eta\Big),$$

then

$$\varphi(h)n/l = O(n^{-g\theta+\frac{\delta}{3+\delta}+\eta}) \leq cn^{-\delta/2(3+\delta)}. \tag{7.2.16}$$

Combining with (7.2.11), we have

$$\frac{2(3-\delta)}{(3+\varepsilon)\delta}\Big(\frac{3\delta}{2(3+\delta)}+\eta\Big) \leq \theta < \frac{3-\delta}{3+\delta}-\frac{4\eta}{3},$$

that is to say, we need take $\eta$ as follows:

$$0<\eta<\frac{3-\delta}{3+\delta}\frac{3\varepsilon\delta}{2(9+3\delta+2\varepsilon\delta)}.$$

Combining (7.2.9)-(7.2.16) together we obtain

$$P\{\|W_n - W\| \geq cn^{-\delta/2(3+\delta)}\} \leq cn^{-\delta/2(3+\delta)}.$$

Since $L(W_n, W) \leq \inf_\varepsilon(\varepsilon + P\{\|W_n - W\| \geq \varepsilon\})$, (7.2.7) holds true. The proof of Theorem 7.2.2 is completed.

**Remark 7.2.1.** Utev (1984) discussed the rate of weak convergence for an $\alpha$-mixing sequence and obtained the following result:

Let $\{X_n, n \geq 1\}$ be a strictly stationary $\alpha$-mixing sequence with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{2+\delta} < \infty$, $0 < \delta < 3$. Assume that

$$0 < \sigma^2 := EX_1^2 + 2\sum_{n=1}^{\infty} EX_1X_{n+1} < \infty$$

and

$$\alpha(n) \leq cn^{-g}, \qquad n = 1, 2, \cdots$$

where

$$g > \max(\delta_1^{-1} j(u)(j(u)+\delta_1), \delta_1^{-1} j(2(2+\delta))(j(2(2+\delta))+\delta_1)),$$

$$u = \max\Big(2(2+\delta), \frac{12+5\delta}{2(3-\delta)}\Big), \quad \delta_1 = \frac{\delta}{(2+\delta)(3+\delta)},$$

$j(u)$ is defined as in Theorem 7.2.1. Then

$$L(P_n, W) \leq cA_\delta n^{-\delta/(2(3+\delta))},$$

where $c = c(A, g, \delta)$.

If $\delta = 1$, i.e., assume that $E|X_1|^3 < \infty$, $\alpha(n) = O(n^{-(438+\varepsilon)})$, then $L(P_n, W) = O(n^{-1/8})$.

Gorodezkii (1983) has also given a similar result.

**Remark 7.2.2.** Yoshihara (1979) discussed the rate of the weakly invariance principle for the strictly stationary absolutely regular sequence $\{X_n, n \geq 1\}$, and proved that if $EX_1 = 0$, $E|X_1|^{4+\delta} < \infty$ for some $\delta > 0$ and $\sum_{n=1}^{\infty} n\beta(n)^{\delta/(4+\delta)} < \infty$, then

$$L(P_n, W) = O(n^{-1/8}(\log n)^{1/2}).$$

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# Part III Almost Sure Convergence and Strong Approximations

In this part, we study the almost sure convergence and the strong approximations of the partial sums of a mixing dependent sequence. Since 1960s, the almost sure convergence has been discussed by some authors, e.g., Iosifescu and Theodorescu (1969) obtained the 0-1 law, the strong law of large numbers and the convergence of random series of a $\varphi$-mixing sequence, etc. The complete convergence of various mixing sequences has been studied deeply by Shao et al. in the past ten years, from which the elegant results of the strong laws of large numbers follow. We shall discuss these in Chapter 8.

The strong approximations of the partial sums $S_n = \sum_{k=1}^{n} X_k$ for a mixing sequence $\{X_n, n \geq 1\}$ by a Wiener process were done by Philipp and Stout (1975) et al., and were improved comprehensively by Shao and Lu; the limiting behaviour of the increments of partial sums for a mixing sequence was obtained by Lin et al. Those elegant theorems will be discussed in Chapter 9 and Chapter 10 respectively.

The strong approximation of a mixing sequence with set-indexed will be introduced in Chapter 11.

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# Chapter 8 Laws of Large Numbers and Complete Convergence

We shall introduce the Borel-Cantelli lemma, the weak law of large numbers and the strong law of large numbers for a $\varphi$-mixing sequence in the first two sections. Since the concept of complete convergence was raised by Hsu and Robbins (1947), this subject has attracted the attention of many mathematicians. The complete convergence of weakly dependent sequences has been obtained by some mathematicians. We shall introduce the complete convergence of a $\varphi$-mixing sequence, a $\rho$-mixing sequence and an $\alpha$-mixing sequence in Sections 8.3-8.5 respectively. At last, we discuss three problems which are posed by Prohorov for the complete convergence of $\rho$-mixing sequence in Section 8.6.

## 8.1 Weak law of large numbers

Theorem 10.1.1 of Chow and Teicher (1978) gave a sufficient and necessary condition for the weak law of large numbers for an array of independent random variables $\{X_{nj}, 1 \le j \le k_n\}$. Du (1993) generalized this result to the $\varphi$-mixing case and proved the following theorem. Denote $S_n = \sum_{j=1}^{k_n} X_{nj}$.

**Theorem 8.1.1.** *Let $\{X_{nj}, 1 \le j \le k_n \to \infty\}$ be a random variable array which is $\varphi$-mixing in each row. There exists an integer $M$ such that $\varphi(M) < 1/2$. Then for some real numbers $A_n$*

$$S_n - A_n \xrightarrow{P} 0, \tag{8.1.1}$$

$$P\Big\{\max_{1\le j\le k_n} |X_{nj}| \ge \varepsilon\Big\} \to 0 \qquad \text{for any} \quad \varepsilon > 0, \tag{8.1.2}$$

*as* $n \to \infty$ *iff*

$$\sum_{j=1}^{k_n} P\{|X_{nj}| \geq \varepsilon\} \to 0 \qquad \text{for any} \quad \varepsilon > 0, \tag{8.1.3}$$

$$\text{Var}\Big(\sum_{j=1}^{k_n} X_{nj} I(|X_{nj}| < 1)\Big) \to 0, \tag{8.1.4}$$

*in which we can take*

$$A_n = \sum_{j=1}^{k_n} E X_{nj} I(|X_{nj}| < 1) + o(1).$$

The proof of Theorem 8.1.1 will needs the following lemmas.

**Lemma 8.1.1.** *Let* $\{X_n, n \geq 1\}$ *be a* $\varphi$*-mixing sequence. Denote by* $m(Y)$ *the median of a random variable* $Y$. *Then we have*

$$\begin{aligned} P\{\max_{1\leq j\leq n} |S_j - m(S_j - S_{n+k-1} + S_j(k-1))| \geq \varepsilon\Big\} \\ \leq \frac{2}{1-2C} P\Big\{(k-1)\max_{1\leq j\leq n+k-1}|X_j| + |S_{n+k-1}| \geq \varepsilon\Big\} \end{aligned} \tag{8.1.5}$$

*for large* $k$ *and some* $C$, $0 < C < 1/2$, *where* $S_j(k) = \sum_{i=j+1}^{j+k} X_i$.

**Proof.** Take $k$ so large that $C := \varphi(k) < 1/2$. Let

$$T = \begin{cases} \min\limits_{1\leq j\leq n} \{j : S_j - m(S_j - S_{n+k-1} + S_j(k-1)) \geq \varepsilon\}, \\ n+1, \qquad \text{if the above set is empty.} \end{cases}$$

Denote

$$\begin{aligned} B_j = \{&S_j - S_{n+k-1} + S_j(k-1) \\ &\leq m(S_j - S_{n+k-1} + S_j(k-1))\}, \quad 1 \leq j \leq n. \end{aligned}$$

It is clear that $P\{B_j\} \geq 1/2$. Since $\{T = j\} \in \mathcal{F}_1^j = \sigma\{X_1, \cdots, X_j\}, B_j \in \mathcal{F}_{j+k}^{n+k-1}$, we have

$$\begin{aligned} \bigcup_{j=1}^{n} (B_j \bigcap (T = j)) &\subset \{S_{n+k-1} - S_j(k-1) \geq \varepsilon\} \\ &\subset \Big\{(k-1)\max_{1\leq j\leq n+k-1}|X_j| + S_{n+k-1} \geq \varepsilon\Big\}. \end{aligned}$$

From the $\varphi$-mixing property it follows that

$$\begin{aligned}
&P\Big\{(k-1)\max_{1\le j\le n+k-1}|X_j| + S_{n+k-1} \ge \varepsilon\Big\}\\
&\ge \sum_{j=1}^{n} P\{B_j, T=j\}\\
&\ge \sum_{j=1}^{n}(P(B_j)-\varphi(k))P\{T=j\}\\
&\ge (\frac{1}{2}-C)P\{1\le T\le n\}\\
&= (\frac{1}{2}-C)P\Big\{\max_{1\le j\le n}(S_j\\
&\quad - m(S_j - S_{n+k-1} + S_j(k-1)) \ge \varepsilon\Big\}.
\end{aligned} \tag{8.1.6}$$

By the same way, we have another inequality when $X_j$ is replaced by $-X_j$. (8.1.5) is proved.

Denote

$$B = \Big\{\max_{0\le l\le n}|S_l(n-l)| > \varepsilon\Big\},$$

$$B_j = \Big\{|S_j(n-j)| > \varepsilon,\ \max_{j<k\le n}|S_k(n-k)| \le \varepsilon\Big\},\quad 0\le j<n,$$

$$B_n = \{|X_n| > \varepsilon\},\quad F_j = \bigcup_{i=j}^{n} B_i,\ 0\le j\le n \quad F_{n+1}=\emptyset.$$

**Lemma 8.1.2.** *Let $\{X_n, n\ge 1\}$ be a $\varphi$-mixing sequence with $|X_j| \le r < \infty$, $j=1,2,\cdots,n$, and $\varphi(M) < 1/6$ for some $M>0$. Then*

$$P(B) \ge \frac{(1-6\varphi(M))E\{\max_{1\le l\le n} S_l^2\} - 4\varepsilon^2}{3E\{\max_{1\le l\le n} S_l^2\} + 5M^2r^2 + 12(\varepsilon+r)^2 - 2\varepsilon^2}. \tag{8.1.7}$$

**Proof.** For $1\le i<k\le n$ we have

$$\max_{1\le i\le k}|S_i| \le |S_k| + \max_{0\le i\le k}|S_i(k-i)| \le 2\max_{0\le i\le k}|S_i(k-i)|. \tag{8.1.8}$$

If $0<M<n, 0<i<n-M$, we have

$$\max_{1\le l\le j}|S_l| \le \max_{1\le l\le i}|S_l| + Mr, \quad i<j\le i+M, \tag{8.1.9}$$

and

$$\max_{i+M\le l\le j}|S_l| \le |S_i| + Mr + \max_{i+M<l\le j}|S_{i+M}(l-i-M)| \tag{8.1.10}$$

for $i + M < l < j \le n$. From (8.1.8)-(8.1.10) it follows that

$$\begin{aligned}\max_{1\le l\le j}|S_l| &= \max\Big\{\max_{1\le k\le i+M}|S_k|, \max_{i+M<l\le j}|S_l|\Big\}\\ &\le \max_{1\le k\le i}|S_k| + Mr + \max_{i+M<l\le j}|S_{i+M}(l-i-M)|\\ &\le \max_{1\le k\le i}|S_k| + Mr + 2\max_{i+M\le l\le j}|S_l(j-l)|\\ &\le \max_{1\le k\le i}|S_k| + Mr + 4\max_{i+M\le l\le n}|S_l(n-l)|\end{aligned}$$

for $j = 1, 2, \cdots, n$. Therefore for any given $i, 1 \le i \le n - M$ we have

$$\max_{1\le l\le n}|S_l| \le \max_{1\le k\le i}|S_k| + Mr + 4\max_{i+M\le l\le n}|S_l(n-l)|. \tag{8.1.11}$$

Note that

$$\begin{aligned}&\max_{j\le l\le n}|S_l(n-l)|I_{B_j}\\ &\quad\le |X_{j+1}|I_{B_j} + \max_{j+1\le l\le n}|S_l(n-l)|I_{B_j} \le (r+\varepsilon)I_{B_j}\end{aligned}$$

for $j = 1, 2, \cdots, n$ and $(a+b+c)^2 \le 3(a^2+b^2+c^2)$. By (8.1.11) we obtain

$$\begin{aligned}E\max_{1\le l\le n}S_l^2 I_B &= \sum_{j=0}^{n} E\max_{1\le l\le n}S_l^2 I_{B_j}\\ &\le \sum_{j=0}^{M} E\max_{1\le l\le n}S_l^2 I_{B_j} + \sum_{j=M+1}^{n} E\Big\{\max_{1\le k\le j-M}|S_k|\\ &\quad + Mr + 4\max_{j\le l\le n}|S_l(n-l)|\Big\}^2 I_{B_j}\\ &\le \sum_{j=0}^{M} E\max_{1\le l\le n}S_l^2 I_{B_j} + 3\sum_{j=M+1}^{n} E\max_{1\le k\le j-M}S_k^2 I_{B_j}\\ &\quad + (3M^2r^2 + 12(r+\varepsilon)^2)P(B).\end{aligned} \tag{8.1.12}$$

Put $Y_j = \max_{1\le k\le j-M}|S_k|$, $M < j \le n$. By Lemma 1.2.10 and noting

that $P(F_i)$ is non-increasing for $i$, we get

$$\begin{aligned}
&\sum_{j=M+1}^{n} E \max_{1\le k\le j-M} S_k^2 I_{B_j} \\
&\quad = \sum_{j=M+1}^{n} EY_j^2 I_{B_j} \\
&\quad = \sum_{j=M+1}^{n} (EY_j^2 I_{F_j} - EY_j^2 I_{F_{j+1}}) \\
&\quad = EY_{M+1}^2 I_{F_{M+1}} + \sum_{j=M+2}^{n} E(Y_j^2 - Y_{j-1}^2) I_{F_j} \\
&\quad \le EY_{M+1}^2 \{P(F_{M+1}) + 2\varphi(M)\} \\
&\qquad + \sum_{j=M+2}^{n} E(Y_j^2 - Y_{j-1}^2)\{P(F_j) + 2\varphi(M)\} \\
&\quad \le EY_n^2 \{P(F_{M+1}) + 2\varphi(M)\}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\sum_{j=0}^{M} E \max_{1\le l\le n} S_l^2 I_{B_j} &\le (Mr+\varepsilon)^2 P(B) \\
&\le 2(M^2r^2+\varepsilon^2)P(B).
\end{aligned}$$

Inserting these into (8.1.12) we have

$$\begin{aligned}
&E \max_{1\le l\le n} S_l^2 I_B \\
&\qquad \le 3E \max_{1\le l\le n} S_l^2 (P(B) + 2\varphi(M)) \\
&\qquad\quad + (5M^2r^2 + 12(r+\varepsilon)^2 + 2\varepsilon^2)P(B). \qquad (8.1.13)
\end{aligned}$$

On the other hand

$$\begin{aligned}
E \max_{1\le l\le n} S_l^2 I_B &= E \max_{1\le l\le n} S_l^2 - E \max_{1\le l\le n} S_l^2 I_{B^c} \\
&\ge E \max_{1\le l\le n} S_l^2 - 4\varepsilon^2(1-P(B)). \qquad (8.1.14)
\end{aligned}$$

Combining (8.1.14) with (8.1.13) yields (8.1.7).

**Lemma 8.1.3.** *Let $\{X_n, n\ge 1\}$ be as in Lemma 8.1.2. If $S_n$ almost surely converge to a random variable as $n\to\infty$, then $E\max_{1\le k\le n} S_k^2$ are convergent.*

**Proof.** Denote $D_{n,n+m} = \bigcup_{i=n}^{n+m}\{|S_i(n+m-i)| > \varepsilon\}$. If $\varphi(M) < 1/6$, by Lemma 8.1.2

$$P\{D_{n,n+m}\} \\ \geq \frac{(1-6\varphi(M))E\{\max_{n\leq l\leq n+m} S_l^2\} - 4\varepsilon^2}{3E\{\max_{n\leq l\leq n+m} S_l^2\} + 5M^2r^2 + 12(\varepsilon+r)^2 - 2\varepsilon^2}.$$

Using the reduction to absurdity, if $E\max_{1\leq k\leq n} S_k^2$ are divergent, then for any given $n \geq 1$

$$E\max_{n\leq l\leq n+m} S_l^2 \geq E\max_{1\leq l\leq n+m} S_l^2 - E\max_{1\leq l\leq n} S_l^2 \geq \delta_0 > 0 \tag{8.1.15}$$

for large $m$. On the other hand, from $S_n \xrightarrow{\text{a.s.}} S$ we have

$$\begin{aligned} P\{D_{n,n+m}\} &= P\Big\{\max_{n\leq i\leq n+m} |S_i(n+m-i)| > \varepsilon\Big\} \\ &\leq P\Big\{\max_{0<l\leq m} |S_n(l)| > \varepsilon/2\Big\} \\ &\quad + P\{|S_n(m)| > \varepsilon/2\} \to 0, \end{aligned}$$

as $n, m \to \infty$.The contradiction to (8.1.15) implies that Lemma 8.1.3 holds true.

**Lemma 8.1.4.** *Let* $B = \Big\{\max_{1\leq j\leq n}|X_j| \geq \varepsilon\Big\}, T_n = \sum_{i=1}^n I(|X_i| \geq \varepsilon)$. *If* $\varphi(M) < 1/2$, *we have*

$$P(B) \geq \frac{(1-2\varphi(M))ET_n}{ET_n + M + 1}. \tag{8.1.16}$$

**Proof.** Denote

$$\begin{aligned} &T_0 = 0,\ T_n(m) = T_{n+m} - T_n, \\ &B_n = \{|X_n| \geq \varepsilon\}, \\ &B_j = \{\max_{j<i\leq n} |X_i| < \varepsilon, |X_j| \geq \varepsilon\}, \\ &F_j = \bigcup_{i=j}^{n} B_i,\ j = 1, \cdots, n,\ F_{n+1} = \emptyset. \end{aligned}$$

We have

$$\begin{aligned}
ET_nI_B &= \sum_{j=1}^{M} E(T_{j-1} + T_{j-1}(n-j+1))I(B_j) \\
&\quad + \sum_{j=M+1}^{n} E(T_{j-M-1} + T_{j-M-1}(M) \\
&\quad + T_{j-1}(n-j-1))I(B_j) \\
&\le \sum_{j=1}^{M} MP(B_j) + \sum_{j=1}^{M} P(B_j) \\
&\quad + \sum_{j=M+1}^{n} (ET_{j-M+1}I(B_j) + MP(B_j) + P(B_j)) \\
&\le \sum_{j=M+1}^{n} ET_{j-M-1}I(B_j) + (M+1)P(B).
\end{aligned}$$

By Lemma 1.2.10

$$\begin{aligned}
&\sum_{j=M+1}^{n} E(T_{j-M-1}I(B_j)) \\
&\qquad = \sum_{j=M+1}^{n} E(T_{j-M-1}(I(F_j) - I(F_{j+1}))) \\
&\qquad = ET_1I(F_{M+2}) + \sum_{j=M+3}^{n} EI(|X_{j-M-1}| \ge \varepsilon)I(F_j) \\
&\qquad \le \Big\{\sum_{i=1}^{n-M} EI(|X_i| \ge \varepsilon)\Big\}\{P(F_{M+1}) + 2\varphi(M)\} \\
&\qquad \le ET_n\{P(B) + 2\varphi(M)\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
ET_n &= ET_nI(B) \\
&\le ET_n\{P(B) + 2\varphi(M)\} + (M+1)P(B),
\end{aligned}$$

as desired.

**Proof of Theorem 8.1.1.**

The part of "*if*". It is clear that (8.1.3) implies (8.1.2). In order to show (8.1.1), denote

$$Y_{nj} = X_{nj}I(|X_{nj}| < 1), \quad V_n = \sum_{j=1}^{k_n} Y_{nj}.$$

By (8.1.4) we have $V_n - EV_n \xrightarrow{P} 0$. From (8.1.3) it follows that

$$P(V_n \neq S_n) \le \sum_{j=1}^{k_n} P\{|X_{nj}| \ge 1\} = o(1), \tag{8.1.17}$$

i.e., $S_n - EV_n \xrightarrow{P} 0$. (8.1.1) holds true for $A_n = EV_n + o(1)$.

The part of "*only if*". From Lemma 8.1.4 we have

$$P\Big\{\max_{1\le j\le k_n} |X_{nj}| \ge \varepsilon\Big\} \ge \frac{(1-2\varphi(M))E\sum_{j=1}^{k_n} I(|X_{nj}| \ge \varepsilon)}{E\sum_{j=1}^{k_n} I(|X_{nj}| \ge \varepsilon) + M + 1},$$

so that (8.1.2) implies (8.1.3). In order to show (8.1.4), let $Y'_{nj}$ be an independent copy of $Y_{nj}$ and put

$$Y^*_{nj} = Y_{nj} - Y'_{nj}, \quad V^*_n = \sum_{j=1}^{k_n} Y^*_{nj}.$$

From (8.1.17) and (8.1.1) it follows that $V_n - A_n \xrightarrow{P} 0$. Therefore $V^*_n \xrightarrow{P} 0$. Denote $V^*_{nk} = \sum_{j=1}^{k} Y^*_{nj}$. By Lemma 8.1.1

$$\begin{aligned} &P\Big\{\max_{1\le k\le k_n - M} |V^*_{nk}| \ge \varepsilon\Big\} \\ &\quad \le \frac{2}{1-2C} P\Big\{(M-1)\max_{1\le j\le k_n} |Y^*_{nj}| + |V^*_n| \ge \varepsilon\Big\} \\ &\quad = o(1). \end{aligned} \tag{8.1.18}$$

Hence by (8.1.2) and (8.1.18)

$$\begin{aligned} &P\Big\{\max_{1\le k\le k_n} |V^*_{nk}| \ge \varepsilon\Big\} \\ &\quad \le P\Big\{\max_{1\le k\le k_n - M} |V^*_{nk}| \ge \varepsilon\Big\} \\ &\quad\quad + \sum_{j=k_n-M+1}^{k_n} P\{|Y^*_{nj}| \ge \varepsilon/M\} = o(1). \end{aligned}$$

Using the same method as in the proof of Lemma 8.1.3 we get that $\mathrm{Var}V^*_n$ tends to 0. Thus $\mathrm{Var}V_n = \mathrm{Var}V^*_n/2$ approaches 0 as well. Theorem 8.1.1 is proved.

## 8.2 Strong laws of large numbers

First we show the Borel-Cantelli lemma for a $\varphi$-mixing sequence.

**Theorem 8.2.1.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence and let $\{\mathcal{F}_n = \sigma(X_n), n \geq 1\}$ be a sequence of $\sigma$-fields. Then for any given $A_n \in \mathcal{F}_n$, $P\{A_n, \text{ i.o.}\} = 0$ iff*

$$\sum_{n=1}^{\infty} P\{A_n\} < \infty. \tag{8.2.1}$$

*Furthermore, that $\sum_{n=1}^{\infty} P\{A_n\} = \infty$ implies $P\{A_n, i.o.\} = 1$.*

**Proof.** For the first part, we need only to show that $P\{A_n, \text{ i.o.}\} = 0$ implies $\sum_{n=1}^{\infty} P\{A_n\} < \infty$. Otherwise, if $\sum_{n=1}^{\infty} P\{A_n\} = \infty$. By the assumption, there exists an $l > 0$ such that $\delta := \varphi(l) < 1$, and there is some $j, 0 \leq j \leq l-1$ such that $\sum_{n=1}^{\infty} P\{A_{nl+j}\} = \infty$. We have

$$\begin{aligned} P\Big\{\bigcup_{i=m}^{n} A_{il+j}\Big\} =& P\{A_{nl+j}\} + P\{A_{nl+j}^c \cap A_{(n-1)l+j}\} + \cdots \\ &+ P\{A_{nl+j}^c \cap \cdots \cap A_{(m+1)l+j}^c \cap A_{ml+j}\}. \end{aligned}$$

By the $\varphi$-mixing property we get

$$\begin{aligned} &P\Big\{\bigcup_{i=m}^{n} A_{il+j}\Big\} \\ &\quad \geq P\{A_{nl+j}\} + P\{A_{(n-1)l+j}\}(P\{A_{nl+j}^c\} - \delta) + \cdots \\ &\qquad + P\{A_{ml+j}\}(P\{A_{(m+1)l+j}^c \cap \cdots \cap A_{nl+j}^c\} - \delta) \\ &\quad \geq \sum_{i=m}^{n} P\{A_{il+j}\}\Big(P\Big\{\Big(\bigcup_{i=m+1}^{n} A_{il+j}\Big)^c\Big\} - \delta\Big). \end{aligned}$$

From $\sum_{k=1}^{\infty} P\{A_{kl+j}\} = \infty$ it follows that $P\Big\{\Big(\bigcup_{i=m}^{\infty} A_{il+j}\Big)^c\Big\} \leq \delta$, i.e., $P\Big\{\bigcup_{i=m}^{\infty} A_{il+j}\Big\} \geq 1 - \delta$, therefore $P\{A_n, \text{ i.o.}\} \geq 1 - \delta$, which contradict with $P\{A_n, \text{ i.o.}\} = 0$.

Secondly, if $\sum_{n=1}^{\infty} P\{A_n\} = \infty$, it follows from the above discussion that

$$P\Big\{\limsup_n A_n\Big\} = P\{A_n, \text{ i.o.}\} \neq 0.$$

It is clear that $\limsup A_n \in \bigcap_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \mathcal{F}_i$. We now show that $\bigcap_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \mathcal{F}_i$ is a trivial $\sigma$-field. Otherwise, there exists a set $B \in \bigcap_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \mathcal{F}_i$, $0 < P(B) < 1$. By the $\varphi$-mixing property, for every $A \in \bigvee_{i=1}^{n} \mathcal{F}_i$ we have

$$|P(AB) - P(A)P(B)| < \eta P(A), \tag{8.2.2}$$

where $\eta < 1/2$. It is easy to check that the class of sets which satisfies (8.2.2) includes the $\sigma$-field $\bigvee_{i=1}^{\infty} \mathcal{F}_i$, so that taking $A = B$ we obtain

$$P(B) - P(B)^2 \leq \eta P(B),$$

which implies $P(B) > 1/2$. On the other hand, we have also $B^c \in \bigcap_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \mathcal{F}_i$. A contradiction leads $P\{A_n, \quad \text{i.o.}\} = 1$ as desired.

**Corollary 8.2.1.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence of identically distributed random variables. Put $S_n = \sum_{k=1}^{n} X_k$. If $S_n/n \to b$ a.s., where $b$ is a finite constant, then $E|X_1| < \infty$.*

**Proof.** From $S_n/n \to b$ (a.s.) we have

$$\frac{X_n}{n} = \frac{S_n}{n} - \frac{S_{n-1}}{n-1} \cdot \frac{n-1}{n} \to 0 \qquad \text{a.s.}$$

Therefore $P\{|X_n/n| \geq \varepsilon, \quad \text{i.o.}\} = 0$ for any $\varepsilon > 0$. By Theorem 8.2.1

$$E|X_1| \leq \sum_{n=0}^{\infty} P\{|X_1| \geq n\} < \infty.$$

For the $\varphi$-mixing sequence of identically distributed random variables, we have the following Marcinkiewicz strong law of large numbers.

**Theorem 8.2.2.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing ($\rho$-mixing) sequence of identically distributed random variables with*

$$\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty \ (\sum_{n=1}^{\infty} \rho(2^n) < \infty), E|X_1|^r < \infty$$

*for some $1 \leq r < 2$. Then*

$$\frac{1}{n} \sum_{i=1}^{n} (X_i - EX_i) = o(n^{-(1-1/r)}) \qquad a.s. \tag{8.2.3}$$

Theorem 8.2.2 is an immediate consequence of Corollary 8.3.4 (Corollary 8.4.2), so we omit its proof here.

**Corollary 8.2.2.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence of identically distributed random variables with $\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty$. Then $S_n/n \to b$ a.s. iff $E|X_1| < \infty$ and $b = EX_1$.*

**Remark 8.2.1.** Xue (1994) showed that Theorem 8.2.2 is also true for a $\varphi$-mixing sequence $\{X_n, n \geq 1\}$ with $\sum \varphi^{1/2}(2^n) < \infty$, if there exists a random variable $X$ such that for any $x > 0$, $P\{|X_n| \geq x\} \leq P\{|X| \geq x\}$ and $E|X|^r < \infty$ for some $1 \leq r < 2$.

**Remark 8.2.2.** Iosifescu and Theodorescu (1969) have discussed the convergence of a series of a $\varphi$-mixing sequence, for example, they showed that for $\sum X_n$ a.s. convergence is equivalent to convergence in probability under some condition. They have also discussed the following strong law of large numbers: if $a_n \uparrow \infty$, $\sum_{n=1}^{\infty} \mathrm{Var}X_n/a_n^2 < \infty$ and $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ then

$$\frac{1}{a_n}(S_n - ES_n) \to 0 \quad \text{a.s.}$$

**Remark 8.2.3.** Chen and Wu (1989) have proved a strong law of large numbers for an $\alpha$-mixing sequence. Suppose that $\sup_n E|X_n|^p < \infty$ for some $p > 1$, and

$$\alpha(n) = \begin{cases} O(n^{-\frac{p}{2p-2}-\varepsilon}) & if \quad 1 < p < 2, \\ O(n^{-\frac{2}{p}-\varepsilon}) & if \quad p \geq 2. \end{cases}$$

Then $(S_n - ES_n)/n = o(1)$ a.s.

## 8.3 Complete convergence for $\varphi$-mixing sequences

The concept of complete convergence was introduced by Hsu and Robbins (1947). They showed: if $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $EX_1 = 0$, $EX_1^2 < \infty$, then

$$\sum_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} < \infty$$

for any $\varepsilon > 0$. Baum and Katz (1965) showed that $EX_1 = 0$, $E|X_1|^{rt} < \infty$ for $r \geq 1$, $1 \leq t < 2$ iff

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| \geq \varepsilon n^{1/t}\} < \infty$$

for any $\varepsilon > 0$. Bai and Su (1985) obtained:

**Theorem 8.3.1.** *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and $r > 1$, $0 < t < 2$, $h(x)$ be a slowly varying function as $x \to \infty$. Then the following conditions are equivalent:*

*(i)* $E|X_1|^{rt}h(|X_1|^t) < \infty$,

*(ii)* $\sum_{n=1}^{\infty} n^{r-2}h(n)P\{|S_n - nb| \geq \varepsilon n^{1/t}\} < \infty$ *for any* $\varepsilon > 0$,

*(iii)* $\sum_{n=1}^{\infty} n^{r-2}h(n)P\{\sup_{k\geq n} |S_k - kb|/k^{1/t} \geq \varepsilon\} < \infty$ *for any* $\varepsilon > 0$,

*where* $b = EX_1$ *if* $1 \leq t < 2$, *and* $0$ *if* $0 < t < 1$.

Since 1970s, the complete convergence of a mixing sequence has been discussed by some research works. The best result corresponding to that for an i.i.d. sequence is due to Shao (1988a).

Let $l(x)$ and $\beta(x)$ be positive even functions such that

$$\left\{ \begin{array}{l} l(x),\ \beta(x)/x^{\theta}\ (\text{some}\,\theta > 0) \text{ and } x^2/\beta(x) \\ \text{are monotonically nondecreasing} \end{array} \right. \tag{8.3.1}$$

for $x > 0$. Shao (1988a) proved the following theorems.

**Theorem 8.3.2.** *Let $\alpha(x) = \operatorname{inv}\beta(x)$, $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence of identically distributed random variables with $EX_1 = 0$. Suppose that*

$$E\beta(X_1)l(\beta(X_1)) < \infty. \tag{8.3.2}$$

*If one of the following conditions is satisfied:*

*a)* $\beta(x)l(\beta(x))/x \uparrow$, $\beta(x)l(\beta(x))/x^2 \downarrow$ *and there exists a* $r \geq 2$ *such that*

$$\sum_{n=1}^{\infty} \frac{1}{nl^r(n)} < \infty \tag{8.3.3}$$

*and*

$$\sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i) \leq \frac{13}{42} \log l(n); \tag{8.3.4}$$

*b)* $\beta(x)l(\beta(x))/x \uparrow$, $\beta(x)l(\beta(x))/x^2 \downarrow$,

$$\sum_{j=n}^{\infty} \frac{l(j)}{\alpha^2(j)} = O\Big(\frac{nl(n)}{\alpha^2(n)}\Big) \quad \text{as} \quad n \to \infty \tag{8.3.5}$$

$$\sum_{i=1}^{\infty} \varphi^{1/2}(2^i) < \infty; \tag{8.3.6}$$

*c)* $\beta(x)l(\beta(x))/x^2 \uparrow$ *and there exist* $q_1 > q_2 > 2$ *such that*

$$\beta(x)l(\beta(x))/x^{q_2} \downarrow,$$

$$\sum_{n=1}^{\infty}\frac{l(n)}{n}\Big(\frac{n^{1/2}}{\alpha(n)}\Big)^{q_1}\exp\Big\{3q_1\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\Big\}<\infty. \tag{8.3.7}$$

*Then*

$$\sum_{n=1}^{\infty}\frac{l(n)}{n}P\Big\{\max_{1\le i\le n}\beta(S_i)\ge \varepsilon n\Big\}<\infty, \quad \textit{for any} \quad \varepsilon>0. \tag{8.3.8}$$

**Theorem 8.3.3.** *Suppose that $l(x)$ is strictly monotone. Let $\{X_n, n\ge 1\}$ be a $\varphi$-mixing sequence with a common distribution. If (8.3.8) is satisfied then (8.3.2) holds and*

$$\sum_{n=1}^{\infty}\frac{l(n)-l([n/2])}{n}P\Big\{\sup_{i\ge n}\frac{\beta(S_i)}{i}>\varepsilon\Big\}<\infty, \quad \text{for any} \quad \varepsilon>0. \tag{8.3.9}$$

**Remark 8.3.1.** If $l(n)=O(l(n)-l([n/2]))$, (8.3.8) is equivalent to (8.3.9).

Let $l(n)=n^{r-1}, r>1, \beta(n)=n^t, 1\le t<2$ in Theorems 8.3.2 and 8.3.3, we have

**Corollary 8.3.1.** *Let $\{X_n, n\ge 1\}$ be a $\varphi$-mixing sequence with a common distribution. The following results are equivalent:*

*(i)* $E|X_1|^{rt}<\infty$, $EX_1=0$;

*(ii)* $\sum_{n=1}^{\infty}n^{r-2}P\Big\{\max_{1\le i\le n}|S_i|>\varepsilon n^{1/t}\Big\}<\infty$, for any $\varepsilon>0$;

*(iii)* $\sum_{n=1}^{\infty}n^{r-2}P\Big\{\sup_{k\ge n}|S_k|/k^{1/t}>\varepsilon\Big\}<\infty$, for any $\varepsilon>0$.

Choosing $l(n)=\log n$, $\beta(n)=n^t$, $1\le t<2$, we have

**Corollary 8.3.2.** *Let $\{X_n, n\ge 1\}$ be a $\varphi$-mixing sequence with a common distribution and $\varphi(n)\le\frac{1}{49}(\log n)^{-2}$ for large $n$. Then the following results are equivalent:*

*(i)* $E|X_1|^t\log(1+|X_1|)<\infty$, $EX_1=0$;

*(ii)* $\sum_{n=1}^{\infty}\frac{\log n}{n}P\Big\{\max_{1\le i\le n}|S_i|\ge\varepsilon n^{1/t}\Big\}<\infty$, *for any* $\varepsilon>0$.

**Corollary 8.3.3.** *Let $\{X_n, n\ge 1\}$ be a $\varphi$-mixing sequence with a common distribution and $EX_1=0$, $E|X_1|^t<\infty$ $1\le t<2$. If $\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$ then*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\Big\{\max_{1\le i\le n}|S_i|\ge\varepsilon n^{1/t}\Big\}<\infty$$

*for any* $\varepsilon > 0$.

**Remark 8.3.2.** Corollary 8.3.1 can be strengthened as follows: the same result as Theorem 8.3.1 holds true for a $\varphi$-mixing sequence with a common distribution, i.e., if $h(x) > 0$ is a slowly varying function, the following are equivalent:

(i)' $E|X_1|^{rt}h(|X_1|^t) < \infty$, $EX_1 = 0$;

(ii)' $\sum_{n=1}^{\infty} n^{r-2}h(n)P\Big\{\max_{1\le i\le n}|S_i| \ge \varepsilon n^{1/t}\Big\} < \infty$, for any $\varepsilon > 0$;

(iii)' $\sum_{n-1}^{\infty} n^{r-2}h(n)P\Big\{\max_{k\ge n}|S_k|/k^{1/t} \ge \varepsilon\Big\} < \infty$, for any $\varepsilon > 0$.

where $r > 1$, $1 < t < 2$.

Corollary 8.3.2 can be strengthened similarly.

**Proof of Theorem 8.3.2.**

It is easy to see that (8.3.1) implies

$$\alpha(x)/x^{1/2}\uparrow, \quad \alpha(x)/x^{1/\theta}\downarrow. \tag{8.3.10}$$

Put

$$X_{ni} = X_i I\{|X_i| \le \alpha(n)\}, \quad S_{ni} = \sum_{j=1}^{i} X_{nj}.$$

We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{l(n)}{n}P\Big\{\max_{i\le n}\beta(S_i) > \varepsilon n\Big\} \\
&\quad \le \sum_{n=1}^{\infty}\frac{l(n)}{n}P\Big\{\max_{i\le n}|X_i| > \alpha(n)\Big\} \\
&\qquad + \sum_{n=1}^{\infty}\frac{l(n)}{n}P\Big\{\max_{i\le n}|S_{ni}| > \alpha(\varepsilon n)\Big\} \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{8.3.11}$$

It is clear that

$$\begin{aligned}
I_1 &\le \sum_{n=1}^{\infty} l(n)P\{|X_i| > \alpha(n)\} \\
&= \sum_{n=1}^{\infty} l(n)\sum_{j=n}^{\infty} P\{j \le \beta(X_1) < j+1\} \\
&\le \sum_{j=1}^{\infty} jl(j)P\{j \le \beta(X_1) < j+1\} \\
&\le E\beta(X_1)l(\beta(X_1)) < \infty.
\end{aligned} \tag{8.3.12}$$

Now we estimate $I_2$. Without loss of generality we can assume that $0 < \varepsilon < 1/2$. By (8.3.10)

$$\begin{aligned} n|EX_1I\{|X_1| \le \alpha(n)\}| &\le nE|X_1|I\{|X_1| > \alpha(n)\} \\ &\le \frac{\alpha(n)}{l(n)} E\beta(X_1)l(\beta(X_1))I\{|X_1| > \alpha(n)\} \\ &\le \frac{\alpha(\varepsilon n)}{\varepsilon^{1/\theta} l(1)} E\beta(X_1)l(\beta(X_1))I\{|X_1| > \alpha(n)\}. \end{aligned}$$

Noting that $l(1) > 0$, $\lim_{n\to\infty} E\beta(X_1)l(\beta(X_1))I\{|X_1| > \alpha(n)\} = 0$, we have

$$n|EX_1I\{|X_1| \le \alpha(n)\}| \le \frac{1}{2}\alpha(\varepsilon n)$$

for large $n$. Therefore

$$I_2 \le c\sum_{n=1}^{\infty} \frac{l(n)}{n} P\Big\{\max_{i\le n} |S_{ni} - ES_{ni}| > \frac{1}{2}\alpha(\varepsilon n)\Big\}. \tag{8.3.13}$$

By Lemma 2.2.2 and Lemma 2.2.10, for given $q \ge 2$ (specified later on), there exists a $C_q = C(q)$ depending only on $q$ such that

$$\begin{aligned} &P\Big\{\max_{1\le i\le n} |S_{ni} - ES_{ni}| > \frac{1}{2}\alpha(\varepsilon n)\Big\} \\ &\le C_q(\alpha(\varepsilon n))^{-q}\bigg(\Big(n\exp\Big(3\sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i)\Big) \\ &\quad \cdot EX_1^2 I\{|X_1| \le \alpha(n)\}\Big)^{q/2} \\ &\quad + nE|X_1|^q I\{|X_1| \le \alpha(n)\}\bigg). \end{aligned} \tag{8.3.14}$$

If condition a) is satisfied, let $q = 28(r+1)$. From (8.3.10), (8.3.13) and (8.3.14) we have

$$\begin{aligned} I_2 &\le c\sum_{n=1}^{\infty} \frac{l(n)}{n}\Big(\frac{n}{\alpha^2(n)}\exp\Big(3\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\Big)EX_1^2I\{|X_1| \le \alpha(n)\}\Big)^{q/2} \\ &\quad + c\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^q(n)} E|X_1|^q I\{|X_1| \le \alpha(n)\} \\ &=: I_3 + I_4. \end{aligned} \tag{8.3.15}$$

By (8.3.4)

$$\begin{aligned}
I_3 &\leq c\sum_{n=1}^{\infty}\frac{\exp\Big((3q/2)\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\Big)}{nl(n)^{\frac{q-2}{2}}}(E\beta(X_1)l(\beta(X_1)))^{q/2} \\
&\leq c\sum_{n=1}^{\infty}\frac{1}{nl(n)^{\frac{q}{28}-1}} \\
&\leq c\sum_{n=1}^{\infty}\frac{1}{nl(n)^{r}} < \infty. \qquad (8.3.16)
\end{aligned}$$

Since $\beta(x)l(\beta(x))/x^2 \downarrow$ implies $xl(x)/\alpha^2(x) \downarrow$, it follows from (8.3.10) that

$$\begin{aligned}
\sum_{j=n}^{\infty}\frac{l(j)}{\alpha^q(j)} &= \sum_{j=n}^{\infty}\frac{jl(j)}{\alpha^2(j)}\cdot\frac{j^{\frac{q-2}{2}}}{\alpha^{q-2}(j)j^{q/2}} \\
&\leq \frac{n^{q/2}l(n)}{\alpha^q(n)}\sum_{j=n}^{\infty}\frac{1}{j^{q/2}} \\
&= O\Big(\frac{nl(n)}{\alpha^q(n)}\Big). \qquad (8.3.17)
\end{aligned}$$

Therefore

$$\begin{aligned}
I_4 &\leq c\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^q(n)}\sum_{j=0}^{n}E|X_1|^qI(j\leq\beta(X_1)<j+1) \\
&\leq c\sum_{j=0}^{\infty}\Big(\sum_{n=j}^{\infty}\frac{l(n)}{\alpha^q(n)}\Big)E|X_1|^qI(j\leq\beta(X_1)<j+1) \\
&\leq c\sum_{j=1}^{\infty}\frac{jl(j)}{\alpha^q(j)}E|X_1|^qI(j\leq\beta(X_1)<j+1) \\
&\leq cE\beta(X_1)l(\beta(X_1)) < \infty. \qquad (8.3.18)
\end{aligned}$$

From (8.3.15), (8.3.16) and (8.3.18) it follows that $I_2 < \infty$, i.e., Theorem 8.3.2 holds true under condition a).

When condition b) is satisfied, let $q = 2$. By the similar discussion we have $I_2 < \infty$. If condition c) is satisfied, let $q = q_1$. Note that from (8.3.7) we have $I_3 < \infty$. Since $\beta(x)l(\beta(x))/x^{q_2} \downarrow$ implies $xl(x)/\alpha(x)^{q_2} \downarrow$, by (8.3.10) we have

$$\sum_{j=n}^{\infty}\frac{l(j)}{\alpha^{q_1}(j)} \leq \frac{n^{1+\frac{q_1-q_2}{2}}l(n)}{\alpha^{q_1}(n)}\sum_{j=n}^{\infty}j^{-1-\frac{q_1-q_2}{2}} = O\Big(\frac{nl(n)}{\alpha^{q_1}(n)}\Big).$$

It follows that $I_4 < \infty$. Therefore $I_2 < \infty$. The proof of Theorem 8.3.2 is completed.

**Proof of Theorem 8.3.3.**

First we show that (8.3.8) implies (8.3.9). Put $d_n = \sum_{j=2^n}^{2^{n+1}-1} l(j)$. We have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{l(n) - l([n/2])}{n} P\Big\{\sup_{i\geq n} \frac{\beta(S_i)}{i} \geq \varepsilon\Big\} \\
&\leq \sum_{j=1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \frac{l(n) - l([n/2])}{2^j} P\Big\{\sup_{i\geq 2^j} \frac{\beta(S_i)}{i} \geq \varepsilon\Big\} \\
&= \sum_{j=1}^{\infty} 2^{-j} \Big( \sum_{n=2^j}^{2^{j+1}-1} l(n) - 2 \sum_{n=2^{j-1}}^{2^j-1} l(n) \Big) P\Big\{\sup_{i\geq 2^j} \frac{\beta(S_i)}{i} \geq \varepsilon\Big\} \\
&\leq \sum_{j=1}^{\infty} 2^{-j} (d_j - 2d_{j-1}) \sum_{k=j}^{\infty} P\Big\{\max_{2^k\leq i<2^{k+1}} \beta(S_i) > \varepsilon 2^k\Big\} \\
&\leq \sum_{k=1}^{\infty} 2^{-k} d_k P\Big\{\max_{2^k\leq i<2^{k+1}} \beta(S_i) \geq \varepsilon 2^k\Big\} \\
&\leq \sum_{k=1}^{\infty} 4 \sum_{2^{k+1}\leq n<2^{k+2}-1} \frac{l(n)}{n} P\Big\{\max_{i\leq 2^{k+1}} \beta(S_i) \geq \varepsilon 2^k\Big\} \\
&\leq 4 \sum_{n=1}^{\infty} \frac{l(n)}{n} P\Big\{\max_{i\leq n} \beta(S_i) > \frac{\varepsilon n}{2}\Big\} < \infty.
\end{aligned} \tag{8.3.19}$$

This proves that (8.3.9) holds true.

Now we show that (8.3.2) holds true. From (8.3.19) it follows that

$$\sum_{k=1}^{\infty} l(2^k) P\Big\{\max_{i\leq 2^k} |X_i| \geq \alpha(\varepsilon 2^k)\Big\} < \infty \tag{8.3.20}$$

for any $\varepsilon > 0$, which implies

$$P\Big\{\max_{i\leq 2^k} |X_i| \geq \alpha(\varepsilon 2^k)\Big\} \longrightarrow 0, \quad \text{as } k \to \infty. \tag{8.3.21}$$

Since $\varphi(p) \to 0$ as $p \to \infty$, there exist $k_0$ and $p_0$ such that for any $k \geq k_0$

$$\varphi(p_0) + P\Big\{\max_{i\leq 2^k} |X_i| \geq \alpha(\varepsilon 2^k)\Big\} < 1/2.$$

Then, as in (3.2) of Lai (1977) we have

$$P\Big\{\max_{1\leq i\leq 2^k} |X_i| \geq \alpha(\varepsilon 2^k)\Big\} \geq \frac{2^k}{2p_0} P\{|X_1| \geq \alpha(\varepsilon 2^k)\}.$$

Whence

$$\sum_{k=1}^{\infty} l(2^k)2^k P\{|X_1| \geq \alpha(\varepsilon 2^k)\}$$
$$\leq c\sum_{k=1}^{\infty} l(2^k)P\Big\{\max_{1\leq i\leq 2^k} |X_i| \geq \alpha(\varepsilon 2^k)\Big\} < \infty. \tag{8.3.22}$$

On the other hand

$$E\beta(X_1)l(\beta(X_1))$$
$$\leq l(1) + \sum_{j=1}^{\infty} 2^j l(2^j)P\{2^{j-1} < \beta(X_1) \leq 2^j\}$$
$$\leq l(1) + \sum_{j=1}^{\infty} 2^j l(2^j)P\{|X_1| \geq \alpha(2^{j-1})\}.$$

From (8.3.22) we obtain $E\beta(X_1)l(\beta(X_1)) < \infty$. (8.3.2) is proved. This completes the proof of Theorem 8.3.3.

## 8.4 Complete convergence for $\rho$-mixing sequences

The conclusions of complete convergence for a $\rho$-mixing sequence have not arrived completely as for the $\varphi$-mixing case, but some ideal sufficient conditions can be given.

**Definition 8.4.1.** A function $l(x) > 0\,(x > 0)$ is said to be quasi-monotone non-decreasing, if

$$\limsup_{x\to\infty} \sup_{0<t\leq x} l(t)/l(x) < \infty.$$

A function $l(x)$ is said to be quasi-monotone non-increasing, if

$$\limsup_{x\to\infty} \sup_{t\geq x} l(t)/l(x) < \infty.$$

Throughout this section, we assume that $l(x)$ is a positive, even and quasi-monotone non-decreasing function, $\beta(x)$ is a positive even function and $\beta(x)/x^\theta$, $x^2/\beta(x)$ are monotone non-decreasing for some $\theta > 0$. Denote $\alpha(x) = \text{inv}\beta(x)$.

Shao (1989c) proved the following theorems.

**Theorem 8.4.1.** *Suppose that $\beta(x)l(\beta(x))$ and $x^{2-\varepsilon_0}/\beta(x)l(\beta(x))$ for some $0 < \varepsilon_0 < 1$ are quasi-monotone non-decreasing functions, $\{X_n, n \geq 1\}$ is a $\rho$-mixing sequence with a common distribution and $EX_1 = 0$, $E\beta(X_1)l(\beta(X_1)) < \infty$. If*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty, \tag{8.4.1}$$

*then for any $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P\Big\{\max_{1\leq i\leq n} \beta(S_i) \geq \varepsilon n\Big\} < \infty. \tag{8.4.2}$$

**Theorem 8.4.2.** *Suppose that there exist $q \geq 2$, $q_0 \geq 2$ and $\delta > 0$ such that $x^q/\beta(x)l(\beta(x))$, $\beta(x)l(\beta(x))/x^2$ are quasi-monotone non-decreasing functions, and $l(x) \geq x^\delta (x > 0)$,*

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} \Big(\frac{n^{1/2}}{\alpha(n)}\Big)^{q_0} < \infty. \tag{8.4.3}$$

*Let $\{X_n\}$ be a $\rho$-mixing sequence with a common distribution and $EX_1 = 0$, $E\beta(X_1)l(\beta(X_1)) < \infty$. If*

$$\sum_{n=1}^{\infty} \rho^{2/r}(2^n) < \infty \tag{8.4.4}$$

*for some $r > q$, then (8.4.2) holds.*

**Corollary 8.4.1.** *Let $\{X_n, n \geq 1\}$ be a $\rho$-mixing sequence with a common distribution and $EX_1 = 0$, $E|X_1|^p h(|X_1|^{1/\alpha}) < \infty$ for $p \geq 1$, $p\alpha > 1$, $\alpha > 1/2$ and a slowly varying function $h(x) > 0$. If*

$$\sum_{n=1}^{\infty} \rho^{2/r}(n) < \infty \tag{8.4.5}$$

*for $r = 2$ when $1 \leq p < 2$; $r > p$, when $p \geq 2$, then*

$$\sum_{n=1}^{\infty} n^{p\alpha-2} h(n) P\Big\{\max_{1\leq i\leq n} |S_i| \geq \varepsilon n^\alpha\Big\} < \infty \tag{8.4.6}$$

*for any $\varepsilon > 0$.*

An immediate consequence of the above complete convergence result is the following Marcinkiewicz-Zygmund law of large numbers.

**Corollary 8.4.2.** *Let $\{X_n, n \geq 1\}$ be a $\rho$-mixing sequence with a common distribution and $EX_1 = 0, E|X_1|^p < \infty, 1 \leq p < 2$. Assume that*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty.$$

*Then*

$$\lim_{n\to\infty} S_n/n^{1/p} = 0. \qquad a.s.$$

**Corollary 8.4.3.** *Let $\{X_n, n \geq 1\}$ be a $\rho$-mixing sequence with a common distribution and $EX_1 = 0$, $E|X_1|^p h(|X_1|^p) < \infty$, $1 \leq p < 2$, and $h(x)$ be a monotone non-decreasing slowly varying function. Suppose that*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty.$$

*Then for any $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} \frac{h(n)}{n} P\Big\{\max_{1\leq i\leq n} |S_i| \geq \varepsilon n^{1/p}\Big\} < \infty.$$

If $E|X_1|^{p+\delta} < \infty$ for some $\delta > 0$ instead of the condition $E|X_1|^p h(|X_1|^{1/\alpha}) < \infty$ in Corollary 8.4.1, condition (8.4.5) can be deleted. Furthermore Kong and Zhang (1994) pointed out that (8.4.6) also holds true for non-identically distributed random variables.

The proofs of Theorems 8.4.1 and 8.4.2 need the following lemma.

**Lemma 8.4.1.** *Let $\{\xi_n, n \geq 1\}$ be a $\rho$-mixing sequence with $E\xi_n = 0$, $Eg(|\xi_n|) < \infty$, where $g(x)$ is a function for which there exists a constant $0 < C < \infty$ such that $\sup_{x\geq t} \frac{x}{g(x)} \leq C\frac{t}{g(t)}$ for any $t > 0$. Denote $T_k(i) = \sum_{j=k+1}^{k+i} \xi_j$. Then for any $q_1, q_2 \geq 2$ there exists a constant $K$ depending*

*only on* $q_1, q_2$ *and* $\rho(\cdot)$ *such that*

$$
\begin{aligned}
P\Big\{\max_{i\le n}|T_0(i)|\ge x\Big\}
&\le \sum_{i=1}^{n} P\{|\xi_i|\ge A\} + K\Big\{x^{-q_1}\Big(n^{q_1/2}\\
&\quad\cdot \exp\Big(K\sum_{i=1}^{[\log n]}\rho(2^i)\Big)\\
&\quad\cdot \max_{i\le n}\|\xi_i I(|\xi_i|\le B)\|_2^{q_1}\\
&\quad+ n\exp\Big(K\sum_{i=1}^{[\log n]}\rho^{2/q_1}(2^i)\Big)\\
&\quad\cdot \log^{q_1}(2n)\max_{i\le n}E|\xi_i|^{q_1}I(|\xi_i|\le B)\Big)\\
&\quad+ x^{-q_2}\Big(n^{q_2/2}\exp\Big(K\sum_{i=1}^{[\log n]}\rho(2^i)\Big)\\
&\quad\cdot \max_{i\le n}\|\xi_i I(B<|\xi_i|<A)\|_2^{q_2}\\
&\quad+ \Big(\frac{n}{k}\Big)^{q_2/2} k\exp\Big(K\sum_{i=1}^{[\log n]}\rho^{2/q_2}(2^i)\Big)\\
&\quad\cdot \max_{i\le n}E|\xi_i|^{q_2}I(B<|\xi_i|<A)\Big)\\
&\quad+ x^{-2}\cdot n\rho^2(k)\cdot\exp\Big(K\sum_{i=1}^{[\log n]}\rho(2^i)\Big)\\
&\quad\cdot \log^4\Big[\frac{n}{k}\Big]\cdot\max_{i\le n}E\xi_i^2 I(B<|\xi_i|<A)\Big\}
\end{aligned}
$$

*for any* $k$ $(\le n)$, $x>0$ *and* $A\ge B>0$ *which satisfy the following conditions*

$$\frac{4nCA}{g(A)}\max_{1\le i\le n}Eg(|\xi_i|)I(|\xi_i|\ge A)\le x, \tag{8.4.7}$$

$$\frac{48kBC}{g(B)}\max_{1\le i\le n}Eg(|\xi_i|)I(|\xi_i|\ge B)\le x. \tag{8.4.8}$$

**Proof.** For simplicity, we assume that $\{\xi, \xi_i, i\ge 1\}$ have a common

distribution. Put

$$\begin{aligned}
\xi_{i1} &= \xi_i I(|\xi_i| \le B) - E\xi_i I(|\xi_i| \le B); \\
\xi_{i2} &= \xi_i I(B < |\xi_i| < A) - E\xi_i I(B < |\xi_i| < A); \\
\xi_{i3} &= \xi_i I(|\xi_i| \ge A) - E\xi_i I(|\xi_i| \ge A); \\
T_{ik} &= \sum_{j=1}^{i} \xi_{jk}, \qquad k = 1, 2, 3.
\end{aligned}$$

It is clear that

$$\begin{aligned}
&P\Big\{\max_{i\le n} |T_0(i)| \ge x\Big\} \\
&\quad \le P\Big\{\max_{i\le n} |T_{i1}| \ge \frac{x}{4}\Big\} + P\Big\{\max_{i\le n} |T_{i2}| \ge \frac{x}{4}\Big\} \\
&\qquad + P\Big\{\max_{i\le n} |T_{i3}| \ge \frac{x}{2}\Big\} \\
&\quad =: I_1 + I_2 + I_3. \qquad (8.4.9)
\end{aligned}$$

From (8.4.7)

$$\begin{aligned}
I_3 &\le P\Big\{\sum_{i=1}^{n} |\xi_i| I(|\xi_i| \ge A) \ge \frac{x}{2} \\
&\qquad - \sum_{i=1}^{n} E|\xi_i| I(|\xi_i| \ge A)\Big\} \\
&\le P\Big\{\sum_{i=1}^{n} |\xi_i| I(|\xi_i| \ge A) \ge \frac{x}{2} \\
&\qquad - \sum_{i=1}^{n} \frac{CA}{g(A)} Eg(|\xi_i|) I(|\xi_i| \ge A)\Big\} \\
&\le P\Big\{\sum_{i=1}^{n} |\xi_i| I(|\xi_i| \ge A) \ge \frac{x}{4}\Big\} \\
&\le \sum_{i=1}^{n} P(|\xi_i| \ge A). \qquad (8.4.10)
\end{aligned}$$

By Lemma 2.2.5 and Lemma 4.1.2, there exists a constant $K_1$ depending

only on $q_1$ and $\rho(\cdot)$ such that

$$\begin{aligned} I_1 \le K_1 x^{-q_1} \Big( & n^{q_1/2} \|\xi I(|\xi| \le B)\|_2^{q_1} \exp\Big(K_1 \sum_{i=0}^{[\log n]} \rho(2^i)\Big) \\ & + n \log^{q_1}(2n) \|\xi I(|\xi| \le B)\|_{q_1}^{q_1} \\ & \cdot \exp\Big(K_1 \sum_{i=0}^{[\log n]} \rho^{2/q_1}(2^i)\Big)\Big). \end{aligned} \tag{8.4.11}$$

In order to estimate $I_2$, put

$$\begin{aligned} Y_i &= \sum_{j=2ik+1}^{(2i+1)k} \xi_{j2}, \qquad i = 0, 1, \cdots, p_1, \\ Z_i &= \sum_{j=(2i+1)k+1}^{2(i+1)k} \xi_{j2}, \qquad i = 0, 1, \cdots, p_2 \end{aligned} \tag{8.4.12}$$

where

$$p_1 = \Big[(\frac{n}{k} - 1)/2\Big], \; p_2 = \Big[(\frac{n}{k} - 2)/2\Big].$$

Denote

$$W_i = \sum_{j=0}^{i} Y_j, \quad W_i^* = \sum_{j=0}^{i} Z_j.$$

Then

$$\begin{aligned} I_2 \le\; & P\Big\{\max_{0 \le i \le p_1} |W_i| \ge \frac{x}{12}\Big\} + P\Big\{\max_{0 \le i \le p_2} |W_i^*| \ge \frac{x}{12}\Big\} \\ & + P\Big\{\max_{0 \le i \le [n/k]} \max_{ik+1 \le j < (i+1)k} \Big| \sum_{l=ik+1}^{j} \xi_{l2} \Big| \ge \frac{x}{12}\Big\} \\ =:\; & I_4 + I_5 + I_6. \end{aligned} \tag{8.4.13}$$

By Lemma 2.2.5 and (8.4.8) we have

$$
\begin{aligned}
I_6 \le 2\Big[\frac{n}{k}\Big] \max_{0\le i\le [n/k]} P\Big\{ \sum_{j=ik+1}^{(i+1)k} |\xi_j| I(B<|\xi_i|<A) \\
- E|\xi_j| I(B<|\xi_j|<A) \\
\ge \frac{x}{12} - 2\sum_{j=ik+1}^{(i+1)k} E|\xi_j| I(B<|\xi_j|<A)\Big\} \\
\le 2\cdot \Big[\frac{n}{k}\Big] \max_{0\le i\le [n/k]} P\Big\{ \sum_{j=ik+1}^{(i+1)k} |\xi_j| I(B<|\xi_j|<A) \\
- E|\xi_j| I(B<|\xi_j|<A) \ge \frac{x}{24}\Big\} \\
\le K_2 \Big[\frac{n}{k}\Big] x^{-q_2} \Big( k^{q_2/2} \|\xi I(B<|\xi|<A)\|_2^{q_2} \\
\cdot \exp\Big(K_2 \sum_{i=0}^{[\log n]} \rho(2^j)\Big) \\
+ k\|\xi I(B<|\xi|<A)\|_{q_2}^{q_2} \exp\Big(K_2 \sum_{i=0}^{[\log n]} \rho^{2/q_2}(2^j)\Big)\Big) \\
\le K_2 x^{-q_2} \Big( n^{q_2/2} \|\xi I(B<|\xi|<A)\|_2^{q_2} \\
\cdot \exp\Big(K_2 \sum_{i=0}^{[\log n]} \rho(2^j)\Big) \\
+ n\cdot \|\xi I(B<|\xi|<A)\|_{q_2}^{q_2} \\
\cdot \exp\Big(K_2 \sum_{i=0}^{[\log n]} \rho^{2/q_2}(2^j)\Big).
\end{aligned} \tag{8.4.14}
$$

Now we estimate $I_4$. Denote $\mathcal{F}_i = \sigma(\xi_j, j\le 2(i+1)k)$, $i=0,1,\cdots,p_1$, $\mathcal{F}_{-1}$ to be a trivial $\sigma$-field. Put

$$
U_i = Y_i - E(Y_i|\mathcal{F}_{i-1}), \quad G_i = \sum_{j=0}^{i} U_j,
$$

$$
H_i = \sum_{j=1}^{i} E(Y_j|\mathcal{F}_{j-1}), \quad i=0,1,\cdots,p_1.
$$

It is easy to see that

$$
\begin{aligned}
I_4 &\le P\Big\{\max_{0\le i\le p_1} |G_i| \ge \frac{x}{24}\Big\} + P\Big\{\max_{0\le i\le p_1} |H_i| \ge \frac{x}{24}\Big\} \\
&=: I_7 + I_8.
\end{aligned} \tag{8.4.15}
$$

Note that $\{U_i, \mathcal{F}_i, i \geq 0\}$ is a martingale difference sequence. Hence by the maximum velue inequality for a martingale difference sequence, the Marcinkiewicz-Zygmund inequality and Lemma 2.2.5, we have

$$
\begin{aligned}
I_7 &\leq \left(\frac{x}{24}\right)^{-q_2} \left(\frac{q_2}{q_2-1}\right)^{q_2} E|G_{p_1}|^{q_2} \\
&\leq K_3 x^{-q_2} \cdot p_1^{q_2/2} \max_{0 \leq i \leq p_1} E|U_i|^{q_2} \\
&\leq K_3 x^{-q_2} \cdot p_1^{q_2/2} \Big( k^{q_2/2} \|\xi I(B < |\xi| < A)\|_2^{q_2} \\
&\quad \cdot \exp\Big(K_3 \sum_{i=0}^{[\log n]} \rho(2^j)\Big) \\
&\quad + k \|\xi I(B < |\xi| < A)\|_{q_2}^{q_2} \\
&\quad \cdot \exp\Big(K_3 \sum_{i=0}^{[\log n]} \rho^{2/q_2}(2^i)\Big)\Big) \\
&\leq K_3 x^{-q_2} \Big( n^{q_2/2} \|\xi I(B < |\xi| < A)\|_2^{q_2} \\
&\quad \cdot \exp\Big(K_3 \sum_{i=0}^{[\log n]} \rho(2^i)\Big) \\
&\quad + \left(\frac{n}{k}\right)^{q_2/2} k \cdot E|\xi|^{q_2} I(B < |\xi| < A) \\
&\quad \cdot \exp\Big(K_3 \sum_{i=0}^{[\log n]} \rho^{2/q_2}(2^j)\Big). \qquad (8.4.16)
\end{aligned}
$$

By imitating the proof of Lemma 2.2.2 and using Lemma 2.2.5, there exists a constant $K_4$ depending only on $\rho(\cdot)$ such that

$$
\begin{aligned}
&E\Big(\sum_{j=i+1}^{i+m} E(Y_j|\mathcal{F}_{j-1})\Big)^2 \\
&\qquad \leq K_4 m k \rho^2(k) E\xi^2 I(B < |\xi| < A) \\
&\qquad \quad \cdot \log^2(2m) \exp\Big(K_4 \sum_{i=0}^{[\log n]} \rho(2^i)\Big). \qquad (8.4.17)
\end{aligned}
$$

From Lemma 4.1.2 it follows that

$$
\begin{aligned}
E \max_{1 \leq i \leq p_1} H_i^2 \leq & 3K_4 p_1 k \rho^2(k) \log^4(2p_1) E\xi^2 I(B < |\xi| < A) \\
& \cdot \exp\Big(K_4 \sum_{i=0}^{[\log n]} \rho(2^i)\Big). \qquad (8.4.18)
\end{aligned}
$$

Thus we obtain

$$I_8 \le K_4 x^{-2} n\rho^2(k) \log^4\Big[\frac{n}{k}\Big] E\xi^2 I(B < |\xi| < A) \cdot \exp\Big(K_4 \sum_{i=0}^{[\log n]} \rho(2^i)\Big). \tag{8.4.19}$$

(When $[n/k] \le 2$, we have $p_1 = 0$ and hence $I_8 = 0$, i.e., (8.4.19) holds true; when $[n/k] > 2$, (8.4.18) implies (8.4.19).) From (8.4.15), (8.4.16) and (8.4.19) it follows that

$$\begin{aligned} I_4 \le K_5 x^{-q_2} \Big( & n^{q_2/2} \|\xi_i I(B < |\xi_i| < A)\|_2^{q_2} \\ & \cdot \exp\Big(K_5 \sum_{i=0}^{[\log n]} \rho(2^i)\Big) \\ & + \Big(\frac{n}{k}\Big)^{q_2/2} k \|\xi_i I(B < |\xi_i| < A)\|_{q_2}^{q_2} \\ & \cdot \exp\Big(K_5 \sum_{i=0}^{[\log n]} \rho^{2/q_2}(2^i)\Big) \\ & + K_5 x^{-2} n\rho^2(k) \log^4\Big[\frac{n}{k}\Big] \|\xi_i I(B < |\xi_i| < A)\|_{q_2}^{q_2} \\ & \cdot \|\xi_i I(B < |\xi_i| < A)\|_2^2 \\ & \cdot \exp\Big(K_5 \sum_{i=0}^{[\log n]} \rho(2^i)\Big). \end{aligned} \tag{8.4.20}$$

By the same way, we have also (8.4.20) for $I_5$. Lemma 8.4.1 is proved.

**Proof of Theorem 8.4.1.**

From the assumption that $l(x), \beta(x)l(\beta(x))/x, x^{2-\varepsilon_0}/\beta(x)l(\beta(x))$ are quasi-monotone non-decreasing, it follows that there exists a constant $C > 0$ such that for any $t \ge x > 0$

$$l(x) \le Cl(t), \tag{8.4.21}$$

$$\beta(x)l(\beta(x))/x \le C\beta(t)l(\beta(t))/t, \tag{8.4.22}$$

$$x^{2-\varepsilon_0}/\beta(x)l(\beta(x)) \le Ct^{2-\varepsilon_0}/\beta(t)l(\beta(t)). \tag{8.4.23}$$

By (8.4.22), (8.4.23) and the monotonicity of $\beta(x)$ we have

$$\frac{\alpha(x)}{xl(x)} \ge \frac{1}{C}\frac{\alpha(t)}{tl(t)}, \tag{8.4.24}$$

$$\frac{\alpha^{2-\varepsilon_0}(x)}{xl(x)} \le C\frac{\alpha^{2-\varepsilon_0}(t)}{tl(t)}. \tag{8.4.25}$$

Let $C_n = \log^{-8/\varepsilon_0} n$ and take $A = \alpha(n), B = \alpha(nC_n), k = [nC_n], x = \alpha(\varepsilon n), g(x) = \beta(x)l(\beta(x))$ in Lemma 8.4.1. Then (8.4.7) and (8.4.8) are satisfied for large $n$. In fact, for $n \geq 1/\varepsilon$, from (8.4.24) and (8.4.21) we have

$$\begin{aligned}
&\frac{4Cn\alpha(n)}{nl(n)} E\beta(X_1)l(\beta(X_1))I(|X_1| \geq \alpha(n)) \\
&\quad \leq \frac{4C^2\alpha(\varepsilon n)}{\varepsilon l(\varepsilon n)} E\beta(X_1)l(\beta(X_1))I(|X_1| \geq \alpha(n)) \\
&\quad \leq \frac{4C^3\alpha(\varepsilon n)}{\varepsilon l(1)} E\beta(X_1)l(\beta(X_1))I(|X_1| \geq \alpha(n)).
\end{aligned}$$

Noting that $\lim_{n\to\infty} E\beta(X_1)l(\beta(X_1))I(|X_1| \geq \alpha(n)) = 0$, for large $n$ we obtain

$$\frac{4Cn\alpha(n)}{nl(n)} E\beta(X_1)l(\beta(X_1))I(|X_1| \geq \alpha(n)) \leq \alpha(\varepsilon n).$$

This proves that (8.4.7) is satisfied. Similarly we can verify (8.4.8). By Lemma 8.4.1, letting $q_1 = q_2 = 2$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{l(n)}{n} P\Big\{\max_{i\leq n} \beta(S_i) \geq \varepsilon n\Big\} \\
&\quad = \sum_{n=1}^{\infty} \frac{l(n)}{n} P\Big\{\max_{i\leq n} |S_i| \geq \alpha(\varepsilon n)\Big\} \\
&\quad \leq \sum_{n=1}^{\infty} l(n) P\{|X_1| \geq \alpha(n)\} \\
&\qquad + c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} EX_1^2 I(|X_1| \leq \alpha(nC_n)) \log^2 n \\
&\qquad + c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} EX_1^2 I(|X_1| \leq \alpha(n)) \\
&\qquad + c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} \rho^2(nC_n)(\log\log n)^4 EX_1^2 I(|X_1| \leq \alpha(n)) \\
&\quad =: J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{8.4.26}$$

From (8.4.21)

$$\begin{aligned}
J_1 &\le \sum_{n=1}^{\infty} l(n) \sum_{j=n}^{\infty} P\{j \le \beta(X_1) < j+1\} \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{j} l(n) P\{j \le \beta(X_1) < j+1\} \\
&\le c \sum_{j=1}^{\infty} j l(j) P\{j \le \beta(X_1) < j+1\} \\
&\le c E\beta(X_1) l(\beta(X_1)) < \infty. \qquad (8.4.27)
\end{aligned}$$

From (8.4.23), (8.4.25) and the monotonicity of $x/\alpha^\theta(x), \alpha^2(x)/x$ we have

$$\begin{aligned}
J_2 &\le c \sum_{n=1}^{\infty} \frac{l(n) \log^2 n \alpha^2(nC_n)}{\alpha^2(\varepsilon n) n C_n l(nC_n)} E\beta(X_1) l(\beta(X_1)) \\
&\le c\varepsilon^{-2/\theta} \sum_{n=1}^{\infty} \frac{\log^2 n}{n} \frac{\alpha^{\varepsilon_0}(nC_n)}{\alpha^{\varepsilon_0}(n)} E\beta(X_1) l(\beta(X_1)) \\
&\le c \sum_{n=1}^{\infty} \frac{\log^2 n}{n} C_n^{\varepsilon_0/2} \\
&\le c \sum_{n=1}^{\infty} \frac{1}{n \log^2 n} < \infty. \qquad (8.4.28)
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ we have $\rho(n) \le c \log^{-1} n$. Hence

$$\begin{aligned}
J_4 &\le c \sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} \log^{-3/2} n E X_1^2 I(|X_1| \le \alpha(n)) \\
&\le c \sum_{n=1}^{\infty} \frac{l(n)}{\alpha^2(\varepsilon n)} \frac{\alpha^2(n)}{n l(n)} \log^{-3/2} n E\beta(X_1) l(\beta(X_1)) \\
&\le c \sum_{n=1}^{\infty} \frac{1}{n \log^{3/2} n} < \infty. \qquad (8.4.29)
\end{aligned}$$

At last we estimate $J_3$. From (8.4.25) and $x/\alpha^2(x) \downarrow$, we have

$$\begin{aligned}
\sum_{j=n}^{\infty} \frac{l(j)}{\alpha^2(j)} &= \sum_{j=n}^{\infty} \frac{l(j) j}{\alpha^{2-\varepsilon_0}(j)} \frac{j^{\varepsilon_0/2}}{\alpha^{\varepsilon_0}(j)} \frac{1}{j^{1+\varepsilon_0/2}} \\
&\le \frac{c l(n) n^{1+\varepsilon_0/2}}{\alpha^2(n)} \sum_{j=n}^{\infty} j^{-1-\varepsilon_0/2} \\
&\le \frac{c}{\varepsilon_0} \cdot \frac{n l(n)}{\alpha^2(n)}.
\end{aligned}$$

Therefore

$$\begin{aligned} J_3 &\le c\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^2(n)}\alpha^2(1) \\ &\quad+\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^2(n)}\sum_{j=1}^{n}EX_1^2I(j<\beta(X_1)\le j+1) \\ &\le c\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\frac{l(n)}{\alpha^2(n)}EX_1^2I(j<\beta(X_1)\le j+1) \\ &\le c\sum_{j=1}^{\infty}\frac{jl(j)}{\alpha^2(j)}EX_1^2I(j<\beta(X_1)\le j+1) \\ &\le cE\beta(X_1)l(\beta(X_1))<\infty. \end{aligned} \tag{8.4.30}$$

Theorem 8.4.1 follows from (8.4.27)–(8.4.30).

**Proof of Theorem 8.4.2.**

From the assumption that $l(x), x^q/\beta(x)l(\beta(x))$ and $\beta(x)l(\beta(x))/x^2$ are quasi-monotone non-decreasing it follows that there exists a $C>0$ such that (8.4.21) is satisfied and for any $t\ge x>0$ we have

$$\frac{\alpha^q(x)}{xl(x)}\le C\frac{\alpha^q(t)}{tl(t)},\qquad \frac{\beta(x)l(\beta(x))}{x^2}\le C\frac{\beta(t)l(\beta(t))}{t^2}. \tag{8.4.31}$$

Let $C_n=n^{-\varepsilon_1}$, where $\varepsilon_1=\frac{\delta}{2(1+\delta)}$, let $\beta=\alpha(nC_n), k=n, g(x)=\beta(x)l(\beta(x))$. Take $x=\alpha(\varepsilon n)$ in Lemma 8.4.1. Then (8.4.7) and (8.4.8) are satisfied. In fact, by $l(x)\ge x^\delta$ we have for large $n$

$$\begin{aligned} &\frac{48C\alpha(nC_n)}{C_nl(nC_n)}E\beta(X_1)l(\beta(X_1)) \\ &\qquad\le\frac{48C\alpha(\varepsilon n)}{C_n(nC_n)^\delta}E\beta(X_1)l(\beta(X_1)) \\ &\qquad\le\frac{48C\alpha(\varepsilon n)}{n^{\delta/2}}E\beta(X_1)l(\beta(X_1)). \end{aligned}$$

Thus (8.4.8) is satisfied for large $n$. By the same way (8.4.7) is satisfied as well.

By Lemma 8.4.1 with $q_1 = q_0 + q + 4, q_2 = r$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{l(n)}{n} P\Big\{\max_{i\le n} \beta(S_i) \ge \varepsilon n\Big\} \\
&\le c\sum_{n=1}^{\infty} l(n)P\{|X_1| \ge \alpha(n)\} \\
&+ c\sum_{n=1}^{\infty} \frac{l(n)}{n\alpha^{q_1}(n)}\Big(n^{q_1/2} + n\exp\Big(K\sum_{i=0}^{[\log n]} \rho^{2/q_1}(2^i)\Big) \\
&\quad \cdot \log^{q_1} nE|X_1|^{q_1} I(|X_1| \le \alpha(nC_n))\Big) \\
&+ c\sum_{n=1}^{\infty} \frac{l(n)}{n\alpha^{r}(n)}\Big(n^{r/2}(EX_1^2 I(|X_1| \ge \alpha(nC_n)))^{r/2} \\
&\quad + nE|X_1|^r I(|X_1| \le \alpha(n))\Big) \\
&=: L_1 + L_2 + L_3. \qquad (8.4.32)
\end{aligned}$$

As in the estimation of $J_1$ in the proof of Theorem 8.4.1, we have

$$L_1 < \infty. \qquad (8.4.33)$$

Noting that $\exp\Big(K\sum_{i=1}^{[\log n]} \rho^{2/q_1}(2^i)\Big)\log^{q_1} n$ is a slowly varying function, $\frac{\alpha^2(x)}{x} \uparrow$, and using (8.4.31) and condition (8.4.3), we have

$$\begin{aligned}
L_2 &\le c\sum_{n=1}^{\infty} \frac{l(n)}{\alpha^{q_1}(n)} n^{q_1/2} E|X_1|^q I(|X_1| \le \alpha(nC_n)) \\
&\le c\sum_{n=1}^{\infty} \frac{l(n)n^{q_1/2}\alpha^{q_1}(nC_n)}{\alpha^{q_1}(n)l(nC_n)nC_n} \\
&\le c\sum_{n=1}^{\infty} n^{\varepsilon_1/2-1}\frac{\alpha^{q_1-q}(nC_n)}{\alpha^{q_1-q}(n)} \\
&\le c\sum_{n=1}^{\infty} n^{\varepsilon_1/2-1}C_n^2 \\
&\le c\sum_{n=1}^{\infty} n^{-1-\varepsilon_1} < \infty. \qquad (8.4.34)
\end{aligned}$$

Next we estimate $L_3$. From (8.4.31) it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{l(n)}{n}\Big(\frac{nEX_1^2I(|X_1|\geq\alpha(nC_n))}{\alpha^2(n)}\Big)^{r/2}\\
&\leq c\sum_{n=1}^{\infty}\frac{l(n)}{n}\frac{\alpha^r(nC_n)}{C_n^{r/2}l^{r/2}(nC_n)\alpha^r(n)}\\
&\leq c\sum_{n=1}^{\infty}\frac{l(n)n}{\alpha^r(n)}\frac{\alpha^r(nC_n)}{nC_nl(nC_n)}\frac{1}{nC_n^{r/2-1}l^{r/2-1}(nC_n)}\\
&\leq c\sum_{n=1}^{\infty}\frac{1}{nC_n^{r/2-1}(nC_n)^{(r/2-1)\delta}}\\
&\leq c\sum_{n=1}^{\infty}n^{-1-(\frac{r-1}{4})\delta}<\infty. \qquad (8.4.35)
\end{aligned}$$

By (8.4.35) and (8.4.31) we obtain

$$\begin{aligned}
L_3&\leq c\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^r(n)}E|X_1|^rI(|X_1|\leq\alpha(n))\\
&\leq c\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^r(n)}\cdot\alpha^r(1)\\
&\quad+\sum_{n=1}^{\infty}\frac{l(n)}{\alpha^r(n)}\cdot\sum_{j=1}^{n}E|X_1|^rI(j\leq\beta(X_1)<j+1)\\
&\leq c\sum_{n=1}^{\infty}\frac{nl(n)}{\alpha^q(n)}\cdot\frac{n^{\frac{r-q}{2}}}{\alpha^{r-q}(n)}\frac{\alpha^r(1)}{n^{1+(r-q)/2}}\\
&\quad+\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\frac{nl(n)}{\alpha^q(n)}\frac{n^{(r-q)/2}}{\alpha^{r-q}(n)}\frac{1}{n^{1+(r-q)/2}}\\
&\quad\cdot E|X_1|^rI(j\leq\beta(X_1)<j+1)\\
&\leq 1+c\sum_{j=1}^{\infty}\frac{jl(j)}{\alpha^r(j)}E|X_1|^rI(j\leq\beta(X_1)<j+1)\\
&\leq cE\beta(X_1)l(\beta(X_1))<\infty. \qquad (8.4.36)
\end{aligned}$$

Then Theorem 8.4.2 follows from (8.4.32)–(8.4.36).

In order to prove Corollaries 8.4.1 and 8.4.2, we need the following lemma.

**Lemma 8.4.2.** *Let $h(x)$ be a positive slowly varying function. Then $x^{\varepsilon}h(x)$ is a quasi-monotone non-decreasing function for any $\varepsilon>0$.*

**Proof.** By the property of a slowly varying function, we have

$$\lim_{N\to\infty} \sup_{2^N\le x\le 2^{N+1}} \frac{h(x)}{h(2^{N+1})} = \lim_{N\to\infty} \inf_{2^N\le x\le 2^{N+1}} \frac{h(x)}{h(2^{N+1})} = 1.$$

Therefore there exists an $N$ such that for $m \ge N$ we have

$$\sup_{2^m\le x\le 2^{m+1}} \frac{h(x)}{h(2^{m+1})} \le 2^\varepsilon, \quad 2^{-\varepsilon} \le \inf_{2^m\le x\le 2^{m+1}} \frac{h(x)}{h(2^{m+1})}. \tag{8.4.37}$$

For any $t \ge x \ge 2^N$, let $M_1, M_2 \ge N$ such that

$$2^{M_1} \le x < 2^{M_1+1}, \quad 2^{M_2} \le t < 2^{M_2+1}. \tag{8.4.38}$$

From (8.4.37) and (8.4.38) we have

$$\begin{aligned} x^\varepsilon h(x) &\le x^\varepsilon \sup_{2^{M_1}\le s<2^{M_1+1}} \frac{h(s)}{h(2^{M_1+1})} \cdot h(2^{M_1+1}) \\ &\le x^\varepsilon 2^{\varepsilon(M_2-M_1+1)} h(2^{M_2+1}) \\ &\le x^\varepsilon 2^{\varepsilon(M_2-M_1+2)} h(t) \\ &\le t^\varepsilon 2^{\varepsilon(M_2-M_1+1)} \cdot 2^{\varepsilon(M_2-M_1+2)} h(t) \\ &= 2^\varepsilon \cdot t^\varepsilon h(t). \end{aligned}$$

This proves that $x^\varepsilon h(x)$ is a quasi-monotone nondecreasing function.

**Proof of Corollary 8.4.1.** From $p\alpha > 1$ and Lemma 8.4.2 it follows that $n^{p\alpha-1}h(n)$ is quasi-monotone nondecreasing. If $1 \le p < 2$, $x^p h(x), x^{2-p}/h(x)$ are quasi-monotone nondecreasing. Then Corollary 8.4.1 follows from Theorem 8.4.1. If $p > 2$, $x^{p-2}h(x)$ and $x^{r-p}/h(x)$ are quasi-monotone non-decreasing. Then Corollary 8.4.1 follows from Theorem 8.4.2. If $p = 2$, $\lim_{n\to\infty} n^{-\varepsilon} EX_1^2 \cdot I(|X_1| \le n^\alpha) = 0$ for any $\varepsilon > 0$, and hence Corollary 8.4.1 can be obtained by repeating the proof of Theorem 8.4.2.

**Proof of Corollary 8.4.2.** Obviously $x^{p-1}h(x)$ is increasing. By Lemma 8.4.2, $x^{2-p}/h(x)$ is quasi-monotone non-decreasing. Hence Corollary 8.4.2 follows from Theorem 8.4.1.

## 8.5 Complete convergence for $\alpha$-mixing sequences

For the complete convergence of $\alpha$-mixing sequences, Hipp (1979) obtained the following result:

**Theorem 8.5.0.** *Let*

$$\frac{1}{2} < \alpha \le 1, 2 < r \le \infty, 1/\alpha < p < r, \{X_n, n \ge 1\}$$

*be a strictly stationary $\alpha$-mixing sequence of random variables with $EX_1 = 0, E|X_1|^r < \infty$. Assume that*

$$\sum_{n=1}^{\infty} \alpha^{1/\theta}(n) < \infty \quad \textit{for some} \quad \theta > \left[2 + \frac{r}{r-p}\right] \cdot \frac{p\alpha}{p\alpha - 1}. \tag{8.5.1}$$

*Then*

$$\sum_{n=1}^{\infty} n^{\alpha-2} P\Big\{ \max_{1\le i\le n} |S_i| \ge \varepsilon n^{\alpha} \Big\} < \infty \quad \textit{for any} \quad \varepsilon > 0. \tag{8.5.2}$$

However, an example contrary to Hipp's conclusion was given by Berbee (1987) when $r = \infty$, i.e., in the case of $X_1$ bounded.

Shao (1993c) proved the following theorem.

**Theorem 8.5.1.** *Let $1/2 < \alpha \le 1$, $1/\alpha \le p < r \le \infty$, $\{X_n, n \ge 1\}$ be an $\alpha$-mixing sequence of random variables with $EX_n = 0$, $\sup_{n\ge 1} \|X_n\|_r < \infty$. Assume that*

$$\alpha(n) = O(n^{-r(p-1)/(r-p)} \log^{-\beta} n) \quad \textit{for some} \quad \beta > rp/(r-p). \tag{8.5.3}$$

*Then (8.5.2) holds true.*

An immediate consequence of Theorem 8.5.1 with $p = \alpha = 1$ is:

**Corollary 8.5.1.** *Let $1 < r \le \infty$, $\{X_n, n \ge 1\}$ be an $\alpha$-mixing sequence of random variables with $EX_n = 0, \sup_{n\ge 1} \|X_n\|_r < \infty$. Assume that*

$$\alpha(n) = O(\log^{-\beta} n) \quad \textit{for some} \quad \beta > r/(r-1).$$

*Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\Big\{ \max_{1\le i\le n} |S_i| \ge \varepsilon n \Big\} < \infty \quad \textit{for any} \quad \varepsilon > 0.$$

*Particularly, we have*

$$S_n/n \to 0 \quad a.s.$$

The proof of Theorem 8.5.1 will needs the following lemmas.

**Lemma 8.5.1.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $EX_n = 0$ for every $n \geq 1$. Then*

$$P\Big\{\max_{1\leq i\leq n} |S_i| \geq x\Big\} \leq 4x^{-1}\sum_{i=1}^{n} E|X_i|I(|X_i| > c) + 4x^a + 32^3 ncx^{-1}\alpha(k) \tag{8.5.4}$$

*for any $a \geq 1$, $x \geq 1$, $c > 0$ and integer $k$ satisfying*

$$1 \leq k \leq x/(64ac\log x) \tag{8.5.5}$$

*and for some $s \geq 2$,*

$$\Big(\sum_{i=1}^{n} \|X_i I(|X_i| \leq c)\|_s^2\Big) \sum_{i=0}^{k} \alpha^{1-2/s}(i) \leq x^2/(32^3 a \log x). \tag{8.5.6}$$

**Proof.** Let

$$\overline{X}_i = X_i I(|X_i| \leq c) - EX_i I(|X_i| \leq c),$$

$$Y_{i,1} = \sum_{j=1+2ik}^{(2i+1)k\wedge n} \overline{X}_j, \quad i = 0, 1, \cdots, q_1 := \Big[\frac{n}{2k} - \frac{1}{2}\Big],$$

$$Y_{i,2} = \sum_{j=1+(2i+1)k}^{2(i+1)k\wedge n} \overline{X}_j, \quad i = 0, 1, \cdots, q_2 := \Big[\frac{n}{2k} - 1\Big],$$

$$\overline{S}_i = \sum_{j=1}^{i} \overline{X}_j, \qquad T_{i,1} = \sum_{j=0}^{i} Y_{j,1} \qquad T_{i,2} = \sum_{j=0}^{i} Y_{j,2}.$$

It is easy to see that

$$\max_{1\leq i\leq n} |S_i| \leq \max_{1\leq i\leq n} |\overline{S}_i| + \sum_{i=1}^{n} |X_i|I(|X_i| > c) + \sum_{i=1}^{n} E|X_i|I(|X_i| > c)$$

and

$$P\Big\{\max_{1\leq i\leq n} |S_i| \geq x\Big\} \leq P\Big\{\max_{1\leq i\leq n} |\overline{S}_i| \geq x/2\Big\} + 4x^{-1}\sum_{i=1}^{n} E|X_i|I(|X_i| > c). \tag{8.5.7}$$

Since

$$\begin{aligned}\max_{1\le i\le n}|\overline{S}_i| &\le \max_{0\le i\le q_1}|T_{i,1}| + \max_{0\le i\le q_2}|T_{i,2}| + 2kc \\ &\le \max_{0\le i\le q_1}|T_{i,1}| + \max_{0\le i\le q_2}|T_{i,2}|x + x/32\end{aligned}$$

by (8.5.5), we have

$$\begin{aligned}&P\Big\{\max_{1\le i\le n}|\overline{S}_i| \ge x/2\Big\} \\ &\quad\le P\Big\{\max_{0\le i\le q_1}|T_{i,1}| \ge x/8\Big\} + P\Big\{\max_{0\le i\le q_2}|T_{i,2}| \ge x/8\Big\} \\ &\quad=: I_1 + I_2. \end{aligned} \tag{8.5.8}$$

We first estimate $I_1$. The estimation of $I_2$ is completely similar. Put

$$\begin{aligned}&G_{-1} = \sigma(\Omega), \quad G_i = \sigma(X_j, 1 \le j \le (2i+1)k), \\ &u_i = Y_{i,1} - E(Y_{i,1}|G_{i-1}), \quad U_i = \sum_{j=0}^{i} u_j, \quad i = 0, 1, \cdots.\end{aligned}$$

Then

$$\begin{aligned}I_1 &\le P\Big\{\sum_{i=0}^{q_1}|E(Y_{i,1}|G_{i-1})| \ge x/16\Big\} \\ &\quad + P\Big\{\max_{0\le i\le q_1}|U_i| \ge x/16\Big\} \\ &=: I_3 + I_4\end{aligned} \tag{8.5.9}$$

and $\{U_i, G_i, i \ge 0\}$ is a martingale with $|u_i| \le 2kc$ for every $i \ge 0$. Noting that for each real $t$ and $i \ge 1$,

$$\begin{aligned}E(e^{tu_i}|G_{i-1}) &= 1 + \sum_{l=2}^{\infty} E\Big(\frac{(tu_i)^l}{l!}\Big|G_{i-1}\Big) \\ &\le 1 + t^2 E(u_i^2|G_{i-1}) \sum_{l=0}^{\infty} \frac{(|t|2kc)^l}{l!} \\ &\le \exp(t^2 e^{2|t|kc} E(u_i^2|G_{i-1})) \\ &\le \exp(t^2 e^{2|t|kc} E(Y_{i,1}^2|G_{i-1})),\end{aligned}$$

we find that $\Big\{\exp\Big(tU_i - t^2 e^{2|t|kc}\sum_{j=0}^{i} E(Y_{j,1}^2|G_{j-1})\Big), G_i, i \ge 0\Big\}$ is a non-negative supermartingale for every $t$ and hence

$$P\Big\{\max_{0\le i\le q_1}\exp\Big(tU_i - t^2 e^{2|t|kc}\sum_{j=0}^{i} E(Y_{j,1}^2|G_{j-1})\Big) \ge y\Big\} \le \frac{1}{y} \tag{8.5.10}$$

for $y > 0$, by the maximum inequality of the non-negative supermartingale. Take $t = (32a\log x)/x$ in (8.5.10). By (8.5.10) and (8.5.5), we have

$$
\begin{aligned}
P\Big\{&\max_{0\le i\le q_1} U_i \ge x/16\Big\}\\
&\le P\Big\{\max_{0\le i\le q_1}\exp\Big(tU_i - t^2e^{2|t|kc}\sum_{j=0}^{i}E(Y_{j,1}^2|G_{j-1})\Big)\\
&\qquad \ge \exp\Big(\frac{xt}{16} - t^2e^{2|t|kc}\sum_{j=0}^{i}E(Y_{j,1}^2|G_{j-1})\Big)\Big\}\\
&\le P\Big\{\sum_{j=0}^{q_1}E(Y_{j,1}^2|G_{j-1}) \ge \frac{x^2}{4(32)^2a\log x}\Big\}\\
&\quad + P\Big\{\max_{0\le i\le q_1}\exp\Big(tU_i - t^2e^{2|t|kc}\sum_{j=0}^{i}E(Y_{j,1}^2|G_{j-1})\Big)\\
&\qquad \ge \exp\Big(\frac{xt}{16} - \frac{t^2e^{2|t|kc}x^2}{4(32)^2a\log x}\Big)\Big\}\\
&\le P\Big\{\sum_{j=0}^{q_1}E(Y_{j,1}^2|G_{j-1}) \ge \frac{x^2}{4(32)^2a\log x}\Big\}\\
&\quad + \exp\Big(-\frac{xt}{16} + \frac{t^2e^{2|t|kc}x^2}{4(32)^2a\log x}\Big)\\
&\le P\Big\{\sum_{j=0}^{q_1}E(Y_{j,1}^2|G_{j-1}) \ge \frac{x^2}{4(32)^2a\log x}\Big\} + x^{-a}.
\end{aligned}
\tag{8.5.11}
$$

Using Lemma 1.2.4, one can obtain

$$
\begin{aligned}
\sum_{j=0}^{q_1}EY_{j,1}^2 &\le 4\Big(\sum_{i=0}^{k}\alpha^{1-2/s}(i)\Big)\sum_{j=0}^{q_1}\sum_{i=2jk+1}^{(2j+1)k}\|X_iI(|X_i|\le c)\|_s^2\\
&\le 4\Big(\sum_{i=0}^{k}\alpha^{1-2/s}(i)\Big)\sum_{i=0}^{n}\|X_iI(|X_i|\le c)\|_s^2\\
&\le \frac{x^2}{8(32)^2a\log x}
\end{aligned}
$$

by (8.5.6). Hence

$$
\begin{aligned}
P\Big\{&\sum_{j=0}^{q_1}E(Y_{j,1}^2|G_{j-1}) \ge \frac{x^2}{4(32)^2a\log x}\Big\}\\
&\le \frac{8(32)^2a\log x}{x^2}\sum_{j=0}^{q_1}E|E(Y_{j,1}^2|G_{j-1}) - EY_{j,1}^2|.
\end{aligned}
\tag{8.5.12}
$$

Write $\xi_j = E(Y_{j,1}^2|G_{j-1}) - EY_{j,1}^2$. We find that

$$E|\xi_j| = E(Y_{j,1}^2 - EY_{j,1}^2)\mathrm{sgn}\xi_j \le 4(kc)^2\alpha(k) \tag{8.5.13}$$

by Lemma 1.2.4 again. Inserting (8.5.13) into (8.5.12), we obtain

$$\begin{aligned} &P\Big\{\sum_{j=0}^{q_1} E(Y_{j,1}^2|G_{j-1}) \ge \frac{x^2}{4(32)^2 a\log x}\Big\} \\ &\le \frac{(32)^3 anc^2 k\alpha(k)\log x}{x^2} \le \frac{(32)^2 nc\alpha(k)}{x} \end{aligned} \tag{8.5.14}$$

by (8.5.5). Now a combination of (8.5.11) and (8.5.14) yields

$$P\Big\{\max_{0\le i\le q_1} U_i \ge \frac{x}{16}\Big\} \le x^{-a} + (32)^2 ncx^{-1}\alpha(k).$$

Similarly, we have

$$P\Big\{\max_{0\le i\le q_1} U_i \le -\frac{x}{16}\Big\} \le x^{-a} + (32)^2 ncx^{-1}\alpha(k).$$

Hence

$$I_4 \le 2x^{-a} + 2(32)^2 ncx^{-1}\alpha(k). \tag{8.5.15}$$

Also, one can get that

$$E|E(Y_{i,1}|G_{i-1})| = EY_{i,1}\mathrm{sgn}(E(Y_{i,1}|G_{i-1})) \le 4kc\alpha(k)$$

and hence

$$I_3 \le 64ncx^{-1}\alpha(k). \tag{8.5.16}$$

It follows from (8.5.9), (8.5.15) and (8.5.16) that

$$I_1 \le 2x^{-a} + 3(32)^2 ncx^{-1}\alpha(k). \tag{8.5.17}$$

Similarly, we have

$$I_2 \le 2x^{-a} + 3(32)^2 ncx^{-1}\alpha(k). \tag{8.5.18}$$

This proves (8.5.4) by (8.5.7), (8.5.8), (8.5.17) and (8.5.18).

**Lemma 8.5.2.** *Let $\{X_n, n \ge 1\}$ be an $\alpha$-mixing sequence with*

$$EX_i = 0, \qquad \|X_i\|_\nu \le D \quad \text{and} \quad \alpha(i) \le C_0 i^{-\tau}\log^{-\lambda} i$$

*for $i \geq 1$ and for some $1 < \nu \leq \infty$, $C_0 \geq 1$, $\tau \geq 0$ and real $\lambda$. Then there exists a finite positive constant $K$ depending only on $\nu$, $\tau$, $\lambda$ and $C_0$ such that*

$$P\Big\{\max_{1\leq i\leq n}|S_i| \geq x\Big\} \leq Kn(D/x)^{\nu(\tau+1)/(\nu+\tau)}\log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)}(x/D) \tag{8.5.19}$$

*for any $x \geq KDn^{1/2}\log^{1+|\lambda|/2} n$.*

**Proof.** We assume , without loss of generality, that $D = 1$. It suffices to show that there exists a constant $K$ such that

$$P\Big\{\max_{1\leq i\leq n}|S_i| \geq x\Big\} \\ \leq Knx^{-\nu(\tau+1)/(\nu+\tau)}\log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)} x \tag{8.5.20}$$

for any $x \geq Kn^{1/2}\log^{1+|\lambda|/2} n$. Take

$$C = 2x^{\tau/(\nu+\tau)}\log^{(\lambda-\tau)/(\nu+\tau)} x,$$
$$a = \tau + 2, \quad k = [x/(64ac\log x)]$$

in Lemma 8.5.1. Assume that

$$Knx^{-\nu(\tau+1)/(\nu+\tau)}\log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)} x \leq 1. \tag{8.5.21}$$

Otherwise, (8.5.20) is trivial. If the conditions of Lemma 8.5.1 are satisfied, then (8.5.20) will follow from (8.5.4) immediately. So we need only to verify (8.5.6). In what follows we denote by $K_1$ the finite positive constant depending only on $\nu$, $\tau$, $\lambda$ and $C_0$, whose value may be different from line to line. If $1 < \nu \leq 2$, then

$$\begin{aligned}
&\Big(\sum_{i=1}^{n}\|X_iI(|X_i| \leq c)\|_2^2\Big)\sum_{i=0}^{k}\alpha^{1-2/\nu}(i) \\
&\leq 2nkc^{2-\nu} \leq xnc^{1-\nu}/(32a\log x) \\
&= \frac{x^2}{(32)^3a\log x}(32)^2 2^{1-\nu}nx^{-\nu(\tau+1)/(\nu+\tau)} \\
&\quad \cdot \log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)} x \\
&\leq \frac{x^2}{(32)^3a\log x}
\end{aligned} \tag{8.5.22}$$

by (8.5.21). When $\nu > 2$, we have

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} \|X_i I(|X_i| \le c)\|_\nu^2\Big) \sum_{i=0}^{k} \alpha^{1-2/\nu}(i) \\
&\le n\Big(1 + C_0 \sum_{i=1}^{k} i^{-\tau(1-2/\nu)} \log^{-\lambda(1-2/\nu)} i\Big) \\
&\le K_1 n(\log^{1+|\lambda|(1-2/\nu)} k + k^{-\tau(1-2/\nu)+1} \log^{-\lambda(1-2/\nu)} k) \\
&\le K_1 n(\log^{1+|\lambda|} x + (x/(c\log x))^{-\tau(1-2/\nu)+1} \log^{-\lambda(1-2/\nu)} x) \\
&\le \frac{x^2}{(32)^3 a \log x} (K_1 n x^{-2} \log^{2+|\lambda|(1-2/\nu)} x \\
&\quad + K_1 n x^{-\nu(\tau+1)/(\nu+\tau)} \log^{(\nu-1)(\tau-\lambda)/(\nu+\tau)} x) \\
&\le \frac{x^2}{(32)^3 a \log x}
\end{aligned} \tag{8.5.23}$$

by (8.5.21) and $x \ge K n^{1/2} \log^{1+|\lambda|/2} n$. Now we conclude from (8.5.22) and (8.5.23) that (8.5.6) is satisfied. This completes the proof of the lemma.

**Proof of Theorem 8.5.1.** By Lemma 8.5.2, we have that for any $\varepsilon > 0$, there exists a positive constant $K$ such that

$$\begin{aligned}
&P\Big\{\max_{1\le i\le n} |S_i| \ge \varepsilon n^\alpha\Big) \\
&\le K n^{1-\alpha r(1+r(p-1)/(r-p))/(r+r(p-1)/(r-p))} \\
&\quad \cdot \log^{(1-r)(\beta-r(p-1)/(r-p))/(r+r(p-1)/(r-p))} n \\
&= K n^{1-p\alpha} \log^{-1-(r-p)(\beta-rp/(r-p))/r} n,
\end{aligned}$$

which yields (8.5.5) immediately by (8.5.6), as desired.

The example below shows that Theorem 8.5.1 does not remain valid, if the assumption $\beta > rp/(r-p)$ is replaced by $\beta \ge r/(r-p)$.

**Example 8.5.1.** Let $1 < p < r$, $1/p \le \alpha \le 1$. Put

$$a = \frac{\alpha(r-p)}{r(1-\alpha)+p\alpha-1}, \quad b = \frac{a(p-1)}{r-p}, \quad d = \frac{-1}{r-1-\alpha(r-p)},$$

$$g(x) = x^a \log^d x, \quad x \ge 0,$$

$$G(0) = 0, \quad G(n) = \sum_{i=1}^{n} [g(i)], \; n = 1, 2, \cdots,$$

$$f(x) = (g(x))^{r(p-1)/(r-p)} (\log g(x))^{r/(r-p)} \log\log g(x), \quad x \ge 0.$$

Let $\{Y_n, n \geq 1\}$ be an independent sequence of random variables with

$$P\{Y_n = \pm f^{1/r}(n)\} = \frac{1}{2f(n)},$$
$$P\{Y_n = 0\} = 1 - \frac{1}{f(n)}.$$

Define a sequence $\{X_n, n \geq 1\}$ by $X_n = Y_j$ for $G(j-1) < n \leq G(j)$. Then $\{X_n, n \geq 1\}$ has the following properties.

$$EX_n = 0, \qquad E|X_n|^r = 1, \tag{8.5.24}$$
$$\alpha(n) = O(n^{-r(p-1)/(r-p)} \log^{-r/(r-p)} n(\log\log n)^{-1}), \tag{8.5.25}$$
$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\{|S_n| \geq \varepsilon n^{\alpha}\} = \infty \quad \text{for any} \quad \varepsilon > 0. \tag{8.5.26}$$

**Example 8.5.2.** Let $r > 1$. Put

$$a = (r-1)/r, \quad g(n) = [n^{a-1}\exp(n^a)], \quad G(n) = \sum_{i=1}^{n} g(i).$$

Let $\{Y_n, n \geq 1\}$ be an independent sequence of random variables with

$$P\{Y_n = \pm n^{1/r} \log^{1/r} n\} = \frac{1}{2n\log n}, \quad P\{Y_n = 0\} = 1 - \frac{1}{n\log n}.$$

Define $X_n = Y_j$ for $G(j-1) < n \leq G(j)$. Then $\{X_n, n \geq 1\}$ has the following properties.

$$EX_n = 0, \qquad E|X_n|^r = 1,$$
$$\alpha(n) = O(\log^{-r/(r-1)}(\log\log n)^{-1}),$$
$$\sum_{n=1}^{\infty} \frac{1}{n} P\{|S_n| \geq \varepsilon n\} = \infty, \quad \text{for any} \quad \varepsilon > 0.$$

**Remark 8.5.1.** We now can discuss whether Theorem 8.5.0 is true or not in the case of $r < \infty$. Assume that $1 < p \leq r/2$. Then (8.5.1) will be satisfied with $\theta > 8$ provided $p$ large enough. But Example 8.5.1 says that the mixing rate $n^{-(p-1)/2}$ is required at least. This means that Theorem 8.5.0 is quite possibly not true. Unfortunately, $\{X_n, n \geq 1\}$ in Example 8.5.1 is not strictly stationary. Shao (1993c) conjectured that there is a strictly stationary $\alpha$-mixing sequence satisfying (8.5.24), (8.5.26) and $\alpha(n) = O(n^{-r(p-1)/(r-p)} \log^{-1} n)$. He also conjectured that the assumption $\beta > rp/(r-p)$ in Theorem 8.5.1 can be replaced by $\beta > r/(r-p)$.

## 8.6 A further discussion on the complete convergence for partial sums of a mixing sequence

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d.r.v.'s, $S_n = \sum_{i=1}^n X_i$. Assume that positive functions $H(t)$ and $\psi(t)$ are defined on $(0, \infty)$ and $H(t) \uparrow \infty (t \to \infty)$, $\widehat{\psi}(t) = \int_0^t \psi(u) du$, $t > 0$. Denote

$$\begin{aligned}
\nu(\varepsilon) &= \sum_{n=1}^{\infty} I((|S_n| \geq \varepsilon H(n)), \\
\eta(\varepsilon) &= \sup_{n \geq 1} \{(|S_n| - \varepsilon H(n))^+ / \varepsilon\}, \\
\chi(\varepsilon) &= \sup\{n \geq 1 : |S_n| \geq \varepsilon H(n)\},
\end{aligned}$$

where $\varepsilon$ is an arbitrary positive number. Three problems posed by Prohorov are as follows:

(i) Is

$$\sum_{n=1}^{\infty} \psi(n) P\{|S_n| \geq \varepsilon H(n)\} < \infty, \quad \forall \varepsilon > 0 \tag{8.6.1}$$

equivalent to

$$E\hat{\psi}(\nu(\varepsilon)) < \infty, \quad \forall \varepsilon > 0?$$

(ii) Is

$$\sum_{n=1}^{\infty} \psi(n) P\{\max_{1 \leq k \leq n} |S_k| \geq \varepsilon H(n)\} < \infty, \quad \forall \varepsilon > 0, \tag{8.6.2}$$

equivalent to

$$E\hat{\psi}(\mathrm{inv} H(\eta(\varepsilon))) < \infty, \quad \forall \varepsilon > 0?$$

(iii) Is

$$\sum_{n=1}^{\infty} \psi(n) P\{\sup_{k \geq n} |S_k| / H(k) \geq \varepsilon\} < \infty, \quad \forall \varepsilon > 0, \tag{8.6.3}$$

equivalent to

$$E\hat{\psi}(\chi(\varepsilon)) < \infty, \quad \forall \varepsilon > 0?$$

Let

$$\begin{aligned}
M_1 = \{\psi : \psi(x) > 0, x \in [1, \infty), \exists \delta_1, \delta_2 > 0, \quad &\text{such that} \\
x^{1-\delta_1} \psi(x) \uparrow, \quad e^{-\delta_2 x} \psi(x) \downarrow, \quad &x \to \infty\}.
\end{aligned}$$

Sirazgimov and Gafrov (1987) discussed these problems and showed that under the following assumptions:

$$\limsup_{x\to\infty} H(Cx)/H(x) < \infty, \quad \forall C > 1; \tag{8.6.4}$$

$$\liminf_{n\to\infty} P\{S_n \geq -\varepsilon H(n)\} > 0, \quad \forall \varepsilon > 0 \tag{8.6.5}$$

and

$$\psi \in M_1 \quad \text{or} \quad \psi(x), \quad \text{as} \quad x \to \infty,$$

the following three conclusions are equivalent

$$P(\psi, H, \varepsilon) := \sum_{n=1}^{\infty} \psi(n) P\{S_n \geq \varepsilon H(n)\} < \infty, \quad \forall \varepsilon > 0, \tag{8.6.6}$$

$$M(\psi, H, \varepsilon) := \sum_{n=1}^{\infty} \psi(n) P\{\max_{1\leq k\leq n} S_k \geq \varepsilon H(n)\} < \infty, \quad \forall \varepsilon > 0, \tag{8.6.7}$$

$$S(\psi, H, \varepsilon) := \sum_{n=1}^{\infty} \psi(n) P\{\sup_{k\geq n} S_k/H(k) \geq \varepsilon\} < \infty, \quad \forall \varepsilon > 0. \tag{8.6.8}$$

Su (1989) studied these problems and gave the following result. Suppose that the following conditions are satisfied:

(A) There exist constants $C > 0$, $t_0 > 0$ such that $\psi(t) \leq C\hat{\psi}(t)$ for any $t \geq t_0$;

(B) $t\psi(t) \uparrow \infty (t \to \infty)$ and there exist $C' > 0, t_0 > 0$ such that for any $t \geq t_0$

$$\int_0^t u\psi(u)du \geq C' t^2 \psi(t);$$

(C) $0 < \beta_1 \leq H(t)/t \leq \beta_2 < +\infty$ or $H(t)/t \uparrow \infty (t \to \infty)$ or $H(t)/t \downarrow 0$ and $H^2(t)/t \uparrow \infty (t \to \infty)$;

(D) There exist a positive integer $N$ and $\delta_2 > 0$ such that for every integer $k > 0$

$$\sum_{n=1}^{N^{k+1}} \psi(n) \geq (1 + \delta_2) \sum_{n=1}^{N^k} \psi(n).$$

Then (8.6.1) $\Longleftrightarrow$ (8.6.2) $\Longleftrightarrow$ (8.6.3) $\Longleftrightarrow E\hat{\psi}(\chi(\varepsilon)) < \infty$.

Wang (1993) discussed Prohorov's three problems for a strictly stationary $\rho$-mixing sequence. Denote

$$\varphi^*(1) = \sup_k \sup_{A\in\mathcal{F}_1^k, B\in\mathcal{F}_{k+1}^\infty, P(A)P(B)>0} \cdot \max\{|P(B|A)-P(B)|, |P(A|B)-P(A)|\} \quad (8.6.9)$$

$$\psi^*(x) = \frac{1}{x}\Big(x\psi(x) - \Big[\frac{x}{2}\Big]\psi\Big(\Big[\frac{x}{2}\Big]\Big)\Big), \quad x \geq 1, \quad (8.6.10)$$

$$M_1^* = \{\psi : \psi(x) > 0, x \geq 1, n\psi(n) \to \infty, n\psi(n) > [n/2]\psi([n/2])\}.$$

Wang (1993) proved the following theorem.

**Theorem 8.6.1.** *Suppose $\{X_n, n \geq 1\}$ is a strictly stationary $\rho$-mixing sequence satisfying*

$$\liminf_{n\to\infty} P\{S_n \geq -\varepsilon H(n)\} > \varphi^*(1) \quad \textit{for any } \varepsilon > 0. \quad (8.6.11)$$

*Suppose that $\psi \in M_1^*$, $H(n)$ satisfies (8.6.4) and*

$$\sum_{n=1}^\infty \psi(n)\rho(n) < \infty \quad (8.6.12)$$

*and for $\psi \in M_1^* - M_1$*

$$P\Big\{\sup_{k\geq n}\frac{S_k}{H(k)} \geq \varepsilon\Big\} \geq CP\Big\{\inf_{k\geq n}\frac{S_k}{H(k)} \leq -C\varepsilon\Big\}, \quad n = 1, 2, \cdots, \quad (8.6.13)$$

*for some $C > 0$. Then for any $\varepsilon > 0$* (8.6.6) $\Longleftrightarrow$ (8.6.7) $\Longleftrightarrow$

$$S(\psi^*, H, \varepsilon) < \infty. \quad (8.6.8')$$

The proof of Theorem 8.6.1 will be given by several lemmas.

**Lemma 8.6.1.** *If for any $x > 0$*

$$\liminf_{n\to\infty} \inf_{k\leq n-1} P\{S_n - S_k \geq -x\} > \varphi^*(1), \quad (8.6.11')$$

*then there exists a constant $C > 0$ such that for any $y$ and $n \in \mathbb{N}$*

$$P\Big\{\max_{k\leq n} S_k \geq y\Big\} \leq CP\Big\{S_n \geq y - x\Big\}. \quad (8.6.14)$$

**Proof.** By the definition of $\varphi^*(1)$ and (8.6.11′) we have

$$\begin{aligned}
&P\{S_n \geq y - x\} \\
&\quad \geq P\Big\{S_n \geq y - x, \max_{k\leq n} S_k \geq y\Big\} \\
&\quad = \sum_{k=1}^{n} P\Big\{S_n \geq y - x, S_k \geq y, \max_{i\leq k-1} S_i < y\Big\} \\
&\quad \geq \sum_{k=1}^{n} P\Big\{S_n - S_k \geq -x, S_k \geq y, \max_{i\leq k-1} S_i < y\Big\} \\
&\quad \geq \sum_{k=1}^{n} \Big(P\{S_n - S_k \geq -x\} P\Big\{S_k \geq y, \max_{i\leq k-1} S_i < y\Big\} \\
&\qquad - \varphi^*(1) P\Big\{S_k \geq y, \max_{i\leq k-1} S_i < y\Big\}\Big) \\
&\quad \geq C P\Big\{\max_{k\leq n} S_k \geq y\Big\}, \quad n = 1, 2, \cdots.
\end{aligned}$$

**Lemma 8.6.2.** *Let* $\{X_n, n \geq 1\}$ *be a strictly stationary sequence satisfying (8.6.11) and* $n\psi(n) \geq C$, $n = 1, 2, \cdots$. *Then* (8.6.6) $\Longleftrightarrow$ (8.6.7).

**Proof.** We need only to prove (8.6.6) $\Longrightarrow$ (8.6.7). Take $y = \varepsilon H(n), x = \varepsilon H(n)/2$ for any $\varepsilon > 0$ in (8.6.14). It is enough to prove (8.6.11′). First we have

$$P\{S_n \geq \varepsilon H(n)\} \longrightarrow 0, \quad \text{as} \quad n \to \infty. \tag{8.6.15}$$

Otherwise there exist $\{n_i\}, r$ and $\varepsilon_0 > 0$, such that

$$P\{S_{n_i} \geq \varepsilon_0 H(n_i)\} \geq r > 0.$$

Without loss of generality, we assume that $n_{i+1} \geq 3n_i$, $i = 1, 2, \cdots$. So by (8.6.11) for any $2n_i \leq n \leq 3n_i$, $i = 1, 2, \cdots$ we have

$$\begin{aligned}
&P\Big\{S_n \geq \frac{\varepsilon_0}{2} H(n)\Big\} \\
&\quad \geq P\Big\{S_{n_i} \geq \varepsilon_0 H(n), S_n - S_{n_i} \geq -\frac{\varepsilon_0}{2} H(n)\Big\} \\
&\quad \geq P\{S_{n_i} \geq \varepsilon_0 H(n)\}\Big(P\Big\{S_n - S_{n_i} \geq -\frac{\varepsilon_0}{2} H(n)\Big\} - \varphi^*(1)\Big) \\
&\quad \geq P\{S_{n_i} \geq \varepsilon_0 H(n)\}\Big(P\Big\{S_{n-n_i} \geq -\frac{\varepsilon_0}{2} H(n - n_i)\Big\} - \varphi^*(1)\Big) \\
&\quad \geq r_1 > 0
\end{aligned}$$

for some $r_1 > 0$. It follows from $n\psi(n) \geq C > 0,\ n = 1, 2, \cdots$ that

$$\begin{aligned} P(\psi, H, \varepsilon_0/2) &\geq \sum_{i=1}^{\infty} \sum_{m=2ni}^{3ni} \psi(m) P\{S_m \geq \varepsilon_0 H(m)/2\} \\ &\geq r_1 \sum_{i=1}^{\infty} \sum_{m=2ni}^{3ni} m\psi(m)/m \\ &\geq r_1 C \sum_{i=1}^{\infty} \sum_{m=2ni}^{3ni} \frac{1}{m} = \infty, \end{aligned}$$

which contradicts with (8.6.6). Therefore from (8.6.15) and (8.6.11) we obtain that for any $\varepsilon > 0$

$$\begin{aligned} &\liminf_{n\to\infty} \inf_{k\leq n-1} P\{S_n - S_k \geq -2\varepsilon H(n)\} \\ &\qquad \geq \liminf_{n\to\infty} \inf_{k\leq n-1} P\{S_n \geq -\varepsilon H(n), S_k < \varepsilon H(n)\} \\ &\qquad \geq \liminf_{n\to\infty} \inf_{k\leq n-1} (P\{S_n \geq -\varepsilon H(n)\} - P\{S_k \geq \varepsilon H(n)\}) \\ &\qquad > \varphi^*(1). \end{aligned}$$

Lemma 8.6.2 is proved.

**Lemma 8.6.3.** *Let $\{X_n\}$ be a sequence of random variables, $H(x)$ satisfy (8.6.4) and $n\psi(n) > [n/2]\psi([n/2])$, $n = 1, 2, \cdots$. Then (8.6.7) $\Longrightarrow$ (8.6.8′).*

**Proof.** Noting that $n\psi(n) \uparrow$ and $H(n)/H(2n) \geq \delta$ we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n\psi(n) - [n/2]\psi([n/2])}{n} P\Big\{\sup_{i\geq n} \frac{S_i}{H(i)} \geq \varepsilon\Big\} \\ &\quad \leq \sum_{j=1}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \frac{n\psi(n) - [n/2]\psi([n/2])}{n} P\Big\{\sup_{i\geq 2^{j-1}} \frac{S_i}{H(i)} \geq \varepsilon\Big\} \end{aligned}$$

$$\le \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{n=2^j}^{2^{j+1}-1} (n\psi(n) - [n/2]\psi([n/2]))$$
$$\cdot \sum_{k=j}^{\infty} P\Big\{ \sup_{2^{k-1}\le i<2^k} \frac{S_i}{H(i)} \ge \varepsilon \Big\}$$
$$\le \sum_{j=1}^{\infty} \Big( \frac{1}{2^j} \sum_{n=2^j}^{2^{j+1}-1} n\psi(n) - \frac{1}{2^{j-1}} \sum_{n=2^{j-1}}^{2^j-1} n\psi(n) \Big)$$
$$\cdot \sum_{k=j}^{\infty} P\Big\{ \sup_{i\le 2^k} S_i \ge \varepsilon H(2^{k-1}) \Big\}$$
$$\le \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{n=2^k}^{2^{k+1}-1} n\psi(n) P\Big\{ \sup_{i\le 2^k} S_i \ge \varepsilon H(2^{k-1}) \Big\}$$
$$\le 2 \sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \psi(n) P\Big\{ \sup_{i\le 2^k} S_i \ge \varepsilon\delta^2 H(2^{k+1}) \Big\}$$
$$\le 2 \sum_{n=2}^{\infty} \psi(n) P\Big\{ \sup_{i\le n} S_i \ge \varepsilon\delta^2 H(n) \Big\}.$$

By the arbitrariness of $\varepsilon$ and (8.6.7) one gets Lemma 8.6.3.

**Lemma 8.6.4.** *Under the conditions of Theorem 8.6.1, we have (8.6.8′) $\Longrightarrow$ (8.6.7).*

**Proof.** For any given $\varepsilon > 0$, put $\varepsilon_1 = \max\{\varepsilon, C\varepsilon\}$. From (8.6.13) we have

$$\begin{aligned}(1+C^{-1}) & P\Big\{ \sup_{i\ge 2^j} S_i/H(i) \ge \varepsilon \Big\} \\ & \ge P\Big\{ \sup_{i\ge 2^j} S_i/H(i) \ge \varepsilon \Big\} + P\Big\{ \inf_{i\ge 2^j} S_i/H(i) \le -C\varepsilon \Big\} \\ & \ge P\Big\{ \sup_{i\ge 2^j} |S_i|/H(i) \ge \varepsilon_1 \Big\}. \end{aligned} \tag{8.6.16}$$

By (8.6.4), there exists a $\delta, 0 < \delta < 1$, such that for every $x \ge 1$

$$\delta H(2x) \le H(x). \tag{8.6.17}$$

Hence, we have

$$\begin{aligned}
&\Big\{\sup_{i\geq 2^j}|S_i|/H(i)\geq\varepsilon_1\Big\}\\
&\quad=\bigcup_{i=j}^{\infty}\bigcup_{k=2^i}^{2^{i+1}-1}\{|S_k|\geq\varepsilon_1 H(k)\}\\
&\quad\supset\bigcup_{i=j}^{\infty}\bigcup_{k=2^i}^{2^{i+1}-1}\Big\{|S_k|\geq\frac{\varepsilon_1}{\delta}H(2^i)\Big\}\bigcup_{i=j}^{\infty}\Big\{|S_{2^i}|\geq\frac{\varepsilon_1}{\delta}H(2^i)\Big\}\\
&\quad\supset\bigcup_{i=j}^{\infty}\bigcup_{k=2^i}^{2^{i+1}-1}\Big\{|S_k-S_{2^i}|\geq\frac{2\varepsilon_1}{\delta}H(2^i)\Big\}\\
&\quad=\Big\{\sup_{i\geq j}\max_{2^i\leq k<2^{i+1}}|S_k-S_{2^i}|\geq\frac{2\varepsilon_1}{\delta}H(2^i)\Big\},
\end{aligned}$$

Combining it with (8.6.16), we obtain

$$\begin{aligned}
&P\Big\{\sup_{i\geq 2^j}S_i/H(i)\geq\varepsilon_1\delta\Big\}\\
&\quad\geq(1+c^{-1})^{-1}P\Big\{\sup_{i\geq j}\max_{2^i\leq k<2^{i+1}}|S_k-S_{2^i}|/H(2^i)\geq 2\varepsilon_1\Big\}.
\end{aligned}\tag{8.6.18}$$

Denote

$$\begin{aligned}
B_i&=\Big\{\max_{2^i\leq k<2^{i+1}}\frac{S_k-S_{2^i}}{H(2^i)}\geq 2\varepsilon_1\Big\},\\
A_i&=\bigcap_{l=i+1}^{\infty}B_l^c,\\
B_i^*&=\Big\{\max_{2^i\leq k<2^{i+1}-2^{i-1}}\frac{S_k-S_{2^i}}{H(2^i)}\geq 2\varepsilon_1\Big\}.
\end{aligned}$$

It is easy to check that any mixing sequence obeys the 0-1 law. Since $\{B_i, i.o.\}\in\bigcap_{n=1}^{\infty}\sigma(X_k, k\geq n)$, we have $P(B_i, i.o.)=1$ or 0. If $P(B_i, i.o.)=1$, then

$$P\Big\{\sup_{i\geq j}\max_{2^i\leq k\leq 2^{i+1}}\frac{S_k-S_{2^i}}{H(2^i)}\geq 2\varepsilon_1\Big\}=P\Big\{\bigcup_{i=j}^{\infty}B_i\Big\}=1,\quad j=1,2,\cdots.$$

By this equality, (8.6.18) and the property of $M_1^*$, we have

$$\begin{aligned}S(\psi^*,H,\varepsilon_0) &\ge \sum_{j=2}^{\infty} 2^{-j}\Big(\sum_{n=2^{j-1}}^{2^j-1} n\psi(n) \\ &\quad - 2\sum_{n=2^{j-2}}^{2^{j-1}-1} n\psi(n)\Big)P\Big\{\sup_{i\ge 2^j}\frac{S_i}{H(i)}\ge\varepsilon_0\Big\} \\ &\ge c\sum_{j=2}^{\infty}\Big(2^{-j}\sum_{n=2^{j-1}}^{2^j-1} n\psi(n) - 2^{-(j-1)}\sum_{n=2^{j-2}}^{2^{j-1}-1} n\psi(n)\Big) \\ &= c\lim_{N\to\infty}\Big(2^{-N}\sum_{n=2^{N-1}}^{2^N-1} n\psi(n) - \frac{1}{2}\psi(1)\Big) = \infty. \qquad (8.6.19)\end{aligned}$$

In fact, for any given $M > 0$ there exists an $N_0$, when $N \ge N_0, n \ge 2^{N_0-1}$, we have $n\psi(n) \ge 2M$, so that $2^{-N}\sum_{n=2^{N-1}}^{2^N-1} n\psi(n) \ge M$. (8.6.19) contradicts with (8.6.8′). Therefore we have

$$P(B_i,\ i.o.) = 0,$$

i.e. $P\Big(\bigcup_{l=i+1}^{\infty} B_l\Big) \to 0$, as $i\to\infty$, so that $P(A_i)\to 1$ as $i\to\infty$.

Combining it with (8.6.17), stationarity and the definition of $\rho$-mixing, for large $j$ we have

$$\begin{aligned}P\Big\{&\sup_{i\ge j}\max_{2^i\le k\le 2^{i+1}}\frac{S_k - S_{2^i}}{H(2^i)}\ge 2\varepsilon_1\Big\} \\ &= P\Big\{\bigcup_{i=j}^{\infty} B_i\Big\} \\ &\ge P\Big\{\bigcup_{i=j}^{\infty} B_iA_i\Big\} = \sum_{i=j}^{\infty} P(B_iA_i) \ge \sum_{i=j}^{\infty} P(B_i^*A_i) \\ &\ge \sum_{i=j}^{\infty}\{P(B_i^*)P(A_i) - \rho(2^{i-1})\} \\ &\ge c\sum_{i=j}^{\infty}\{P(B_i^*) - c\rho(2^{i-1})\} \\ &\ge c\sum_{i=j}^{\infty}\Big\{P\Big\{\max_{k\le 2^{i-1}} S_k \ge \frac{2\varepsilon_1}{\delta}H(2^{i-1})\Big\} - c\rho(2^{i-1})\Big\}. \qquad (8.6.20)\end{aligned}$$

Since $n\psi(n) \ge [n/2]\psi([n/2])$, $\sum_{2^{j-1}}^{2^j-1} n\psi(n) \ge \sum_{2^{j-2}}^{2^{j-1}-1} n\psi(n)$, combining

it with (8.6.18), (8.6.20) and (8.6.12) for any given $\varepsilon > 0$, we have

$$
\begin{aligned}
\infty &> S(\psi^*, H, \varepsilon\delta) \\
&\geq c\sum_{j=2}^{\infty} 2^{-j}\Big(\sum_{n=2^{j-1}}^{2^j-1} n\psi(n) - 2\sum_{n=2^{j-2}}^{2^{j-1}-1} n\psi(n)\Big) \\
&\quad \cdot P\Big\{\sup_{i\geq j}\max_{2^i\leq k<2^{i+1}} \frac{S_k}{H(k)} \geq \varepsilon_1\delta\Big\} \\
&\geq c\sum_{j=2}^{\infty} 2^{-j}\Big(\sum_{n=2^{j-1}}^{2^j-1} n\psi(n) - 2\sum_{n=2^{j-2}}^{2^{j-1}-1} n\psi(n)\Big) \\
&\quad \cdot P\Big\{\sup_{i\geq j}\max_{2^i\leq k<2^{i+1}} \frac{S_k - S_{2^i}}{H(2^i)} \geq 2\varepsilon_1\Big\} \\
&\geq c\sum_{i=2}^{\infty}\sum_{j=2}^{i}\Big(2^{-j}\sum_{n=2^{j-1}}^{2^j-1} n\psi(n) \\
&\quad - 2^{-(j-1)}\sum_{n=2^{j-2}}^{2^{j-1}-1} n\psi(n)\Big)P(B_i^* A_i) \\
&\geq c\sum_{i=2}^{\infty} 2^{-i}\sum_{n=2^{i-2}}^{2^{i-1}-1} n\psi(n)P(B_i^* A_i) \\
&\geq c\sum_{i=2}^{\infty}\sum_{n=2^{i-2}}^{2^{i-1}-1} \psi(n)\Big(P\Big\{\max_{k\leq 2^{i-1}} S_k \geq \frac{2\varepsilon_1}{\delta}H(2^{i-1})\Big\} - c\rho(2^{i-1})\Big) \\
&\geq c\sum_{n=1}^{\infty} \psi(n)P\Big\{\max_{k\leq n} S_k \geq \frac{2\varepsilon_1}{\delta}H(n)\Big\} - c\sum_{n=1}^{\infty}\psi(n)\rho(n) \\
&= cM(\psi, H, 2\varepsilon_1/\delta) - c\sum_{n=1}^{\infty}\psi(n)\rho(n).
\end{aligned}
$$

Lemma 8.6.4 is proved.

Theorem 8.6.1 follows from Lemmas 8.6.1 – 8.6.4.

**Remark 8.6.1.** The condition (8.6.13) is only used in the proof of Lemma 8.6.4. It can be removed when one considers the convergence of two-sided tail probability series.

Further discussion of Prohorov's problem for a sequence of independent random variables were given by Su (1989) and Shao (1991), etc.

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# Chapter 9 Strong Approximations

The law of the iterated logarithm and the strong invariance principle for a sequence of mixing random variables have been discussed by some mathematicians in the sixties. Applying Strassen's martingale embedding method to strong approximations for partial sums of dependent random variables by a Wiener process, Philipp and Stout (1975) established the almost sure invariance principle for $\alpha$-mixing sequences, $\varphi$-mixing sequences, lacunary trigonometric series, a class of Gaussian sequences and additive functional of Markov chains. The results for strictly stationary $\varphi$-mixing and $\rho$-mixing sequences have been improved by Berkes and Philipp (1979), Dabrowski (1982) and Bradley (1985) respectively. These results are improved essentially by Lu and Shao. In Shao and Lu (1987), they got a better approximation order for a $\varphi$-mixing sequence. This will be introduced in Section 9.1. For a sequence of stationary $\varphi$-mixing random variables $\{X_n,\ n \geq 1\}$ with $EX_1 = 0$, $EX_1^2 < \infty$ and $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$, Heyde and Scott (1973) first gave that the order of strong approximation is $o((n \log\log n)^{1/2})$. Shao (1989a, 1993b) improved this result and gave that the same order of strong approximations for $\varphi$-mixing and $\rho$-mixing sequences when the rate of mixing is $O((\log n)^{-1-\varepsilon})$. These results imply the law of the iterated logarithm. When the $(2+\delta)$-th moment is finite, he gives the further results which imply the Chung law of the iterated logarithm. We shall discuss these results for the $\varphi$-mixing and $\rho$-mixing cases in Section 9.2. Shao and Lu(1987) and Shao (1989a) have also studied the strong approximations for an $\alpha$-mixing sequence. These will be given in Section 9.3.

## 9.1 Strong approximations for a $\varphi$-mixing sequence

Let $\{X_n,\ n \geq 1\}$ be a sequence of $\varphi$-mixing random variables with $EX_n = 0$. Put $S_n = \sum_{k=1}^{n} X_k$, $S(t) = S_{[t]}$ $(t > 0)$. In this section, we first discuss how fast is the rate of strong approximations for a sequence of $\varphi$-mixing random variables when applying Strassen's martingale embedding

method. The rate of strong approximations gotten by Shao and Lu (1987) approaches the ideal order $O((n\log\log n)^{1/4}(\log n)^{1/2})$ for a sequence of i.i.d.r.v.'s. The following theorem is proved.

**Theorem 9.1.1.** (Shao, Lu 1987) *Let $\{X_n,\ n \ge 1\}$ be a sequence of $\varphi$-mixing random variables with $EX_n = 0$. Suppose that*

*(i)* $\sigma_n^2 = ES_n^2 \ge Cn$ *for some* $C > 0$,

*(ii)* $\sup_n EX_n^4 < \infty$,

*(iii)* $\varphi(n) = O(1/n)$.

*Then without changing the distribution of $\{S(t),\ t \ge 0\}$, we can redefine the process $\{S(t),\ t \ge 0\}$ on a richer probability space together with a standard Wiener process $\{W(t),\ t \ge 0\}$ such that for any $\varepsilon > 0$*

$$S(t) - W(\sigma_t^2) = O(\sigma_t^{1/2}(\log\sigma_t)^{9/4+\varepsilon}) \quad a.s. \tag{9.1.1}$$

*as $t \to \infty$, where $\sigma_t^2 = \sigma_{[t]}^2$.*

From Theorem 9.1.1 we have the following corollary immediately.

**Corollary 9.1.1.** *Let $\{X_n,\ n \ge 1\}$ be a sequence of stationary $\varphi$-mixing random variables with $EX_1 = 0$. If*

$$\sigma^2 = EX_1^2 + 2\sum_{k=2}^{\infty} EX_1X_k \tag{9.1.2}$$

*is absolutely convergent, by assuming $\sigma = 1$ without loss of generality, then under the conditions (ii) and (iii), we have*

$$S(t) - W(t) = O(t^{1/4}(\log t)^{9/4+\varepsilon}) \quad a.s. \tag{9.1.3}$$

In order to prove Theorem 9.1.1, we first give a fundamental proposition. The proof of the proposition points out how to use the Strassen's martingale embedding method for a mixing sequence to get the results of strong approximations.

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\{\mathcal{F}_n, n \ge 0\}$ be a sequence of non-decreasing $\sigma$-fields. Assume that $X_n$ is $\mathcal{F}_n$-measurable, $n = 1, 2, \cdots$. Define

$$\begin{aligned} Y_n &= \sum_{k=0}^{2^n}\{E(X_{n+k}|\,\mathcal{F}_n) - E(X_{n+k}|\mathcal{F}_{n-1})\} \\ &= X_n + u_n - u_{n-1} - v_n \end{aligned} \tag{9.1.4}$$

for every $n \geq 1$, where

$$u_n = \sum_{k=1}^{2^n} E(X_{n+k} | \mathcal{F}_n), \quad v_n = \sum_{k=2^{n-1}}^{2^n} E(X_{n+k} | \mathcal{F}_{n-1}). \tag{9.1.5}$$

$\{Y_n, \mathcal{F}_n, n \geq 1\}$ is a martingale difference sequence.

**Proposition 9.1.1.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $EX_n = 0$, $\sup_n E|X_n|^{2+\delta} < \infty\,(0 < \delta \leq 2)$. Let $\mathcal{F}_n = \sigma\{X_k, 1 \leq k \leq n\}$ be a natural $\sigma$-field sequence. If the following conditions are satisfied:*

*(a)* $\sigma_n^2 = ES_n^2 \geq Cn$ *for some* $C > 0$,

*(b)* $\|u_n\|_{2+\delta} = O(1)$, $\sum_{n=1}^{\infty} \|v_n\|_{2+\delta} < \infty$,

*(c)* *for some* $\lambda \geq 0$

$$E|\sum_{m<k\leq m+n} (Y_k^2 - EY_k^2)|^{(2+\delta)/2} = O(n(\log n)^{\lambda}), \tag{9.1.6}$$

*then without changing the distribution of $\{S(t), t \geq 0\}$, we can redefine the process $\{S(t), t \geq 0\}$ on a richer probability space together with a standard Wiener process $\{W(t), t \geq 0\}$ such that for any $\varepsilon > 0$*

$$S(t) - W(\sigma_t^2) = O\Big(\sigma_t^{2/(2+\delta)} (\log \sigma_t)^{1+\varepsilon+(1+\lambda)/(2+\delta)}\Big) \quad a.s. \tag{9.1.7}$$

*as $t \to \infty$. Particularly, for $\delta = 2$ we have*

$$S(t) - W(\sigma_t^2) = O\Big(\sigma_t^{1/2} (\log \sigma_t)^{(5+\lambda)/4+\varepsilon}\Big) \quad a.s. \tag{9.1.8}$$

**Proof.** 1) We first prove that for each $\varepsilon > 0$,

$$S(t) - \sum_{k\leq t} Y_k = O(t^{1/(2+\delta)} (\log t)^{(1+\varepsilon)/(2+\delta)}\Big) \quad \text{a.s.} \tag{9.1.9}$$

In fact, noting that

$$S(t) - \sum_{k\leq t} Y_k = \sum_{k\leq t} v_k - u_{[t]},$$

using condition $(b)$ and the Borel-Cantelli lemma, we conclude that (9.1.9) holds.

We now apply a martingale version of the Skorohod-Strassen representation theorem. There exists a probability space, on which a standard

Wiener process and a sequence of non-negative stopping time $T_n$ are defined such that

$$\Big\{W\Big(\sum_{j\le n} T_j\Big),\, n\ge 1\Big\} \quad \text{and} \quad \Big\{\sum_{j\le n} Y_j,\, n\ge 1\Big\}$$

have the same distribution. Hence without loss of generality, on this new probability space we can redefine $Y_n$ by

$$Y_n = W\Big(\sum_{j\le n} T_j\Big) - W\Big(\sum_{j<n} T_j\Big),$$

and keep the same notation. Now, write $\mathcal{F}_n = \sigma\{W(\sum_{j\le k} T_j);\ 1\le k\le n\}$, and $\mathcal{G}_n = \sigma\{W(t);\ 0\le t\le \sum_{j\le n} T_j\}$. It is easy to see that $\mathcal{F}_n \subseteq \mathcal{G}_n$, $T_n$ is $\mathcal{G}_n$-measurable and for every $n\ge 1$,

$$E(T_n|\mathcal{G}_{n-1}) = E(Y_n^2|\mathcal{G}_{n-1}) = E(Y_n^2|\mathcal{F}_{n-1}) \quad \text{a.s.} \tag{9.1.10}$$

Moreover, for each $1<p\le 2$,

$$E|T_n|^p \le E|Y_n|^{2p}.$$

2) Write

$$S(t) - W(\sigma_t^2) = S(t) - \sum_{k\le t} Y_k + W\Big(\sum_{k\le t} T_k\Big) - W(\sigma_t^2). \tag{9.1.11}$$

In order to estimate the second difference of the right hand side of (9.1.11), we have to estimate $\sum_{k\le t} T_k - \sigma_t^2$. Write

$$\begin{aligned}\sum_{k\le t} T_k - \sigma_t^2 \\ &= \sum_{k\le t}\{T_k - E(T_k|\mathcal{G}_{k-1})\} \\ &\quad + \sum_{k\le t}\{E(Y_k^2|\mathcal{F}_{k-1}) - Y_k^2\} + \Big\{\sum_{k\le t} Y_k^2 - \sigma_t^2\Big\}.\end{aligned} \tag{9.1.12}$$

We shall show that

$$\sum_{k\le t} T_k - \sigma_t^2 = O(\sigma_t^{4/(2+\delta)}(\log\sigma_t)^{1+\varepsilon+2(1+\lambda)/(2+\delta)}) \quad \text{a.s.} \tag{9.1.13}$$

Put $R_j = Y_j^2 - E(Y_j^2|\mathcal{F}_{j-1})$. Then $\{R_j, \mathcal{F}_j\}$ is a martingale difference sequence. We obtain from condition $(b)$ that

$$E|R_j|^{(2+\delta)/2} \le 16E|Y_j|^{2+\delta} = O(1).$$

Hence by the fundamental theorem on the martingale (cf. Chow 1965), we have

$$\sum_{k\le t} R_k = O(t^{2/(2+\delta)}(\log t)^{(2+\varepsilon)/(2+\delta)}) \quad \text{a.s.} \tag{9.1.14}$$

Similarly, we have

$$\sum_{k\le t}(T_k - E(T_k|\mathcal{G}_{k-1})) = O(t^{2/(2+\delta)}(\log t)^{(2+\varepsilon)/(2+\delta)}) \quad \text{a.s.} \tag{9.1.15}$$

For the third term of the right hand side of (9.1.12), by the condition $(c)$, Móricz's theorem and the Borel-Cantelli lemma, it is easy to show

$$\sum_{k\le t}(Y_k^2 - EY_k^2) = O(t^{2/(2+\delta)}(\log t)^{1+\varepsilon+2(1+\lambda)/(2+\delta)}) \quad \text{a.s.} \tag{9.1.16}$$

Noting that $\{Y_k, \mathcal{F}_k\}$ is a martingale difference sequence, from the conditions $(a), (b)$ and the Schwarz inequality, we obtain

$$\begin{aligned}\sum_{k\le t} EY_k^2 - \sigma_t^2 &= E\Big(\sum_{k\le t} Y_k\Big)^2 - \sigma_t^2 \\ &= 2E\Big(\sum_{k\le t} X_k\Big)\Big(u_t - \sum_{k\le t} v_k\Big) + E\Big(u_t - \sum_{k\le t} v_k\Big)^2 \\ &= O(\sigma_t).\end{aligned} \tag{9.1.17}$$

Hence by (9.1.16), (9.1.17), and condition $(a)$, we have

$$\sum_{k\le t} Y_k^2 - \sigma_t^2 = O(\sigma_t^{4/(2+\delta)}(\log \sigma_t)^{1+\varepsilon+2(1+\lambda)/(2+\delta)}) \quad \text{a.s.} \tag{9.1.18}$$

Equality (9.1.13) follows from (9.1.14), (9.1.15) and (9.1.18).

We now conclude

$$W\Big(\sum_{k\le t} T_k\Big) - W(\sigma_t^2) = O(\sigma_t^{2/(2+\delta)}(\log \sigma_t)^{1+\varepsilon+(1+\lambda)/(2+\delta)}) \quad \text{a.s.}$$

by (9.1.13) and the proof of Theorem 3.2B in Hanson and Russo (1983). Combining (9.1.11) with (9.1.9) yields the proposition.

**Proof of Theorem 9.1.1.** We now verify that the conditions $(b)$ *and* $(c)$ of proposition 9.1.1 are satisfied. By Lemma 1.2.8 and Lemma

2.2.8, we have

$$
\begin{aligned}
\|u_n\|_{2+\delta}^{2+\delta} &= E|u_n|^{2+\delta} = E(u_n(\mathrm{sgn} u_n)|u_n|^{1+\delta}) \\
&= E(X_{n+1}|u_n|^{1+\delta}\mathrm{sgn} u_n) + \sum_{i=0}^{n-1} E\Big(\sum_{k=2^i+1}^{2^{i+1}} X_{n+k}|u_n|^{1+\delta}\mathrm{sgn} u_n\Big) \\
&= O\Big(\|u_n\|_{2+\delta}^{1+\delta} + \sum_{i=0}^{n-1} \varphi^{(1+\delta)/(2+\delta)}(2^i)2^{i/2}\|u_n\|_{2+\delta}^{1+\delta}\Big) \\
&= O(\|u_n\|_{2+\delta}^{1+\delta}),
\end{aligned}
$$

which implies that

$$\|u_n\|_{2+\delta} = O(1). \tag{9.1.19}$$

Similarly, we have

$$\|v_n\|_{2+\delta} = O(\varphi^{(1+\delta)/(2+\delta)}(2^{n-1})2^{(n-1)/2}).$$

Hence

$$\sum_{n=1}^{\infty} \|v_n\|_{2+\delta} < \infty. \tag{9.1.20}$$

Therefore condition $(b)$ is satisfied.

For condition $(c)$, put

$$T_m(n) = \sum_{m<k\le m+n} (Y_k^2 - EY_k^2), \quad \tau_n = \sup_m \|T_m(n)\|_2. \tag{9.1.21}$$

It is easy to obtain

$$\|T_m(n)\|_2 \le \|T_m([n/2]) + T_{m+[n/2]+l}([n/2])\|_2 + 2\tau_l + 2\tau_1, \tag{9.1.22}$$

where $l = [2n(\log(2n))^{-2-\varepsilon}]$. Note that

$$
\begin{aligned}
&E(T_m([n/2]) + T_{m+[n/2]+l}([n/2]))^2 \\
&\quad \le 2\tau_{[n/2]}^2 + 2E\{T_m([n/2])T_{m+[n/2]+l}([n/2])\}.
\end{aligned} \tag{9.1.23}
$$

By an elementary calculation, from (9.1.4) and Lemma 2.2.8 we obtain

$$
\begin{aligned}
&ET_m([n/2])\,T_{m+[n/2]+l}([n/2]) \\
&\quad \le 2\varphi^{1/2}(l)\,\|T_m([n/2])\|_2\Big\|\sum_k X_k\Big\|_4^2 \\
&\qquad + \|T_n([n/2])\|_2\Big(\|u_{N_1}\|_4 + \|u_{N_2}\|_4 + \sum_k \|v_k\|_4\Big)\Big\|\sum_k X_k\Big\|_4 \\
&\qquad + \|T_m([n/2])\|_2\Big(\|u_{N_1}\|_4^2 + \|u_{N_2}\|_4^2 + \sum_k \|v_k\|_4^2\Big),
\end{aligned}
$$

where $\sum_k$ is extended over $m+[n/2]+l<k\le m+2[n/2]+l$ and

$$N_2=m+[n/2]+l,\quad N_1=m+2[n/2]+l.$$

Using Lemma 2.2.8, (9.1.12), (9.1.13) and conditions (i) , (ii), we conclude that there exists a constant $C$ such that for each $m,n\ge 1$,

$$ET_m([n/2])T_{m+[n/2]+l}([n/2])\le C\tau([n/2])\,n^{1/2}\,(\log n)^{1+\varepsilon}. \tag{9.1.24}$$

From (9.1.22), (9.1.23) and (9.1.24), we obtain

$$\begin{aligned}\|T_m(n)\|_2\le 2^{1/2}\tau([n/2])+C\,n^{1/2}\,(\log n)^{1+\varepsilon}\\ +\tau([2n(\log n)^{-2-\varepsilon}]).\end{aligned} \tag{9.1.25}$$

Finally, from (9.1.25) and by induction we have

$$\tau_n\le C_0n^{1/2}(\log n)^{2+\varepsilon},$$

with $C_0=\max(\exp(2^{2/\varepsilon}),2C)$. This shows that condition $(c)$ holds for $\lambda=4+\varepsilon$, and Theorem 9.1.1 follows.

**Remark 9.1.1.** Let $\{X_n;\,n\ge 1\}$ be a $\varphi$-mixing sequence with $EX_n=0$. Suppose that condition $(i)$ of Theorem 9.1.1 is satisfied and for some $0<\delta\le 2$,

$$\sup_n E|X_n|^{2+\delta}<\infty$$

and $\varphi(n)$ goes to zero with the polynomial rate. Shao and Lu (1987) have also given the following results:

1) If $0<\delta<2$ and $\varphi(n)=O(n^{-\alpha})$ for some $\alpha>1$, then

$$S(t)-W(\sigma_t^2)=O(\sigma_t^{2/(2+\delta)}(\log\sigma_t)^{1+\varepsilon+(1+\lambda)/(2+\delta)})\quad\text{a.s.}$$

for any $\varepsilon>0$, where $\lambda=2\log 3/\log\theta^{-1},\theta=1-2(\alpha-1)/(\alpha(2+\delta))>0$.

2) If $0<\delta\le 2$ and $\varphi(n)=O(n^{-\alpha}),(2+\delta)/(2(1+\delta))<\alpha\le 1$, then

$$S(t)-W(\sigma_t^2)=O(\sigma_t^{1-\alpha\delta/(2+\delta)+\varepsilon})\quad\text{a.s.}$$

for any $\varepsilon>0$.

## 9.2 Strong approximation for a $\rho$-mixing sequence

In the introduction of this chapter, we have mentioned an interesting problem: what the rate of strong approximation of partial sum process $S(t)$

for a mixing sequence by a Wiener process is when we only assume that $\sup_n EX_n^2 < \infty$ and $\varphi(n) = O((\log n)^{-a})$, $a > 1$ (or $\rho(n) = O((\log n)^{-a})$, $a > 1$) ? In this section, we deal with a $\rho$-mixing sequence, and the strong approximation order $O((n \log\log n)^{1/2})$ is given. With this result the law of the iterated logarithm holds true for this $\rho$-mixing sequence. In order to get the Chung law of iterated logarithm for a $\rho$-mixing sequence, we need the assumption $\sup_n E|X_n|^{2+\delta} < \infty$.

**Theorem 9.2.1.**(Shao 1989a, 1993b) *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence with $EX_1 = 0$, $EX_1^2 < \infty$. Assume that*

*(i)* $\sigma_n^2 = ES_n^2 \to \infty$ as $n \to \infty$,

*(ii)* $\rho(n) = O((\log n)^{-1-\varepsilon'})$ for some $\varepsilon' > 0$.

*Then without changing the distribution of $\{S(t), t \geq 0\}$, we can redefine the process $\{S(t), t \geq 0\}$ on a richer probability space together with the standard Wiener process $\{W(t), t \geq 0\}$ such that*

$$S(t) - W(\sigma_t^2) = o(\sigma_t(\log\log t)^{1/2}) \quad a.s. \tag{9.2.1}$$

*as $t \to \infty$.*

**Theorem 9.2.2.**(Shao 1989a, 1993b) *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\varphi$-mixing sequence with $EX_1 = 0, EX_1^2 < \infty$. Assume that*

*(i)'* $\sigma_n^2 \to \infty$ *as $n \to \infty$,*

*(ii)* $\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty$. *Then (9.2.1) is also true.*

With the help of the law of the iterated logarithm for a Wiener process, from the previous theorems we have the following corollary.

**Corollary 9.2.1.** *Under the conditions of Theorem 9.2.1 or 9.2.2, we have*

$$\limsup_{n\to\infty} |S_n|/\sqrt{2\sigma_n^2 \log\log \sigma_n^2} = 1 \quad a.s. \tag{9.2.2}$$

The following strong approximation theorem implies the Chung law of the iterated logarithm.

**Theorem 9.2.3.**(Shao 1989a, 1993b) *Let $\{X_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence with $EX_1 = 0$, $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. Assume that*

*(i)* $\sigma_n^2 \to \infty$, as $n \to \infty$,

*(ii)* $\rho(n) = O((\log n)^{-r})$ *for some $r > 1/2$.*

*Then for any* $0 < \theta < r/2 - 1/4$, *we have*

$$S(t) - W(\sigma_t^2) = o(\sigma_t(\log t)^{-\theta}) \quad a.s. \tag{9.2.3}$$

**Corollary 9.2.2.** *Under the conditons of Theorem 9.2.3, we have (9.2.2) and*

$$\liminf_{N\to\infty}\Big(\frac{8\log\log N}{\pi^2\sigma_N^2}\Big)^{1/2}\max_{1\le n\le N}|S_n| = 1 \quad a.s. \tag{9.2.4}$$

**Remark 9.2.1.** The results of Theorem 9.2.3 and Corollary 9.2.2 hold true for a stationary $\varphi$-mixing sequence with $EX_1 = 0, E|X_1|^{2+\delta} < \infty$, $\sigma_n^2 \to \infty$ and $\varphi(n) = O((\log n)^{-r})$ for some $r > (2+\delta)/(2(1+\delta))$.

**Remark 9.2.2.** The stationarity condition in these theorems and corollaries can been replaced by the identical distribution condition.

We point out in passing that the results in this section improve those of Bradley(1985), Dabrowski(1982) and Theorem 4 of Berkes and Philipp(1979).

In the proof of these theorems , we shall use both the Bernstein divided-section method and the Strassen martingale embedding method. Put

$$\overline{X}_k = X_k I(X_k^2 \ge k) - EX_k I(X_k^2 \ge k), \tag{9.2.5}$$

$$\widehat{X}_k = X_k I(X_k^2 < k) - EX_k I(X_k^2 < k). \tag{9.2.6}$$

Let

$$\overline{S}(n) = \sum_{k=1}^{n}\overline{X}_k, \qquad \widehat{S}(n) = \sum_{k=1}^{n}\widehat{X}_k,$$

$$\overline{S}_k(n) = \sum_{i=k+1}^{k+n}\overline{X}_i \qquad \widehat{S}_k(n) = \sum_{i=k+1}^{k+n}\widehat{X}_i.$$

We shall first prove

$$\overline{S}(n) = o(n^{1/2}) \qquad \text{a.s.} \tag{9.2.7}$$

Define blocks of integers $H_1, I_1, H_2, I_2, \cdots$ by requiring that $H_k$ contains $h_k$ and $I_k$ contains $i_k$ consecutive integers and that there are no gaps between consecutive blocks, where

$$\begin{aligned} h_k &= \text{Card } H_k = [ak^{a-1}\exp(k^a)], \\ i_k &= \text{Card } I_k = [ak^{a-1}\exp(k^a/2)] \end{aligned} \tag{9.2.8}$$

and $0 < a < 1$ will be chosen later on. Put

$$\begin{aligned}
N_k &:= \sum_{j\le k} \mathrm{Card}(H_j \cup I_j) \sim \exp(k^a),\\
u_k &:= \sum_{j\in H_k} \widehat{X}_j, \qquad v_k := \sum_{j\in I_k} \widehat{X}_j,\\
\xi_k &:= u_k - E(u_k|\,\mathcal{F}_{k-1}) \quad \text{where } \mathcal{F}_k = \sigma(X_i, i \le N_{k-1} + h_k),\\
m_n &:= \{k : n \in H_k \cup I_k\} - 1 \sim (\log n)^{1/a}.
\end{aligned}$$

It is clear that

$$S_n = \widehat{S}(n) + \overline{S}(n). \tag{9.2.9}$$

$$\begin{aligned}
\widehat{S}(n) &= \sum_{i=1}^{m_n} \xi_i + \sum_{i=1}^{m_n} E(u_i|\,\mathcal{F}_{i-1})\\
&\quad + \sum_{i=1}^{m_n} v_i + \sum_{i=N_{m_n}+1}^{n} \widehat{X}_i.
\end{aligned} \tag{9.2.10}$$

In order to prove Theorem 9.2.1, by this two representations , we shall show at first that one needs only to check (9.2.7) and to show that the following relations hold true:

$$\sum_{i=1}^{n} v_i = o(\exp(n^a/2)) \quad \text{a.s.} \tag{9.2.11}$$

$$\max_{N_n < j \le N_{n+1}} \Big| \sum_{i=N_n+1}^{j} \widehat{X}_i \Big| = o(\exp(n^a/2)) \quad \text{a.s.} \tag{9.2.12}$$

$$\sum_{i=1}^{n} E(u_i|\,\mathcal{F}_{i-1}) = o(\exp(n^a/2)) \quad \text{a.s.} \tag{9.2.13}$$

$$\sum_{i=1}^{\infty} e^{-(1+\delta_1)i^a} E|\xi_i|^{2(1+\delta_1)} < \infty \quad \text{for some } \delta_1 \in (0,1) \tag{9.2.14}$$

$$\sum_{i=1}^{n} (E(\xi_i^2|\,\mathcal{F}_{i-1}) - E\xi_i^2) = o(\exp(n^a)) \quad \text{a.s.} \tag{9.2.15}$$

$$\sum_{i=1}^{m_n} E\xi_i^2 - ES_n^2 = o(n) \quad \text{a.s.} \tag{9.2.16}$$

**Proof of Theorem 9.2.1.**

Assume that (9.2.7) and (9.2.11)–(9.2.13) hold. Denote $\tilde{\sigma}_n^2 = \sum_{i=1}^{m_n} E\xi_i^2$. Then

$$S(n) - \sum_{i=1}^{m_n} \xi_i = o(n^{1/2}) \quad \text{a.s.} \tag{9.2.17}$$

By the Strassen martingale embedding method, for the martingale difference sequence $\{\xi_k, \mathcal{F}_k, k \geq 1\}$ there exist the stopping times $\{T_n, n \geq 1\}$ such that

$$\Big\{W(\sum_{i=1}^{n} T_i),\ n \geq 1\Big\} \overset{\mathrm{d}}{=} \Big\{\sum_{i=1}^{n} \xi_i,\ n \geq 1\Big\}.$$

Redefine

$$\xi_1 = W(T_1), \quad \xi_n = W\Big(\sum_{i=1}^{n} T_i\Big) - W\Big(\sum_{i=1}^{n-1} T_i\Big), \quad n \geq 2.$$

As in (9.1.10), we have

$$E(T_n | \mathcal{G}_{n-1}) = E(\xi_n^2 | \mathcal{G}_{n-1}) = E(\xi_n^2 | \mathcal{F}_{n-1}).$$

Write

$$\sum_{i=1}^{m_n} \xi_i - W(\tilde{\sigma}_n^2) = W\Big(\sum_{i \leq m_n} T_i\Big) - W\Big(\sum_{i \leq m_n} ET_i\Big) \tag{9.2.18}$$

and

$$\begin{aligned}\sum_{i=1}^{n}(T_i - ET_i) &= \sum_{i=1}^{n}(T_i - E(T_i | \mathcal{G}_{i-1})) \\ &\quad + \sum_{i=1}^{n}(E(\xi_i^2 | \mathcal{F}_{i-1}) - \xi_i^2) + \sum_{i=1}^{n}(\xi_i^2 - E\xi_i^2).\end{aligned} \tag{9.2.19}$$

By a martingale result of Chow (1965) and (9.2.14), the series

$$\sum_{i=1}^{\infty} e^{-n^a} \xi_n^2 < \infty \quad \text{a.s.}$$

implies that the series

$$\sum_{i=1}^{\infty} e^{-n^a} (\xi_n^2 - E\xi_n^2)$$

is also almost surely convergent. It follows from the Kronecker lemma that

$$\sum_{i\le n}(\xi_i^2 - E\xi_i^2) = o(\exp(n^a)) \quad \text{a.s.}$$

By the same way, we have $\sum_{i\le n}(T_i - E(T_i|\,\mathcal{G}_{i-1})) = o(\exp(n^a))$. Hence, in combination with (9.2.15), we have

$$\sum_{i=1}^{n}(T_i - ET_i) = o(\exp(n^a)) \quad \text{a.s.}$$

Using the Hanson-Russo Theorem for the lag increments of a Wiener process (see Hanson and Russo 1983 Theorem 3.2.B) and (9.2.18), we get

$$\sum_{i=1}^{m_n}\xi_i - W(\tilde{\sigma}_n^2) = o((n\log\log n)^{1/2}) \quad \text{a.s.} \tag{9.2.20}$$

Similarly, from (9.2.16), we have $W(\tilde{\sigma}_n^2) - W(\sigma_n^2) = o((n\log\log n)^{1/2})$ a.s. Combining it with (9.2.20) and (9.2.17), the conclusion of Theorem 9.2.1 follows.

Now in order to complete the proof of Theorem 9.2.1, it is sufficient to prove (9.2.7), (9.2.11)–(9.2.16). They will be given by a series of lemmas in the following. At first, we give three lemmas for any sequence of random variables. Lemma 9.2.1 implies (9.2.7).

**Lemma 9.2.1.** *Let $\{X_n\}$ be a sequence of strictly stationary random variables, $EX_1^2 < \infty$. Then*

$$\sum_{i=1}^{n}\{|X_i|I(|X_i| \ge i^{1/2}) + E|X_i|I(|X_i| \ge i^{1/2})\} = o(n^{1/2}) \quad a.s. \tag{9.2.21}$$

**Proof.** We have

$$\begin{aligned}
&\sum_{i=1}^{\infty} i^{-1/2}E|X_i|I(X_i^2 \ge i)\\
&\le \sum_{j=1}^{\infty}\sum_{i=1}^{j} i^{-1/2}E|X_1|I(j-1 < X_1^2 \le j)\\
&\le 4\sum_{j=1}^{\infty}(j-1)^{1/2}E|X_1|I(j-1 < X_1^2 \le j)\\
&\le 4\sum_{j=1}^{\infty} EX_1^2 I(j-1 < X_1^2 \le j)\\
&= 4EX_1^2 < \infty.
\end{aligned}$$

Therefore $\sum_{i=1}^{\infty} i^{-1/2}|X_i|I(X_i^2 \geq i) < \infty$ a.s., then (9.2.21) follows from the Kronecker lemma.

**Lemma 9.2.2.** *If $E|X| < \infty$, then for any $0 < b \leq 1, \varepsilon > 0$*

$$\sum_{k=1}^{\infty} k^{b-1} e^{-\varepsilon k^b} E|X|^{1+\varepsilon} I\Big(|X| < e^{k^b}\Big) < \infty.$$

**Proof.** It is easy to see that the left hand side of the above inequality equals to

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^{b-1} e^{-\varepsilon k^b} E|X|^{1+\varepsilon} I(|X| < 1) \\
&\qquad + \sum_{k=1}^{\infty} k^{b-1} e^{-\varepsilon k^b} \sum_{l=1}^{k} E|X|^{1+\varepsilon} I\Big(e^{(l-1)^b} \leq |X| < e^{l^b}\Big) \\
&\qquad \leq c + c \sum_{l=1}^{\infty} E|X|^{1+\varepsilon} I\Big(e^{(l-1)^b} \leq |X| < e^{l^b}\Big) e^{-\varepsilon l^b} \\
&\qquad \leq cE|X| < \infty.
\end{aligned}$$

**Lemma 9.2.3.** *Let $\{c_n,\ n \geq 1\}$ be a sequence of nonincreasing positive numbers. Then, for any real sequence $\{\eta_n,\ n \geq 1\}$, we have*

$$\max_{1 \leq i \leq n} \Big|\sum_{j=1}^{i} \eta_j\Big| c_i \leq 2 \max_{1 \leq i \leq n} \Big|\sum_{j=1}^{i} c_j \eta_j\Big|. \tag{9.2.22}$$

**Proof.** Denote $D_i = \sum_{j=1}^{i} c_j \eta_j$. We have

$$\eta_j = (D_j - D_{j-1})/c_j = \sum_{i=1}^{j} \Big(\frac{1}{c_i} - \frac{1}{c_{i-1}}\Big)(D_j - D_{j-1}),$$

where $1/c_0 = 0$. Therefore

$$\begin{aligned}
\sum_{j=1}^{k} \eta_j &= \sum_{j=1}^{k} \sum_{i=1}^{j} \Big(\frac{1}{c_i} - \frac{1}{c_{i-1}}\Big)(D_j - D_{j-1}) \\
&= \sum_{i=1}^{k} \Big(\frac{1}{c_i} - \frac{1}{c_{i-1}}\Big)(D_k - D_{i-1}).
\end{aligned}$$

It follows that

$$\max_{1\le k\le n} c_k\Big|\sum_{j=1}^{k}\eta_j\Big| \le \max_{1\le k\le n}\max_{1\le i\le k}|D_k - D_{i-1}|$$
$$\le 2\max_{1\le k\le n}\Big|\sum_{j=1}^{k}c_j\eta_j\Big|. \tag{9.2.23}$$

The Lemma is proved.

**Lemma 9.2.4.** *Let $\{X_n,\ n\ge 1\}$ be a $\rho$-mixing sequence with $EX_n = 0$, $EX_n^2<\infty$. Let $u_k, v_k$ be as above. Denote $u_k(n)=\sum_{i=k+1}^{k+n}u_i$, $v_k(n)=\sum_{i=k+1}^{k+n}v_i$. Suppose that*

$$\sum_{n=1}^{\infty}\rho(2^n)<\infty.$$

*Then there exists a constant $c=c(\rho(\cdot))$ such that for any $k\ge 0, n\ge 1$*

$$Eu_k^2(n)\le c\Big(\sum_{i=k+1}^{k+n}Eu_i^2\Big), \tag{9.2.24}$$

$$Ev_k^2(n)\le c\Big(\sum_{i=k+1}^{k+n}Ev_i^2\Big). \tag{9.2.25}$$

**Proof.** We only give the proof of (9.2.24). It is obvious for $n=1$. For $n\ge 2$, denote $n_1=n-[n/2]$, $n_2=[n/2]$. Then by the definition of $\rho(\cdot)$ we have

$$Eu_k^2(n)=Eu_k^2(n_1)+Eu_{k+n_1}^2(n_2)+2Eu_k(n_1)u_{k+n_1}(n_2)$$
$$\le (Eu_k^2(n_1)+Eu_{k+n_1}^2(n_2))(1+\rho(i_{n_1})). \tag{9.2.26}$$

Let $c_1=1, c_n=c_{n_1}(1+\rho(i_{n_1}))\ (n\ge 2)$. It is easy to see that $c_n$ is non-decreasing. It follows from (9.2.26) that for any $k\ge 0$, $n\ge 1$

$$Eu_k^2(n)\le c_n\Big(\sum_{i=k+1}^{k+n}Eu_i^2\Big).$$

Now, we prove that $\{c_n\}$ is a bounded sequence. Note that

$$c_{2^m}=c_{2^{m-1}}(1+\rho(i_{2^{m-1}}))$$
$$\le c_{2^{m-1}}\exp(\rho(i_{2^{m-1}}))\le \exp\Big\{\sum_{j=0}^{m-1}\rho(i_{2^j})\Big\}.$$

From the definition (9.2.8) of $i_k$, for any large $j$, we have

$$\rho(i_{2^j}) \le \rho(2^{2^{ja}}),$$

and further

$$\sum_{j=0}^{m-1} \rho(i_{2^j}) \le \sum_{j=0}^{m-1} \rho(2^{2^{ja}}) \le \sum_{j=0}^{m-1} \rho(2^j) < \infty$$

by monotoncity of $\rho(\cdot)$. This proves $c_{2^m} \le c < \infty$ . It follows from the monotoncity of $c_n$ that $\{c_n\}$ is bounded. The lemma is proved.

The following lemma gives (9.2.11).

**Lemma 9.2.5.** *Let $\{X_n, n \ge 1\}$ be as in Theorem 9.2.1, we have*

$$\sum_{j=1}^{n} v_j = o(\exp(n^a/3)) \quad a.s. \tag{9.2.27}$$

**Proof.** Using Lemma 2.2.5 with $q = 2$ and condition (ii), we have

$$\begin{aligned} E(\sum_{j=1}^{n} v_j)^2 &\le cn \max_{1\le j\le n} Ev_j^2 \\ &\le cn^a \exp(n^a/2). \end{aligned}$$

It follows from the Borel-Cantelli lemma that (9.2.27) holds true.

The following lemma gives (9.2.12).

**Lemma 9.2.6.** *Assume that the conditions of Theorem 9.2.1 are satisfied and*

$$0 < a < \varepsilon'/(8+\varepsilon'). \tag{9.2.28}$$

*Then*

$$\max_{N_k<j\le N_{k+1}} \Big| \sum_{i=N_k}^{j} \widehat{X}_i \Big| = o(\exp(k^a/2)) \quad a.s. \tag{9.2.29}$$

**Proof.** By the Borel-Cantelli lemma, we need only to prove that for any $\varepsilon > 0$

$$\sum_{k=1}^{\infty} P\Big\{\max_{N_k<j\le N_{k+1}} \Big| \sum_{i=N_k}^{j} \widehat{X}_i \Big| \ge \varepsilon \exp(k^a/2)\Big\} < \infty. \tag{9.2.30}$$

Put $d_k = N_{k+1} - N_k \sim ak^{a-1}\exp(k^a)$, and let

$$B = k^{-a(16+\varepsilon')/\varepsilon'} \exp(k^a/2), \qquad m = [k^{-a(16+\varepsilon')/\varepsilon'} \exp(k^a)].$$

For any $\varepsilon > 0$, we have

$$\frac{48m}{B} EX_1^2 I(|X_1| \ge B) \le \varepsilon \exp(k^a/2) \tag{9.2.31}$$

for large $k$. It follows from Lemma 4.3.2 that

$$\begin{aligned}
&P\Big\{ \max_{N_k < j \le N_{k+1}} \big|\sum_{i=N_k}^{j} \widehat{X}_i\big| \ge \varepsilon \exp(k^a/2)\Big\} \\
&\le c\Big\{\exp\Big(-\frac{8+\varepsilon'}{8} k^a\Big)\big(d_k^{(8+\varepsilon')/8} \\
&\quad + d_k \log^{2+\varepsilon'/4}(2d_k) E|X_1|^{2+\varepsilon'/4} I(|X_1| \le B) \\
&\quad + d_k E|X_1|^{2+\varepsilon'/4} I(|X_1| \le N_{k+1})\big) \\
&\quad + \exp(-k^a) d_k (1 + \rho^2(m) \log^4[d_k/m])\Big\} \\
&\le c\Big\{k^{-(1-a)(8+\varepsilon')/8} + k^{-1-a} \\
&\quad + k^{a-1} \exp(-\varepsilon' k^a/8) E|X_1|^{2+\varepsilon'/4} I(|X_1|^2 \le 2e^{k^a})\Big\} \\
&\quad + c\Big\{k^{a-1-2a(1+\varepsilon')} \log^4 k\Big\}.
\end{aligned}$$

By Lemma 9.2.2 and (9.2.28), (9.2.30) holds true.

The following lemma gives (9.2.13).

**Lemma 9.2.7.** *Suppose that condition (ii) of Theorem 9.2.1 is satisfied. Then we have*

$$\sum_{k=1}^{n} E(u_k|\mathcal{F}_{k-1}) = o(\exp(n^a/2)) \quad a.s. \tag{9.2.32}$$

**Proof.** We first prove that for any $k \ge 0, n \ge 1$ and a sequence of real numbers $\{c_n\}$

$$EG_k^2(n) \le c\Big\{ \sum_{j=k+1}^{k+n} \rho^2(i(j/2))\, c_j^2\, Eu_j^2\Big\} \log^2(2n) \tag{9.2.33}$$

where $G_k(n) = \sum_{j=k+1}^{k+n} c_j\, E(u_j|\mathcal{F}_{j-1})$ and $i(x)$ is the linear interpolation function of $i_k$. From (9.2.24) , there exists a constant $c'$ such that for any $k \ge 0, n \ge 1$

$$E\Big(\sum_{i=k+1}^{k+n} c_i\, u_i\Big)^2 \le c' \Big(\sum_{i=k+1}^{k+n} c_i^2\, Eu_i^2\Big). \tag{9.2.34}$$

Put $c = 100c' \log^{-2}(3/2)$. Apply the induction on $n$. When $n = 1$

$$\begin{aligned} EG_k^2(1) &= c_{k+1}^2 E(E(u_{k+1}|\mathcal{F}_k))^2 \\ &\le c_{k+1}^2 \rho(i_k)\|u_{k+1}\|_2 \|E(u_{k+1}|\mathcal{F}_k)\|_2 \end{aligned}$$

i.e.

$$EG_k^2(1) \le c_{k+1}\rho^2(i_k)Eu_{k+1}^2.$$

This proves (9.2.33) for $n = 1$. Suppose that (9.2.33) holds true for any integer less than $n$. We prove that (9.2.33) remains true for $n$. Denote $n_1 = n - [n/2]$, $n_2 = [n/2]$. We have

$$\begin{aligned} EG_k^2(n) &= EG_k^2(n_1) + EG_{k+n_1}^2(n_2) + 2EG_k(n_1)G_{k+n_1}(n_2) \\ &= EG_k^2(n_1) + EG_{k+n_1}^2(n_2) + 2EG_k(n_1)\Big(\sum_{j=k+n_1+1}^{k+n} c_j u_j\Big) \\ &\le EG_k^2(n_1) + EG_{k+n_1}^2(n_2) \\ &\quad + 2\rho(i_{k+n_1})\| G_k(n_1)\|_2 \Big\| \sum_{j=k+n_1+1}^{k+n} c_j u_j\Big\|_2. \end{aligned}$$

By (9.2.34) and the assumption of induction, we have

$$\begin{aligned} EG_k^2(n) &\le c\Big\{ \sum_{j=k+1}^{k+n} \rho^2(i(j/2))c_j^2 Eu_j^2\Big\} \log^2(2n_1) \\ &\quad + 2\sqrt{cc'}\rho(i_{k+n_1})\Big\{ \sum_{j=k+n_1+1}^{k+n} c_j^2 Eu_j^2\Big\}^{1/2} \\ &\quad \cdot \Big\{ \sum_{j=k+1}^{k+n_1} \rho^2(i(j/2))c_j^2 Eu_j^2\Big\}^{1/2} \log(2n_1) \\ &\le (c\,(\log 2n_1)^2 + \sqrt{cc'}\log(2n_1))\Big\{ \sum_{j=k+1}^{k+n} \rho^2(i(j/2))c_j^2 Eu_j^2\Big\} \\ &\le c\Big\{ \sum_{j=k+1}^{k+n} \rho^2(i(j/2))c_j^2 Eu_j^2\Big\} \log^2(2n). \end{aligned}$$

(9.2.33) is proved.

It follows from (9.2.33), $\sum_{i=1}^{\infty} \rho(2^i) < \infty$ and Lemma 2.2.2 that

$$
\begin{aligned}
P\Big\{\Big|\sum_{i=1}^{n} E(u_i|\mathcal{F}_{i-1})\Big| \ge \varepsilon \exp(n^a/2)\Big\} & \\
&\le c\,e^{-n^a}\Big\{\sum_{i=1}^{n} i^{-2a} E u_i^2\Big\} \log^2(2n) \\
&\le c\,e^{-n^a}\Big\{\sum_{i=1}^{n} i^{-a-1} e^{i^a}\Big\} \log^2(2n) \\
&\le c\,n^{-2a} \log^2(2n).
\end{aligned}
$$

Let $n_k = [k^{1/a}]$. By the Borel-Cantelli lemma, we have

$$
\sum_{i=1}^{n_k} E(u_i|\,\mathcal{F}_{i-1}) = o\left(\exp(n_k^a/2)\right) \quad \text{a.s.} \tag{9.2.35}
$$

Moreover from Lemma 9.2.3, Lemma 2.2.2, Lemma 4.1.2 and (9.2.33), we have

$$
\begin{aligned}
&P\Big\{\max_{n_k<j\le n_{k+1}} \Big|\sum_{l=n_k}^{j} e^{-j^a/2} E(u_l|\,\mathcal{F}_{l-1})\Big| \ge \varepsilon\Big\} \\
&\le P\Big\{\max_{n_k<j\le n_{k+1}} \Big|\sum_{l=n_k}^{j} e^{-l^a/2} E(u_l|\mathcal{F}_{l-1})\Big| \ge \varepsilon/2\Big\} \\
&\le c\Big\{\sum_{j=n_k+1}^{n_{k+1}} j^{-2a} e^{-j^a} E u_j^2\Big\} \log^4(n_{k+1}-n_k) \\
&\le c\,k^{-2}(\log k)^4,
\end{aligned}
$$

which implies that

$$
\max_{n_k<j\le n_{k+1}} \Big|\sum_{l=n_k}^{j} e^{-j^a/2} E(u_l|\,\mathcal{F}_{l-1})\Big| \to 0 \quad \text{a.s.} \tag{9.2.36}
$$

Thus (9.2.32) follows from (9.2.35) and (9.2.36).

The following lemma gives (9.2.14).

**Lemma 9.2.8.** *Suppose that a satisfies (9.2.28), and condition (ii) of Theorem 9.2.1 is satisfied. Then for* $\delta_1 = \varepsilon'/4$, *we have*

$$
\sum_{k=1}^{\infty} e^{-(1+\delta_1)k^a} E|\xi_k|^{2+2\delta_1} < \infty. \tag{9.2.37}
$$

**Proof.** From Lemma 2.2.5, Lemma 9.2.2 and condition (ii), we have

$$\begin{aligned} E|\xi_k|^{2+2\delta_1} &\le c\,E\,|u_k|^{2+2\delta_1} \\ &\le c\Big\{(k^{a-1}\exp(k^a))^{1+\delta_1} \\ &\quad + k^{a-1}\exp(k^a)\,E|X_1|^{2+2\delta_1}I(X_1^2 \le 2\exp(k^a))\Big\}. \end{aligned}$$

Then, by $0 < a < \varepsilon'/(8+\varepsilon')$ and Lemma 9.2.2, (9.2.37) is proved.

The following lemma gives (9.2.15).

**Lemma 9.2.9.** *Suppose that a satisfies (9.2.28) and condition (ii) of Theorem 9.2.1 is satisfied. Then*

$$\sum_{j=1}^{n}(E(\xi_j^2|\mathcal{F}_{j-1}) - E\xi_j^2) = o(\exp(n^a)) \quad a.s. \tag{9.2.38}$$

**Proof.**

$$\begin{aligned} &\Big|\sum_{j=1}^{n}(E(\xi_j^2|\mathcal{F}_{j-1}) - E\xi_j^2)\Big| \\ &\quad\le \Big|\sum_{j=1}^{n}(E(u_j^2|\mathcal{F}_{j-1}) - Eu_j^2)\Big| \\ &\quad + \sum_{j=1}^{n}(E^2(u_j|\mathcal{F}_{j-1}) + E(E(u_j|\mathcal{F}_{j-1}))^2). \end{aligned} \tag{9.2.39}$$

We first prove that

$$\sum_{j=1}^{n}(E^2(u_j|\mathcal{F}_{j-1}) + E(E(u_j|\mathcal{F}_{j-1}))^2) = o(\exp(n^a)) \quad \text{a.s.} \tag{9.2.40}$$

In fact, by the definition of $\rho(\cdot)$ , we have

$$\begin{aligned} E(E(u_j|\mathcal{F}_{j-1}))^2 &= E(u_jE(u_j|\mathcal{F}_{j-1})) \\ &\le \rho(i_{j-1})\|u_j\|_2\,\|E(u_j|\mathcal{F}_{j-1})\|_2. \end{aligned}$$

Therefore

$$E(E(u_j|\mathcal{F}_{j-1}))^2 \le \rho^2(i_{j-1})Eu_j^2 \le c\,j^{-2a(1+\varepsilon)+a-1}\,e^{j^a},$$

and further

$$\sum_{j=1}^{\infty} E(E(u_j|\mathcal{F}_{j-1}))^2/e^{j^a} \le c\sum_{j=1}^{\infty} j^{-1-a(1+2\varepsilon)} < \infty.$$

By the Kronecker lemma, (9.2.40) holds true.

Secondly, we prove

$$\sum_{j=1}^{n}(E(u_j^2|\mathcal{F}_{j-1}) - Eu_j^2) = o(\exp(n^a)) \quad \text{a.s.} \tag{9.2.41}$$

Denote $\overline{u}_i = u_i^2 I(|u_i| \le e^{i^a/2})$. It is easy to see that

$$\begin{aligned}
&|\sum_{i=1}^{n}(E(u_i^2|\mathcal{F}_{j-1}) - Eu_i^2)| \\
&\quad \le |\sum_{i=1}^{n}(E(\overline{u}_i|\mathcal{F}_{j-1}) - E\overline{u}_i)| \\
&\qquad + \sum_{i=1}^{n}(E(u_i^2 I(|u_i| \ge e^{i^a/2})|\mathcal{F}_{i-1})) \\
&\qquad + Eu_i^2 I(|u_i| \ge e^{i^a/2})).
\end{aligned} \tag{9.2.42}$$

It follows from Lemma 2.2.5 and Lemma 9.2.2 that

$$\begin{aligned}
&\sum_{j=1}^{\infty} E(E(u_j^2 I(|u_j| \ge e^{j^a/2})|\mathcal{F}_{j-1}))/e^{j^a} \\
&\quad = \sum_{j=1}^{\infty} e^{-j^a} Eu_j^2 I(|u_j| \ge e^{j^a/2}) \\
&\quad \le c\sum_{j=1}^{\infty} e^{-(1+\delta_1/2)j^a} E|u_j|^{2+\delta_1} \\
&\quad \le c\sum_{j=1}^{\infty} e^{-(1+\delta_1/2)j^a}\Big\{(j^{a-1}e^{j^a})^{1+\delta_1/2} \\
&\qquad + e^{j^a} j^{a-1} E|X_1|^{2+\delta_1} I(X_1^2 \le 2e^{j^a})\Big\} \\
&\quad < \infty.
\end{aligned}$$

By the Kronecker lemma

$$\begin{aligned}
&\sum_{j=1}^{n}\Big(E(u_j^2 I(|u_j| \ge e^{j^a/2})|\mathcal{F}_{j-1}) + Eu_j^2 I(|u_j| \ge e^{j^a/2})\Big) \\
&\qquad = o(\exp(n^a)) \quad \text{a.s.}
\end{aligned} \tag{9.2.43}$$

On the other hand, by the same way as in the proof of (9.2.33) in Lemma 9.2.7, there exists a constant $C$ such that

$$E\Big(\sum_{j=1}^{n} E(\overline{u}_j - E(\overline{u}_j|\,\mathcal{F}_{j-1}))\Big)^2 \le C\,\Big(\sum_{j=1}^{n} \rho(i(j/2))\, E\overline{u}_j^2\Big) \log^2(2n).$$

Then by Lemma 2.2.2 and condition (ii), we have

$$\begin{aligned}
&P\{|\sum_{j=1}^{n} E(\overline{u}_j - E(\overline{u}_j|\,\mathcal{F}_{j-1}))| \ge \varepsilon e^{n^a}\}\\
&\le c\,e^{-2n^a}\Big(\sum_{j=1}^{n} j^{-2a(1+\varepsilon')} E\overline{u}_j^2\Big)(\log n)^2\\
&\le c\,e^{-2n^a}\Big(\sum_{j=1}^{n} j^{-2a(1+\varepsilon')} e^{j^a}\Big)(\log n)^2\\
&\le c\,n^{-2a(1+\varepsilon')}(\log n)^2.
\end{aligned}$$

By the same discussion as in the proof of second half part of Lemma 9.2.7, it follows that

$$\sum_{j=1}^{n} E(\overline{u}_j - E\overline{u}_j|\,\mathcal{F}_{j-1}) = o\,(\exp(n^a)) \quad \text{a.s.}$$

Combining it with (9.2.42), (9.2.43) yields (9.2.41). Lemma 9.2.9 is proved.
The following lemma gives (9.2.16).

**Lemma 9.2.10.** *If the conditions of Theorem 9.2.1 are satisfied, we have*

$$\sum_{i=1}^{m_n} E\xi_i^2 - ES_n^2 = o(n). \tag{9.2.44}$$

**Proof.** Denote $\widehat{u}_j = \sum_{l\in H_j} X_l$, $\widehat{v}_j = \sum_{l\in I_j} X_l$. We have

$$\begin{aligned}
ES_n^2 &= E\Big(\sum_{j=1}^{m_n}\widehat{u}_j + \sum_{j=1}^{m_n}\widehat{v}_j + \sum_{j=N(m_n)+1}^{n} X_j\Big)^2\\
&= E\Big(\sum_{j=1}^{m_n}\widehat{u}_j\Big)^2 + E\Big(\sum_{j=1}^{m_n}\widehat{v}_j + \sum_{j=N(m_n)+1}^{n} X_j\Big)^2\\
&\quad + 2E\Big(\sum_{j=1}^{m_n}\widehat{u}_j\Big)\Big(\sum_{j=1}^{m_n}\widehat{v}_j + \sum_{j=N(m_n)+1}^{n} X_j\Big).
\end{aligned} \tag{9.2.45}$$

By Lemma 9.2.4 and the definitions of $N_k$ and $m_n$, it follows that

$$E\Big(\sum_{j=1}^{m_n}\widehat{v}_j+\sum_{j=N(m_n)+1}^{n}X_j\Big)^2=O\Big(\sum_{j=1}^{m_n}E\widehat{v}_j^2+(n-N_{m_n})\Big)$$
$$=O\big(n(\log n)^{\frac{a-1}{a}}\big). \qquad (9.2.46)$$

Write

$$E\Big(\sum_{j=1}^{m_n}\widehat{u}_j\Big)^2=E\Big(\sum_{j=1}^{m_n}u_j+\sum_{j=1}^{m_n}(\widehat{u}_j-u_j)\Big)^2.$$

From Lemma 9.2.4 and Lemma 2.2.2 we obtain

$$\begin{aligned}E\Big(\sum_{j=1}^{m_n}(\widehat{u}_j-u_j)\Big)^2&\le c\sum_{j=1}^{m_n}E(\widehat{u}_j-u_j)^2\\&\le c\sum_{j=1}^{m_n}j^{a-1}e^{j^a}EX_1^2I(|X_1|^2\ge N_{j-1})\\&=o(n).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{j=1}^{m_n}E\xi_j^2-E\Big(\sum_{j=1}^{m_n}u_j\Big)^2\\&=E\Big(\sum_{j=1}^{m_n}\xi_j\Big)^2-E\Big(\sum_{j=1}^{m_n}u_j\Big)^2\\&=E\Big(\sum_{j=1}^{m_n}E(u_j|\mathcal{F}_{j-1})\Big)^2+2E\Big(\sum_{j=1}^{m_n}u_j\Big)\sum_{j=1}^{m_n}E(u_j|\mathcal{F}_{j-1}).\end{aligned}$$

It follows from (9.2.33) that

$$\begin{aligned}&E\Big(\sum_{j=1}^{m_n}E(u_j|\mathcal{F}_{j-1})\Big)^2\\&\qquad\le c\Big(\sum_{j=1}^{m_n}j^{-2(1+\varepsilon)a}Eu_j^2\Big)(\log m_n)^2\\&\qquad\le cn(\log n)^{-2(1+\varepsilon)}(\log\log n)^2.\end{aligned}$$

Combining it with the above relations yields (9.2.44) .

Now the proof of Theorem 9.2.1 is completed.

The proof of Theorem 9.2.2 is the exactly same as that of Theorem 9.2.1, provided Lemma 2.2.10 is used instead of Lemma 2.2.5 and 4.3.2. The proof of Theorem 9.2.3 is similar to that of Theorem 9.2.1 as well. We need only to apply the Bernstein divided-section method for $\{X_n\}$ immediately. The details are omitted.

## 9.3 Strong approximations for an $\alpha$-mixing sequence

Shao and Lu (1987) improved, the strong approximation result of Philipp and Stout (1975) for an $\alpha$-mixing sequence, and obtained a better rate of strong approximation when using the Strassen martingale embedding method.

**Theorem 9.3.1.** *Let $\{X_n, n \geq 1\}$ be an $\alpha$-mixing sequence with $EX_n = 0$ and $g(x)$ a function such that $g(x)/x^{2+\delta}$ for some $0 < \delta \leq 2$ is increasing to infinity. Denote*

$$\|X\|_g = \inf\{t > 0,\ Eg(|X|/t) \leq 1\}.$$

*If $\sup_n \|X_n\|_g < \infty$ and the following conditions are satisfied*
*(i) $\sigma_n^2 = ES_n^2 \geq C\,n$, for some $C > 0$,*
*(ii) $\sum\limits_{n=1}^{\infty} \alpha(n)^{\frac{1}{2+\delta}} \operatorname{inv}g\left(\frac{1}{\alpha(n)}\right) < \infty$ ,*
*then*

$$S(t) - W(\sigma_t^2) = O(\sigma_t^{2/(2+\delta)}(\log \sigma_t^2)^{1+(1+\lambda)/(2+\delta)}) \quad a.s. \tag{9.3.1}$$

*where $\lambda = (\log 2)/\log((2+\delta)/\delta) \leq 1$.*

**Proof.** Write

$$\begin{aligned} Y_n &= \sum_{k=0}^{\infty}\{E(X_{k+n}|\,\mathcal{F}_n) - E(X_{k+n}|\,\mathcal{F}_{n-1})\} \\ &= X_n + u_n - u_{n-1}, \end{aligned} \tag{9.3.2}$$

where $u_n = \sum\limits_{k=0}^{\infty} E(X_{k+n}|\mathcal{F}_{n-1})$. Let us verify the conditions of Proposition 9.1.1. Condition (a) is assumed. To check condition (b), denote

$$u_{kn} = E(X_{k+n}|\mathcal{F}_{n-1}).$$

Take $f(x) = x^{(2+\delta)/(1+\delta)}$ in Lemma 1.2.3, we have

$$\begin{aligned} E|u_{kn}|^{2+\delta} &= E(X_{k+n}u_{kn}|u_{kn}|^{\delta}) \\ &\leq c\,\operatorname{inv}g\Big(\frac{1}{\alpha(k)}\Big)\alpha(k)^{1/(2+\delta)}\|X_{k+n}\|_g\|u_{kn}\|_{2+\delta}^{1+\delta}. \end{aligned}$$

Therefore

$$\|u_{kn}\|_{2+\delta} \le c \operatorname{inv} g(1/\alpha(k))\alpha(k)^{1/(2+\delta)}.$$

It follows from (ii) that condition (b) of Proposition 9.1.1 is satisfied.

Denote

$$T_m(n) = \sum_{m<k\le m+n} (Y_k^2 - EY_k^2),$$
$$\tau_n = \sup_m E|T_m(n)|^{(2+\delta)/2}.$$

We shall show

$$\tau_n \le cn(\log n)^\lambda \tag{9.3.3}$$

by induction on $n$, where

$$\lambda = (\log \frac{2+M_\delta}{2})/\log\frac{2+\delta}{\delta} \le 1,$$
$$1 \le M_\delta \le 2,\ 0<\delta\le 2.$$

Put

$$\theta = \delta/(2+\delta),\ n_1 = n-[n^\theta],\ n_2 = [n^\theta].$$

From an lementary inequality

$$|1+x|^p \le 1+px+C_p|x|^p, \quad \text{for } 1<p\le 2,\ 1\le C_p\le 2,$$

we have

$$\begin{aligned}
E|T_m(n)|^{\frac{2+\delta}{2}} &\le E|T_m(n_1)|^{\frac{2+\delta}{2}} + M_\delta E|T_{m+n_1}(n_2)|^{\frac{2+\delta}{2}} \\
&\quad + \Big(\frac{2+\delta}{2}\Big)E|T_m(n_1)|^{\frac{\delta}{2}}|T_{m+n_1}(n_2)| \\
&\le \tau_{n_1} + M_\delta\tau_{n_2} + \Big(\frac{2+\delta}{2}\Big)E\Big(|T_m(n_1)|^{\frac{\delta}{2}} \\
&\quad \cdot\Big\{\Big(\sum_{j=N_2}^{m+n} X_j + u_{N_1} - u_{N_2}\Big)^2 \\
&\quad - E\Big(\sum_{j=N_2}^{m+n} X_j + u_{N_1} - u_{N_2}\Big)^2\Big\}\Big) \\
&=: \tau_{n_1} + M_\delta\tau_{n_2} + \frac{2+\delta}{2}I,
\end{aligned} \tag{9.3.4}$$

where

$$N_1 = m + n + 1,\ N_2 = m + n_1 + 1.$$

Note that

$$\begin{aligned}
I &= E|T_m(n_1)|^{\frac{\delta}{2}} \sum_{j=N_2}^{m+n} (X_j^2 - EX_j^2) \\
&\quad + E|T_m(n_1)|^{\frac{\delta}{2}}((u_{N_1} - u_{N_2})^2 - E(u_{N_1} - u_{N_2})^2) \\
&\quad + 2E|T_m(n_1)|^{\frac{\delta}{2}}\Big\{\Big(\sum_{j=N_2}^{m+n} X_j\Big)(u_{N_1} - u_{N_2}) \\
&\quad - E\Big(\sum_{j=N_2}^{m+n} X_j\Big)(u_{N_1} - u_{N_2})\Big\} \\
&\quad + 2E|T_m(n_1)|^{\frac{\delta}{2}} \sum_{1\le i,j,i+j\le n_2} (X_{j+N_2-1}X_{i+j+N_2-1} \\
&\quad - EX_{j+N_2-1}X_{i+j+N_2-1}) \\
&=: I_1 + I_2 + 2I_3 + 2I_4. \qquad (9.3.5)
\end{aligned}$$

From Lemma 1.2.3 ( take $g_1(x^2) = g(x), f(x) = x^{(2+\delta)/\delta}$) and condition (ii), we have

$$\begin{aligned}
I_1 &\le c\,(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}} \sum_{k=0}^{\infty} \text{inv}g_1\Big(\frac{1}{\alpha(k)}\Big)\text{inv}f\Big(\frac{1}{\alpha(k)}\Big)\alpha(k) \\
&\le c\,(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}} \sum_{k=0}^{\infty}\Big(\text{inv}g\Big(\frac{1}{\alpha(k)}\Big)\alpha(k)^{\frac{1}{2+\delta}}\Big)^2 \\
&\le c\,(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}.
\end{aligned}$$

By the Hölder inequality and condition $(b)$, we have

$$\begin{aligned}
I_2 &\le c\,(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}(\|u_{N_1}\|_{2+\delta}^2 + \|u_{N_2}\|_{2+\delta}^2) \\
&\le c\,(E|T_m(n_1)|^{\frac{2+\delta}{2}})^{\frac{\delta}{2+\delta}}.
\end{aligned}$$

By the Hölder inequality and Lemma 1.2.3, we have

$$
\begin{aligned}
I_3 = E|T_m(n_1)|^{\frac{\delta}{2}}\Big\{ &\sum_{j=N_2+1}^{N_1-1} X_j \sum_{k=1}^{\infty} X_{N_1+k} - \sum_{j=N_2+1}^{N_1-1} u_{N_2} X_j \\
&- E \sum_{j=N_2+1}^{N_1-1} X_j \sum_{k=1}^{\infty} X_{N_1+k} + E \sum_{j=N_2+1}^{N_1-1} u_{N_2} X_j \Big\} \\
\le\ & c\big(E|T_m(n_1)|^{\frac{2+\delta}{2}}\big)^{\frac{\delta}{2+\delta}} \\
&\cdot \sum_j \sum_k \operatorname{inv} f\Big(\frac{1}{\alpha(N_1+k-j)}\Big) \alpha(N_1+k-j)^{\frac{2}{2+\delta}} \\
\le\ & c\big(E|T_m(n_1)|^{\frac{2+\delta}{2}}\big)^{\frac{\delta}{2+\delta}} \\
&\cdot \sum_j \sum_k \Big(\operatorname{inv} g\Big(\frac{1}{\alpha(N_1+k-j)}\Big) \alpha(N_1+k-j)^{\frac{1}{2+\delta}}\Big)^2 \\
\le\ & c\big(E|T_m(n_1)|^{\frac{2+\delta}{2}}\big)^{\frac{\delta}{2+\delta}},
\end{aligned}
$$

where $f(x^{(2+\delta)/2}) = g(x)$, $\operatorname{inv} f(x) = (\operatorname{inv} g(x))^{(2+\delta)/2}$. For $I_4$, we have

$$
\begin{aligned}
I_4 = E|T_m(n_1)|^{\frac{\delta}{2}} \Big(&\sum_{j=1}^{n_2} \sum_{1\le i<j} + \sum_{i=1}^{n_2} \sum_{1\le j<i}\Big) \\
&\cdot \Big(X_{j+N_2-1} X_{i+j+N_2-1} - E X_{j+N_2-1} X_{i+j+N_2-1}\Big) \\
\le c \Big(&\sum_{j=1}^{n_2} \sum_{1\le i<j} + \sum_{i=1}^{n_2} \sum_{1\le j<i}\Big) \alpha(i)^{\frac{2}{2+\delta}} \operatorname{inv} f\Big(\frac{1}{\alpha(i)}\Big) \\
&\cdot \big(E|T_m(n_1)|^{\frac{2+\delta}{2}}\big)^{\frac{\delta}{2+\delta}},
\end{aligned}
$$

where $f(x^2) = g(x)$, $\operatorname{inv} f(x) = (\operatorname{inv} g(x))^2$. It is easy to see that

$$
\begin{aligned}
I_4 \le c\Big(&\sum_{j=1}^{n_2} \sum_{1\le i<j} + \sum_{i=1}^{n_2} \sum_{1\le j<i}\Big) \\
&\cdot \alpha(i)^{\frac{1}{2+\delta}} \operatorname{inv} g\Big(\frac{1}{\alpha(i)}\Big)^2 \big(E|T_m(n_1)|^{\frac{2+\delta}{2}}\big)^{\frac{\delta}{2+\delta}} \\
\le c\big(&E|T_m(n_1)|^{\frac{2+\delta}{2}}\big)^{\frac{\delta}{2+\delta}}.
\end{aligned}
$$

Thus combining these with (9.3.4) and (9.3.5) we have

$$
\tau_n \le \tau_{n_1} + M_\delta \tau_{n_2} + c\,\tau_{n_1}^{\delta/(2+\delta)}.
$$

Hence condition (c) of Proposition 9.1.1 is satisfied and the proof of Theorem 9.3.1 is completed.

**Corollary 9.3.1.** *Let $\{X_n, n \geq 1\}$ be an $\alpha$-mixing sequence, $g(x) = x^r$, $r > 2+\delta$, $0 < \delta \leq 2$. If*

*(i) $\sigma_n^2 = ES_n^2 \geq Cn$, for some $C > 0$,*

*(ii) $\sum_{n=1}^{\infty} \alpha(n)^{\frac{1}{2+\delta}-\frac{1}{r}} < \infty$,*

*then*

$$S_n - W(\sigma_n^2) = O\big(\sigma_n^{\frac{2}{2+\delta}}(\log \sigma_n)^{1+(1+\lambda)/(2+\delta)}\big) \qquad a.s.$$

*Particularly, if $\delta = 2$ and $\lim_n \sigma_n^2/n = \sigma^2$ (without loss of generality, assume that $\sigma^2 = 1$), we have*

$$S_n - W(n) = O(n^{1/4}(\log n)^{3/2}) \qquad a.s.$$

*Furthermore, if $\sup_n |X_n| < \infty$, $\sum \alpha(n)^{1/4} < \infty$ and $\sigma_n^2 = n\sigma^2$, then for any $\varepsilon > 0$*

$$S_n - W(n) = O(n^{1/4}(\log n)^{5/4+\varepsilon}). \qquad a.s.$$

Shao (1989a) gave the following theorem and corollary, which improve the results in Bradley (1983) and Dehling (1983) under the weaker conditions.

**Theorem 9.3.2.** *Let $\{X_n, n \geq 1\}$ be an $\alpha$-mixing sequence with $EX_n = 0$, $\sup_n E|X_n|^{2+\delta} < \infty$ for some $\delta > 0$. Assume that*

$$\sup_{n\geq 1}\sup_{k\geq 0} E|S_k(n)|^{2+\delta}/n^{1+\delta/2} < \infty, \tag{9.3.6}$$

$$\alpha(n) = O((\log n)^{-r}), \qquad r > 1 + 2/\delta. \tag{9.3.7}$$

*Then for any $0 < \theta < \frac{1}{4}(\frac{r\delta}{2+\delta} - 1)$, we have*

$$S(t) - W(\sigma_t^2) = O(t^{1/2}(\log t)^{-\theta}) \qquad a.s.$$

**Corollary 9.3.2.** *Let $\{X_n, n \geq 1\}$ be an $\alpha$-mixing sequence with $EX_n = 0$, $\sup_n E|X_n|^{2+\delta} < \infty (\delta > 0)$. Assume that $\alpha(n) = O(n^{-r})$ for $r > 1 + 2/\delta$ and*

$$\lim_{n\to\infty} \sigma_n^2/n = \sigma^2 > 0.$$

*Then $\{X_n\}$ Obeys the law of the iterated logarithm and the Chung law of the iterated logarithm.*

The proofs of Theorem 9.3.2 and Corollary 9.3.2 are omitted.

# Chapter 10 The Increments of Partial Sums

In Chapter 9, we studied strong approximations of partial sums of a mixing sequence by a Wiener process. But, with the help of these theorems, it is not enough to obtain the ideal increment results, similar to that in the case of i.i.d. sequence (see M.Csörgő and P.Révész 1981). In this chapter, we intend to study the increments of partial sums of a sequence of $\varphi$-mixing random variables, which may be non-stationary, even non-identically distributed, by a direct approach. Under the restriction on the rate of convergence to zero for mixing coefficients, the following results are close to those corresponding to independent random variables. Our method can be used to deal with other kinds of mixing sequences as well.

## 10.1 Some lemmas

In order to show increment results, we need to establish some exponential inequalities. To this end, we first give the next result (cf. Stout 1974, Lemma 5.4.1 and its corollary).

**Lemma 10.1.1.** *Let $\{Z_n, \mathcal{F}_n, n \geq 1\}$ be a supermartingale with $EZ_n = 0$. Let $Z_0 = 0$ and $U_i = Z_i - Z_{i-1}$ for $i \geq 1$. Suppose $U_i \leq C$ a.s. for some $0 \leq C < \infty$ and all $i \geq 1$. Fix $\lambda > 0$ such that $\lambda C \leq 1$. Put*

$$M_n = \exp(\lambda Z_n) \exp\Big\{-(\lambda^2/2)(1 + \lambda C/2) \sum_{i=1}^{n} E(U_i^2 | \mathcal{F}_{i-1})\Big\}$$

*for $n \geq 1$ and $M_0 = 1$ a.s. Then $\{M_n, \mathcal{F}_n, n \geq 0\}$ is a nonnegative supermartingale, and further,*

$$P\Big\{\sup_{n\geq 0} M_n > \alpha\Big\} \leq \alpha^{-1}$$

*for any $\alpha > 0$.*

Let $\{Y_n\}$ be a sequence of $\varphi$-mixing random variables with mixing coefficients $\varphi_n = \varphi(n) \downarrow 0$. Without loss of generality, assume that $EY_n =$

0 for every $n$. Put $T_n = \sum_{i=1}^{n} Y_i$, $\tau_n^2 = ET_n^2$. In this section, we always assume that the following conditions are satisfied:

(a) $|Y_n| \le b_n < \infty$;

(b) there exist $0 < \sigma^2 \le \sigma'^2 < \infty$, such that $\sigma^2 n \le E(Y_{m+1} + \cdots + Y_{m+n})^2 \le \sigma'^2 n$ for every $m \ge 0$ and all large $n$.

Let $p, q, k$ be integers satisfying that $p = p_n \le n, q = q_n = o(p_n)$, $q_n \uparrow \infty$, $k = k_n = [n/(p_n + q_n)]$. And put $b = \max_{1\le i\le n} b_i$.

**Lemma 10.1.2.** *Suppose that*

$$\varphi_q p b^2 = o(1), \quad \sum_{j=1}^{k} \varphi^{1/2}(jp) = O(1). \tag{10.1.1}$$

*Then for $x = x_n$ and small $\varepsilon > 0$ satisfying*

$$\frac{4}{\varepsilon} b\, n\, \varphi_q \le x \le \frac{\varepsilon \tau_n^2}{pb} \tag{10.1.2}$$

*and*

$$x^2/n \to \infty, \tag{10.1.3}$$

*we have*

$$P\Big\{\max_{1\le i\le n} |T_i| \ge x\Big\} \le 3 \exp\Big\{-\frac{(1-6\varepsilon)x^2}{2\tau_n^2}\Big\} \tag{10.1.4}$$

*for all large $n$. If (10.1.2) is replaced by*

$$x > \frac{\varepsilon \tau_n^2}{pb}, \tag{10.1.5}$$

*then*

$$P\Big\{\max_{1\le i\le n} |T_i| \ge x\Big\} \le 3 \exp\Big\{-\frac{\varepsilon(1-5\varepsilon)x}{2pb}\Big\}. \tag{10.1.6}$$

**Proof.** We always assume that $n$ is large enough. First, we prove (10.1.4). Define

$$\xi_i = \sum_{j=i(p+q)+1}^{(i+1)p+iq} Y_j,$$

$$\eta_i = \sum_{j=(i+1)p+iq+1}^{(i+1)(p+q)} Y_j, \ i = 0, 1, \cdots, k-1,$$

$$\eta_k = \sum_{j=k(p+q)+1}^{n} Y_j.$$

Put $\sigma$-fields $\mathcal{F}_{-1}=\{\emptyset,\,\Omega\}, \mathcal{F}_i=\sigma(Y_j,\,j\le(i+1)p+iq),\, i=0,1,\cdots,k-1$. Define martingale differences $\gamma_i=\xi_i-E(\xi_i|\mathcal{F}_{i-1}), i=0,1,\cdots,k-1$. Write

$$\begin{aligned}P\{|T_n|\ge x\}&\le P\Big\{|\sum_{i=0}^{k-1}\xi_i|\ge(1-\frac{\varepsilon}{2})x\Big\}+P\Big\{|\sum_{i=0}^{k}\eta_i|\ge\frac{1}{2}\varepsilon x\Big\}\\&=: J_1+J_2.\end{aligned}\tag{10.1.7}$$

Consider $J_1$. Write

$$\begin{aligned}J_1&\le P\Big\{|\sum_{i=0}^{k-1}\gamma_i|\ge(1-\varepsilon)x\Big\}+P\Big\{|\sum_{i=0}^{k-1}E(\xi_i|\mathcal{F}_{i-1})|\ge\frac{1}{2}\varepsilon x\Big\}\\&=: J_{11}+J_{12}.\end{aligned}\tag{10.1.8}$$

By Lemma 2.2.8 (note $|\xi_i|\le pb$), for any $B_{i-1}\in\mathcal{F}_{i-1}$,

$$|E\,\xi_i I_{B_{i-1}}|\le 2\varphi_q\,p\,b\,P(B_{i-1})\qquad(i=0,1,\cdots,k-1),$$

which implies

$$|E(\xi_i|\mathcal{F}_{i-1})|\le 2\varphi_q pb\quad\text{a.s.}\quad(i=0,1,\cdots,k-1).\tag{10.1.9}$$

Using condition (10.1.2), we obtain

$$J_{12}=0.\tag{10.1.10}$$

Next, we estimate $J_{11}$. Since $\{\gamma_i,\mathcal{F}_i,i=0,1,\cdots,k-1\}$ possesses martingale difference property, by using Lemma 10.1.1 and noting $|\gamma_i|\le(1+\varepsilon)\,p\,b$ a.s. for large $n$, for $0<\lambda\le((1+\varepsilon)\,p\,b)^{-1}$

$$\zeta_j=\exp\Big(\lambda\sum_{i=0}^{j}\gamma_i\Big)\exp\Big\{-\frac{\lambda^2}{2}\Big(1+\frac{1}{2}(1+\varepsilon)pb\lambda\Big)\sum_{i=0}^{j}E(\gamma_i^2|\mathcal{F}_{i-1})\Big\}$$

$$(j=0,1,\cdots,k-1)$$

possess nonnegative supermartingale property. Write

$$\begin{aligned}&P\Big\{\sum_{i=0}^{k-1}\gamma_i\ge(1-\varepsilon)x\Big\}\\&\qquad=P\Big\{\zeta_{k-1}\ge\exp(\lambda(1-\varepsilon)x)\\&\qquad\cdot\exp\Big\{-\frac{\lambda^2}{2}\Big(1+\frac{1}{2}(1+\varepsilon)pb\lambda\Big)\sum_{i=0}^{k-1}E(\gamma_i^2|\mathcal{F}_{i-1})\Big\}\Big\}.\end{aligned}\tag{10.1.11}$$

We are going to estimate $\sum_{i=0}^{k-1} E(\gamma_i^2|\mathcal{F}_{i-1})$. Write

$$\sum_{i=0}^{k-1} E(\gamma_i^2|\mathcal{F}_{i-1}) = \sum_{i=0}^{k-1} E(\xi_i^2|\mathcal{F}_{i-1}) - \sum_{i=0}^{k-1}(E(\xi_i|\mathcal{F}_{i-1}))^2.$$

We have $|E({\xi_i}^2|\mathcal{F}_{i-1}) - E{\xi_i}^2| \le 2\varphi_q(pb)^2$ a.s. by a similar proof to (10.1.9). Hence, by conditions (b) and (10.1.1), we get

$$\sum_{i=0}^{k-1} E({\xi_i}^2|\mathcal{F}_{i-1}) = (1+o(1))\sum_{i=0}^{k-1} E{\xi_i}^2 \quad \text{a.s.}$$

Moreover inequality (10.1.9) implies that

$$\sum_{i=0}^{k-1}(E(\xi_i|\mathcal{F}_{i-1}))^2 \le 4\varphi_q^2 pb^2 n = o(n) \quad \text{a.s.}$$

Thus

$$\sum_{i=0}^{k-1} E({\gamma_i}^2|\mathcal{F}_{i-1}) = (1+o(1))\sum_{i=0}^{k-1} E{\xi_i}^2 \quad \text{a.s.} \tag{10.1.12}$$

Now we estimate $\sum\limits_{i=0}^{k-1} E\xi_i^2$. Write

$$\sum_{i=0}^{k-1} E\xi_i^2 = E\Big(\sum_{i=0}^{k-1}\xi_i\Big)^2 - 2\sum_{0\le i<j\le k-1} E\xi_i\xi_j, \tag{10.1.13}$$

where $|E\xi_i\xi_j| \le 2\varphi((j-i-1)p)pb\sigma' p^{1/2}$ by Lemma 1.2.10 and so for fixed $i$

$$\sum_{j\ge i+2} |E\xi_i\xi_j| \le 2\sigma'\varphi_p^{1/2}p^{3/2}b\sum_{j=1}^{\infty}\varphi^{1/2}(jp) = o(p)$$

by assumptions in (10.1.1). Summing on $i$ leads to $o(n)$. Also

$$|E\xi_i\xi_{i+1}| \le 2\varphi_q pb\sigma' p^{1/2} = o(p)$$

by (10.1.1) and summing leads to $o(n)$ again. Thus

$$\sum_{i=0}^{k-1} E\xi_i^2 = E\Big(\sum_{i=0}^{k-1}\xi_i\Big)^2 + o(n). \tag{10.1.14}$$

Furthermore

$$E\Big(\sum_{i=0}^{k-1}\xi_i\Big)^2 = ET_n^2 - 2ET_n\Big(\sum_{i=0}^{k}\eta_i\Big) + E\Big(\sum_{i=0}^{k}\eta_i\Big)^2. \tag{10.1.15}$$

By condition (b) and the second equality in (10.1.1), we can prove that

$$E\Big(\sum_{i=0}^{k}\eta_i\Big)^2 = O(kq+p).$$

In fact

$$E\Big(\sum_{i=0}^{k}\eta_i\Big)^2 \le 2\,E\Big(\sum_{i=0}^{k-1}\eta_i\Big)^2 + 2\,E\eta_k^2,$$

where $E\eta_k^2 = O(p)$ and

$$\begin{aligned}E\Big(\sum_{i=0}^{k-1}\eta_i\Big)^2 &= \sum_{i=0}^{k-1}E\eta_i^2 + 2\sum_{0\le i<j\le k-1}E\eta_i\eta_j\\ &\le \sigma'^2 kq + 2\sigma'^2 q\sum_{j=1}^{k-1}(k-j)\varphi^{1/2}(jp) = O(kq).\end{aligned}$$

Hence (10.1.15) implies that $E(\sum_{i=0}^{k-1}\xi_i)^2 = (1+o(1))\,\tau_n^2$. Combining it with (10.1.14) and (10.1.12) implies that

$$\sum_{i=0}^{k-1}E(\gamma_i^2|\,\mathcal{F}_{i-1}) = (1+o(1))\tau_n^2 \quad \text{a.s.} \tag{10.1.16}$$

Inserting it into (10.1.11) and using Lemma 10.1.1, we obtain

$$\begin{aligned}P\Big\{\sum_{i=0}^{k-1}\gamma_i \ge (1-\varepsilon)x\Big\}&\\ \le \exp\{-\lambda(1-\varepsilon)x\}&\\ \cdot\exp\Big\{\frac{\lambda^2}{2}\Big(1+\frac{1}{2}(1+\varepsilon)pb\lambda\Big)(1+\varepsilon)\tau_n^2\Big\}&\end{aligned} \tag{10.1.17}$$

for all large $n$. Choosing $\lambda = x/((1+\varepsilon)\tau_n^2)$, we have $\lambda \le \varepsilon((1+\varepsilon)pb)^{-1}$ by (10.1.2). Inserting it into (10.1.17) implies that

$$P\Big\{\sum_{i=0}^{k-1}\gamma_i \ge (1-\varepsilon)x\Big\} \le \exp\Big\{-\frac{(1-4\varepsilon)x^2}{2\tau_n^2}\Big\}. \tag{10.1.18}$$

Replacing $Y_j$ with $-Y_j$, we obtain

$$J_{11} = P\Big\{|\sum_{i=0}^{k-1}\gamma_i| \ge (1-\varepsilon)x\Big\} \le 2\,\exp\Big\{-\frac{(1-4\varepsilon)x^2}{2\tau_n^2}\Big\}.$$

Combining it with (10.1.10), we have

$$J_1 \le 2\exp\Big\{-\frac{(1-4\varepsilon)x^2}{2\tau_n^2}\Big\}.$$

For $J_2$, it is clear that $3^{-1}\exp\{-(1-4\varepsilon)x^2/(2\tau_n^2)\}$ is one of its upper bounds in view of $q_n = o(p_n)$. Then we have proved

$$P\{|T_n| \ge x\} \ge \frac{7}{3}\exp\Big\{-\frac{(1-4\varepsilon)x^2}{2\tau_n^2}\Big\}. \tag{10.1.19}$$

In order to obtain (10.1.4) from (10.1.19), we use Lemma 5.1.2. First of all, since $\varphi_n \downarrow 0$, there exists an interger $n_0$ such that $\varphi_{n_0} \le 10^{-1}$. Using condition (b) and (10.1.3), we obtain

$$\begin{aligned}\max_{1\le i\le n} P\Big\{|T_n - T_i| \ge \frac{\varepsilon x}{2(1+\varepsilon)}\Big\} &\le \max_{1\le i\le n}\frac{4(1+\varepsilon)^2E(T_n-T_i)^2}{\varepsilon^2x^2}\\ &\le \frac{4(1+\varepsilon)^2\sigma'^2 n}{\varepsilon^2x^2} \le \frac{1}{10}\end{aligned}$$

for large $n$. So $\eta$ in Lemma 5.1.2 can be chosen as $5^{-1}$. Furthermore

$$\max_{1\le i\le n}|Y_i| \le b \le \frac{\varepsilon\sigma'^2 n}{px} \le \frac{\varepsilon\sigma'^2 k}{x^2}x = o(x) \tag{10.1.20}$$

by (10.1.2) and (10.1.3). Hence

$$P\Big\{\max_{1\le i\le n}|Y_i| \ge \varepsilon x/(2(1+\varepsilon)(n_0-1))\Big\} = 0$$

for large $n$. Therefore, from the above results and Lemma 5.1.2, we obtain

$$\begin{aligned}&P\Big\{\max_{1\le i\le n}|T_i| \ge x\Big\}\\ &\le \frac{5}{4}P\Big\{|T_n| \ge \frac{x}{1+\varepsilon}\Big\} + \frac{5}{4}P\Big\{\max_{1\le i\le n}|Y_i| \ge \frac{\varepsilon x}{2(1+\varepsilon)(n_0-1)}\Big\}\\ &< 3\exp\Big\{-\frac{(1-6\varepsilon)}{2\tau_n^2}x^2\Big\}.\end{aligned}$$

This is namely (10.1.4).

Now we prove (10.1.6). It is clear that we can assume $x/pb > \delta$ for some $\delta > 0$. If condition (10.1.5) is satisfied, choosing $\lambda = \varepsilon((1+\varepsilon)pb)^{-1}$ in (10.1.17), we have

$$P\Big\{\sum_{i=0}^{k-1}\gamma_i \ge (1-\varepsilon)x\Big\} \le \exp\Big\{-\frac{\varepsilon(1-4\varepsilon)x}{2pb}\Big\}. \tag{10.1.21}$$

By imitating the procedure from (10.1.18) to (10.1.4) and replacing (10.1.20) by $\max_{1\le i\le n} |Y_i| \le b < x/(p\delta) = o(x)$, we obtain (10.1.16) from (10.1.21). The lemma is proved.

**Lemma 10.1.3.** *Suppose that condition (10.1.1) in Lemma 10.1.2 is replaced by*

$$\varphi_p q\, b^2 = o(1), \quad \sum_{j=1}^{k} \varphi^{1/2}(jp) = O(1). \tag{10.1.1'}$$

*Then for any $\nu > 0$, there exists an $\alpha(\nu)$, such that for all large $n$ and $\alpha$ satisfying $\alpha \ge \alpha(\nu)$, we have*

$$P\{T_n \ge \alpha\tau_n\} \ge \frac{1}{2}\exp\{-(1+\nu)\alpha^2/2\}$$

*provided that*

$$\varphi_q = o(\alpha/b) \tag{10.1.22}$$

*and*

$$p\, b\, \alpha\, \tau_n = o(n). \tag{10.1.23}$$

**Proof.** For $0 < \delta < 1$ write

$$\begin{aligned} P\{T_n \ge \alpha\tau_n\} &\ge P\Big\{\sum_{i=0}^{k-1} \xi_i \ge (1+\delta)\alpha\tau_n\Big\} - P\Big\{\sum_{i=0}^{k} \eta_i \le -\delta\alpha\tau_n\Big\} \\ &=: I_1 - I_2 \end{aligned} \tag{10.1.24}$$

and

$$\begin{aligned} I_1 &\ge P\Big\{\sum_{i=0}^{k-1} \gamma_i \ge (1+2\delta)\alpha\tau_n\Big\} - P\Big\{\sum_{i=0}^{k-1} E(\xi_i|\mathcal{F}_{i-1}) \le -\delta\alpha\tau_n\Big\} \\ &=: I_{11} - I_{12}. \end{aligned} \tag{10.1.25}$$

Using condition (10.1.22) and recalling (10.1.9), we also have for large $n$

$$I_{12} = 0. \tag{10.1.26}$$

We are going to estimate $I_{11}$. Let $\varepsilon$ be a small positive number indicated later. Put $v = 3\sqrt{\varepsilon}/(1-3\sqrt{\varepsilon}), u = (1+2\delta)\alpha/(1-v)$. Write

$$\begin{aligned} &E\Big\{\exp\Big[u\Big(\sum_{i=0}^{k-1}\gamma_i\Big)/\tau_n\Big]\Big|\,\mathcal{F}_{k-2}\Big\} \\ &\qquad = \exp\Big\{u\Big(\sum_{i=0}^{k-2}\gamma_i\Big)\Big/\tau_n\Big\}E\{\exp(u\gamma_{k-1}/\tau_n)|\,\mathcal{F}_{k-2}\}. \end{aligned} \tag{10.1.27}$$

Condition (10.1.23) implies

$$upb/\tau_n \to 0. \tag{10.1.28}$$

Hence

$$\begin{aligned}
&E\{\exp(u\gamma_{k-1}/\tau_n)|\,\mathcal{F}_{k-2}\} \\
&\qquad \ge 1+\frac{u^2}{2\tau_n^2}\Big(1-\frac{upb}{\tau_n}\Big)E(\gamma_{k-1}^2|\,\mathcal{F}_{k-2}) \\
&\qquad \ge \exp\Big\{\frac{u^2}{2\tau_n^2}\Big(1-\frac{2upb}{\tau_n}\Big)E(\gamma_{k-1}^2|\,\mathcal{F}_{k-2})\Big\}.
\end{aligned}$$

Inserting it into (10.1.27) we have

$$\begin{aligned}
&E\Big\{\Big[\exp\Big(u\Big(\sum_{i=0}^{k-1}\gamma_i\Big)/\tau_n\Big)\exp\Big(-\frac{u^2}{2\tau_n^2}\Big(1-\frac{2upb}{\tau_n}\Big) \\
&\qquad \cdot E(\gamma_{k-1}^2|\,\mathcal{F}_{k-2})\Big)\Big]\,|\mathcal{F}_{k-2}\Big\} \ge \exp\Big\{u\Big(\sum_{i=0}^{k-2}\gamma_i\Big)/\tau_n\Big\}.
\end{aligned}$$

Thus , by induction,

$$E\Big\{\exp\Big[u\Big(\sum_{i=0}^{k-1}\gamma_i\Big)\Big/\tau_n\Big]\exp\Big[-\frac{u^2}{2\tau_n^2}\Big(1-\frac{2upb}{\tau_n}\Big)\sum_{i=0}^{k-1}E(\gamma_i^2|\,\mathcal{F}_{i-1})\Big]\Big\} \ge 1.$$

Since we also have (10.1.16), the above inequality can be rewritten as

$$\begin{aligned}
E\Big\{\exp\Big[u\Big(\sum_{i=0}^{k-1}\gamma_i\Big)/\tau_n\Big]\Big\} &\ge \exp\Big\{\frac{u^2}{2}\Big(1-\frac{2upb}{\tau_n}-\frac{\varepsilon}{4}\Big)\Big\} \\
&\ge \exp\Big\{\frac{u^2}{2}\Big(1-\frac{\varepsilon}{2}\Big)\Big\}
\end{aligned} \tag{10.1.29}$$

for given $\varepsilon > 0$, provided that $n$ is large enough.

Then the following proof is similar to the corresponding one of Theorem 5.2.2 in Stout (1974) except that some details are different (noting the choice of $v$ and $u$ ). It is omitted.

In addition to Lemmas 10.1.2 and 10.1.3, the following generalization (see Erdös 1959) of the Borel-Cantelli lemma will also be utilized for the proof of our theorem.

**Lemma 10.1.4.** *If $C_1, C_2, \cdots$ are arbitrary events satisfying the conditions*

$$\sum_{n=1}^{\infty} P(C_n) = \infty$$

$$\liminf_{n\to\infty} \sum_{k=1}^{n}\sum_{l=1}^{n} P(C_k C_l)/(\sum_{k=1}^{n} P(C_k))^2 = 1,$$

*then*

$$P\{C_n, \text{i.o.}\} = 1.$$

## 10.2 How big are the increments when the moment generation functions exist?

Let $\{X_n\}$ be a sequence of $\varphi$-mixing random variables with $EX_n = 0$ $(n \geq 1)$. Put $S_n = \sum_{k=1}^{n} X_k$, ${\sigma_k}^2 = EX_k^2$.

**Theorem 10.2.1.** (Lin *1991*) *Suppose that $\{X_n\}$ defined above satisfies the following conditions:*

*(i)* $\underline{\lim}_{n\to\infty} \inf_{m\geq 0} E(X_{m+1} + \cdots + X_{m+n})^2/n > 0$;

*(ii) there exist $t_0, M > 0$, such that $Ee^{tX_k} \leq M$ for every $k$ and any $|t| \leq t_0$;*

*(iii) there exists an $l > 1$, such that $\varphi_n = O(n^{-l})$.*

*Suppose that $\{a_n\}$ is a non-decreasing sequence of positive integers satisfying*

*(a) there exists an $a > 0$ such that $a\log^d n \leq a_n \leq n$ where $d > (3l+1)/(l-1)$. Then, putting*

$$\sigma_{nN}^2 = E(X_{n+1} + \cdots + X_{n+a_N})^2,$$
$$\beta_{nN} = \sigma_{nN}\{2[\log(N/\sigma_{nN}^2) + \log\log N]\}^{1/2},$$
$$S(n,k) = S_{n+k} - S_n,$$

*we have*

$$\limsup_{N\to\infty} \beta_{NN}^{-1}|S(N, a_N)| = 1 \quad a.s. \tag{10.2.1}$$

$$\limsup_{N\to\infty} \max_{1\leq n\leq N} \beta_{nN}^{-1}|S(n, a_N)| = 1 \quad a.s. \tag{10.2.2}$$

$$\limsup_{N\to\infty} \max_{1\leq k\leq a_N} \beta_{NN}^{-1}|S(N, k)| = 1 \quad a.s. \tag{10.2.3}$$

$$\limsup_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \beta_{nN}^{-1}|S(n,k)| = 1 \quad a.s. \tag{10.2.4}$$

*Furthermore, if condition (a) is replaced by*

*(b) there exists an $a > 0$, such that $an^{1/(l+1)} \le a_n \le n$ and*

$$\lim_{n\to\infty} \log(n/a_n)/\log\log n = \infty,$$

*then*

$$\lim_{N\to\infty} \max_{1\le n\le N} \beta_{nN}^{-1}|S(n,a_N)| = 1 \quad a.s. \tag{10.2.5}$$

$$\lim_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \beta_{nN}^{-1}|S(n,k)| = 1 \quad a.s. \tag{10.2.6}$$

**Proof.** From conditions (i)$-$(iii), there exist constants $w_1$ and $w_2$, $0 < w_1 \le w_2 < \infty$ such that

$$w_1 n \le E(X_{m+1} + \cdots + X_{m+n})^2 \le w_2 n \tag{10.2.7}$$

for every integer $m \ge 0$ and $n$ large enough. The right inequality sign is due to that from Lemma 1.2.8

$$\begin{aligned}|EX_iX_j| &\le 2\varphi_{j-i}^{1/l'} E^{1/l'}|X_i|^{l'} E^{1-l/l'}|X_j|^{l'/(l'-1)} \\ &= O((j-i)^{-l/l'})\end{aligned}$$

for $1 < l' < l, \quad i < j$.

First, we prove (10.2.1)$-$(10.2.4). Obviously, it is sufficient to verify

$$\limsup_{N\to\infty} \beta_{NN}^{-1}|S(N,a_N)| \ge 1 \quad \text{a.s.} \tag{10.2.8}$$

$$\limsup_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \beta_{nN}^{-1}|S(n,k)| \le 1 \quad \text{a.s.} \tag{10.2.9}$$

For $B > 1/t_0$, define

$$Y_n = X_n I(|X_n| < B\log n), \quad Y_n' = Y_n - EY_n, \quad T_n = \sum_{k=1}^n Y_k',$$
$$\lambda_{nN}^2 = E(Y_{n+1} + \cdots + Y_{n+a_N})^2, \quad T(n,k) = T_{n+k} - T_n,$$
$$\alpha_{nN} \doteq \lambda_{nN}\{2[\log(N/\lambda_{nN}^2) + \log\log N]\}^{1/2}.$$

From condition (ii), $P\{X_n \ne Y_n,\ i.o.\} = 0$ and for $t'$ such that $t' < t_0$ and $t'B > 1$

$$|EY_n| \le cn^{-t'B},$$

Hence, as $N \to \infty$

$$\max_{1\le n\le N} \max_{1\le k\le a_N} \beta_{nN}^{-1}|S(n,k) - T(n,k)| \to 0 \quad \text{a.s.} \tag{10.2.10}$$

Moreover, it is not difficult to show by (10.2.7) and the definition of $Y_n$ that

$$\sigma_{nN}^2/\lambda_{nN}^2 \to 1 \quad (N \to \infty) \tag{10.2.11}$$

uniformly in $n$.

We are now going to prove (10.2.8), which is equivalent to

$$\limsup_{N\to\infty} \alpha_{NN}^{-1}|T(N, a_N)| \geq 1 \quad \text{a.s.} \tag{10.2.12}$$

Put $h = 2/(3l+1)$. Define $p_n = [n^h]$, $q_n = [\alpha_n p_n]$, where $\{\alpha_n\}$ is a sequence of positive numbers tending to zero slowly enough. By conditions (iii), (a) and the definitions of $p_n$ and $q_n$ , it can be seen that

$$\begin{aligned} \varphi_{q_{a_N}} p_{a_N} (\log N)^2 &\leq c\alpha_{a_N}^{-l} p_{a_N}^{-l+1} (\log N)^2 \\ &\leq c\alpha_{a_N}^{-l} a_N^{-2(l-1)/(3l+1)} (\log N)^2 \to 0, \quad \text{as}\, N \to \infty \end{aligned}$$

provided that $\alpha_n$ tend to zero slowly enough. We also have

$$\sum_{j=1}^{k} \varphi^{1/2}(jp) \leq c\sum_{j=1}^{k} (jp)^{-l/2} \leq ckn^{-l/2} \leq cn^{(3l-1)/(3l+1)-l/2} = o(1)$$

with the help of condition (iii). Hence condition (10.1.1), and further condition (10.1.1'), are satisfied. Similarly, we have

$$\varphi_{q_{a_N}} = o((\log(N/\lambda_{NN}^2) + \log\log N)^{1/2}/\log N)$$

and $p_{a_N}(\log N)\alpha_{nN}/a_N = o(1)$. Hence, we can use Lemma 10.1.3 (choosing $\nu = \varepsilon > 0$), which implies that for large $N$

$$\begin{aligned} P(C_N) &\geq \frac{1}{2}\exp\{-(1+\varepsilon)(1-\varepsilon)^2 \log(N/\lambda_{NN}^2) + \log\log N]\} \\ &\geq c\Big(\frac{a_N}{N\log N}\Big)^{1-\varepsilon}, \end{aligned} \tag{10.2.13}$$

where $C_N = \{\alpha_{NN}^{-1} T(N, a_N) \geq 1 - \varepsilon\}$. Put $n_1 = 1$ and define $n_{k+1} = n_k + 2a_{n_k}$. Noting that the sequence is mixing, we have

$$\Big|\sum_{j=1}^{m}\sum_{k=1}^{m} P(C_{n_j}C_{n_k})\Big/\Big(\sum_{k=1}^{m} P(C_{n_k})\Big)^2 - 1\Big|$$

$$\leq \frac{\sum_{j=1}^{m} P(c_{n_j})(1 - P(C_{n_j})) + 4\sum_{j=1}^{m}\sum_{k=j+1}^{m} P(C_{n_j})\varphi(n_k - n_j - a_{n_j})}{\Big(\sum_{k=1}^{m} P(C_{n_k})\Big)^2}$$

$$\le \Big(1 + 4\sum_{k=2}^{m} \varphi(n_k - n_1 - a_{n_1})\Big) \Big/ \sum_{k=1}^{m} P(C_{n_k}).$$

Using condition (iii) and recalling the definition of $n_k$, one can verify that $\sum\limits_{k=2}^{\infty} \varphi(n_k - n_1 - a_{n_1}) < \infty$. Moreover $\sum_{k=1}^{\infty} P(C_{n_k}) = \infty$ since

$$\sum_{n=n_k+1}^{n_{k+1}} (n \log n)^{-1} \le (n_k \log n_k)^{-1}(n_{k+1} - n_k)$$
$$= (n_k \log n_k)^{-1} 2a_{n_k} \le cP(C_{n_k}).$$

Hence, from Lemma 10.1.4, $P\{C_{n_k}, i.o.\} = 1$, which yields (10.2.12). The proof of (10.2.8) is completed.

Furthermore, we want to prove (10.2.9). From (10.2.10) and (10.2.11), it is sufficient to show that

$$\limsup_{N\to\infty} \max_{1\le n\le N} \max_{1\le k\le a_N} \alpha_{nN} |T(n,k)| \le 1 \quad \text{a.s.} \tag{10.2.14}$$

Let $r = r(\varepsilon) > 1$ be a positive integer indicated later. Put $R = [a_N/r], n_r = R[n/R]$. We have

$$|\lambda_{nN}^2 / E(Y_{n_r+1} + \cdots + Y_{(n+a_N)_r})^2 - 1| \le \frac{2w_2(2r^{1/2}+1)}{w_1(r-1)} \tag{10.2.15}$$

for large $N$ by (10.2.7) and (10.2.11). Write

$$|T_{n+k} - T_n| \le |T_{n+k} - T_{(n+k)_r}| + |T_{(n+k)_r} - T_{n_r}| + |T_{n_r} - T_n|. \tag{10.2.16}$$

Consider the second term of the right hand side of (10.2.16). We have

$$P\Big\{ \max_{1\le n\le N} \max_{1\le k\le a_N} \alpha_{nN}^{-1} |T_{(n+k)_r} - T_{n_r}| \ge 1 + \varepsilon/3 \Big\}$$
$$\le cr\frac{N}{a_N} \max_{1\le n\le N} P\Big\{ \max_{1\le k\le a_N} |T_{(n+k)_r}$$
$$- T_{n_r}| \ge (1+\varepsilon/3)\alpha_{nN} \Big\}. \tag{10.2.17}$$

Obviously, $|(n + a_N)_r - n_r| \le a_N(1 + 1/r)$. We can use (10.1.4) of Lemma 10.1.2, since

$$(\log N) a_N \varphi_{q_{a_N}} = o(\alpha_{nN}),$$
$$\alpha_{nN} = o(E(T_{(n+a_N)_r} - T_{n_r})^2 / p_{a_N} \log N).$$

Noting (10.2.15) and choosing $r$ to be large enough, we know the right hand side of (10.2.17) does not exceed

$$
\begin{aligned}
cr\frac{N}{a_N}\exp\Big\{-(1-\varepsilon/3)(1+\varepsilon/3)^2\alpha_{nN}^2/E(T_{(n+a_N)_r}-T_{n_r})^2\Big\} \\
\leq cr\frac{N}{a_N}\Big(\frac{a_N}{N\log N}\Big)^{1+\varepsilon/4} \\
= cr\Big(\frac{a_N}{N}\Big)^{\varepsilon/4}(\log N)^{-(1+\varepsilon/4)}
\end{aligned}
$$

for large $N$. Let $N_1 = 1$ and define $N_{j+1}$ by $a_{N_{j+1}} = \min\{a_n : a_n \geq [\theta^j]\}(\theta > 1)$. Obviously, $N_{j+1} \geq [\theta^j]$ since $a_N \leq N$. Thus

$$
\sum_{j=1}^{\infty} P\Big\{\max_{1\leq n\leq N_j}\max_{1\leq k\leq a_{N_j}}\alpha_{nN_j}^{-1}|T_{(n+k)_r}-T_{n_r}|\geq 1+\frac{\varepsilon}{3}\Big\}<\infty
$$

which implies

$$
\limsup_{j\to\infty}\max_{1\leq n\leq N_j}\max_{1<k\leq a_{N_j}}\alpha_{nN_j}^{-1}|T_{(n+k)_r}-T_{n_r}|\leq 1+\frac{\varepsilon}{3}\quad\text{a.s.}\tag{10.2.18}
$$

From conditions (i)-(iii) and (10.2.11) and the definition of $N_j$, it is easy to see that there exists a constant $G > 0$, such that

$$
\left|\varliminf_{j\to\infty}\frac{\lambda_{nN_{j+1}}^2}{\lambda_{nN_j}^2}-1\right|\leq G(\theta-1)^{1/2}.\tag{10.2.19}
$$

Now, choosing $\theta$ near enough to 1 and noting that the ranges in the two max's in (10.2.18) enlarge as $j$ increases we have from (10.2.18)

$$
\limsup_{N\to\infty}\max_{1\leq n\leq N}\max_{1\leq k\leq a_N}\alpha_{nN}^{-1}|T_{(n+k)_r}-T_{n_r}|\leq 1+\frac{\varepsilon}{2}\quad\text{a.s.}\tag{10.2.20}
$$

We turn to the first term in the right hand side of (10.2.16). Write

$$
\begin{aligned}
P\Big\{\max_{1\leq n\leq N}\max_{1\leq k\leq a_N}\alpha_{nN}^{-1}|T_{n+k}-T_{(n+k)_r}|\geq\frac{\varepsilon}{6}\Big\} \\
\leq\frac{rN}{a_N}\max_n\max_k P\Big\{\max_{(n+k)_r<j\leq(n+k)_r+R}|T_j \\
-T_{(n+k)_r}|\geq\frac{\varepsilon}{6}\alpha_{nN}\Big\}.
\end{aligned}\tag{10.2.21}
$$

Using a similar proof to (10.2.18), one can also employ (10.1.4) of Lemma 10.1.2. By recalling (10.2.7) and (10.2.11) and choosing $r = r(\varepsilon)$ to be

large enough, the right hand side of (10.2.21) does not exceed

$$
\begin{aligned}
3\frac{rN}{a_N}\exp\Big\{-&\frac{(1-6\varepsilon)\varepsilon^2\lambda_{nN}^2}{36E(T_{(n+k)_r+R}-T_{(n+k)_r})^2}\log\frac{N\log N}{\lambda_{nN}^2}\Big\} \\
&\le \frac{3rN}{a_N}\exp\Big\{-cr\varepsilon^2\log\frac{N\log N}{a_N}\Big\} \\
&\le \frac{3rN}{a_N}\Big(\frac{a_N}{N\log N}\Big)^{cr\varepsilon^2} \le c\log^{-2}N
\end{aligned}
$$

for all large $N$. Thus

$$\limsup_{j\to\infty}\max_{1\le n\le N_j}\max_{1\le k\le a_{N_j}}\alpha_{nN_j}^{-1}|T_{n+k}-T_{(n+k)_r}|\le\frac{\varepsilon}{6}\quad\text{a.s.}$$

By imitating the procedure from (10.2.18) to (10.2.20), we have

$$\limsup_{N\to\infty}\max_{1\le n\le N}\max_{1\le k\le a_N}\alpha_{nN}^{-1}|T_{n+k}-T_{(n+k)_r}|\le\frac{\varepsilon}{4}\quad\text{a.s.}\tag{10.2.22}$$

Obviously, we also have

$$\limsup_{N\to\infty}\max_{1\le n\le N}\alpha_{nN}^{-1}|T_n-T_{n_r}|\le\frac{\varepsilon}{4}\quad\text{a.s.}$$

Therefore, (10.2.14), and further (10.2.9), are proved.

Finally, we prove (10.2.5) and (10.2.6). For this purpose, from (10.2.2) and (10.2.4) proved, it is sufficient to verify that

$$\liminf_{N\to\infty}\max_{1\le n\le N}\beta_{NN}^{-1}|S(n,a_N)|\ge 1\quad\text{a.s.}\tag{10.2.23}$$

By (10.2.10) and (10.2.11), (10.2.23) is equivalent to

$$\liminf_{N\to\infty}\max_{1\le n\le N}\alpha_{nN}^{-1}|T(n,a_N)|\ge 1\quad\text{a.s.}\tag{10.2.24}$$

We have for large $N$

$$
\begin{aligned}
P\Big\{&\max_{1\le n\le N}\alpha_{nN}^{-1}|T(n,a_N)|\le 1-\varepsilon\Big\} \\
&\le P\Big\{\max_{1\le j\le[N/a_N]^{1-\varepsilon/2}}\alpha_{2ja_N,N}^{-1}|T(2ja_N,a_N)|\le 1-\varepsilon\Big\} \\
&\le \prod_{j=1}^{[N/a_N]^{1-\varepsilon/2}} P\{\alpha_{2ja_N,N}^{-1}|T(2ja_N,a_N)|\le 1-\varepsilon\Big\} \\
&\quad + \Big(\frac{N}{a_N}\Big)^{1-\varepsilon/2}\varphi(a_N)
\end{aligned}
$$

(by (10.2.13) and condition (b))

$$\begin{aligned}
&\leq \Big\{1 - c\Big(\frac{a_N}{N\log N}\Big)^{1-\varepsilon}\Big\}^{[N/a_N]^{1-\varepsilon/2}} + c\Big(\frac{N}{a_N}\Big)^{-\varepsilon/2} \\
&\leq 2\exp\Big\{-c\Big(\frac{N}{a_N}\Big)^{-\varepsilon/2}(\log N)^{-(1-\varepsilon)}\Big\} + c\Big(\frac{N}{a_N}\Big)^{-\varepsilon/2} \\
&\leq c(\log N)^{-2}.
\end{aligned}$$

The last inequality is due to condition (b). Thus, if $N_j$ is defined as above, we have

$$\liminf_{j\to\infty} \max_{1\leq n\leq N_j} \alpha_{nN_j}^{-1}|T(n, a_{N_j})| \geq 1 \quad \text{a.s.} \tag{10.2.25}$$

Considering $N_j < N \leq N_{j+1}$, we get

$$\begin{aligned}
&\liminf_{N\to\infty} \max_{1\leq n\leq N} \alpha_{nN}^{-1}|T(n, a_N)| \\
&\qquad \geq \liminf_{j\to\infty} \max_{1\leq n\leq N_j} (\alpha_{nN_j}^{-1}|T(n, a_{N_j})|)(\alpha_{nN_j}\alpha_{nN}^{-1}) \\
&\qquad\quad - \limsup_{N\to\infty} \max_{1\leq n\leq N} \max_{1\leq k\leq a_N - a_{N_j}} \alpha_{nN}^{-1}|T(n, k)|.
\end{aligned}$$

The first term in the right hand side is a.s. $\geq 1 - G'(\theta - 1)^{1/2}$ for some $G' > 0$ by (10.2.25) and (10.2.19). The latter is a.s.$\leq G''(1 - 1/\theta)^{1/2}$ for some $G'' > 0$ by (10.2.14). Letting $\theta \downarrow 1$, we obtain (10.2.24). The theorem is proved.

**Remark 10.2.1.** When $l$ in condition (iii) is large enough, in other words, $\varphi_n$ tends to zero at a great rate, $a_n$ can be $O(\log^{3+\varepsilon} n)$, for any given $\varepsilon > 0$. For an independent sequence, it is required that $a_n/\log n \to \infty$ as $n \to \infty$ (see Lin 1988).

**Remark 10.2.2.** In the theorem, we don't require that $a_n/n$ is non-increasing. But it is assumed even if a sequence is independent (cf. Csörgő and Révész 1981). In fact, this condition is not realistic, since either $a_n = n$ for all $n$ or $a_n = m \bigwedge n$ for some fixed $m$ when the condition is added.

## 10.3 How big are the increments when the moment generating functions do not exist?

Let $\{X_n\}$ be a $\varphi$-mixing sequence mentioned in the beginning of the above section.

**Theorem 10.3.1.**(Lin 1989) *Suppose that $\{X_n\}$ satisfies condition (i) in Theorem 10.2.1 and*

*(ii)′ there exists a non-decreasing continuous function $H(x), x > 0$, such that*

$$\sum_{n=1}^{\infty} P\{H(|X_n|) > \delta n\} < \infty \quad for\ any\ \delta > 0, \tag{10.3.1}$$

$$\overline{\lim_{k\to\infty}} \frac{1}{k} \sum_{j=n+1}^{n+k} E(H(|X_j|))^{\beta} \le M < \infty \tag{10.3.2}$$

*uniformly in $n$ for any $\beta < 1$,*

$$x^{-(2+\gamma)} H(x) \text{ is non-decreasing for some } \gamma > 0, \tag{10.3.3}$$

$$\underline{\lim_{x\to\infty}}\ H(x/2)/H(x) > 0, \tag{10.3.4}$$

*(iii)′ there exists an $l \ge 1 + \frac{4}{\gamma}$ such that $\varphi_n = O(n^{-l})$.*

*Let $\{a_n\}$ be a non-decreasing sequence of positive integers satisfying*

*(a)′ there exists such an $a > 0$ that $a(\text{inv}H(n))^d \le a_n \le n$ where $d = 2(l+1)/(l-1)$.*

*Then the conclusions of Theorem 10.2.1 remain true.*

**Proof.** The proof of the theorem is similar to that of Theorem 10.2.1. We outline the main differences. It is easy to verify that condition (10.3.1) can be replaced by

$$\sum_{n=1}^{\infty} P\{H(|X_n|) > \delta_n n\} < \infty \tag{10.3.5}$$

for some $\delta_n \downarrow 0$. Define $Y_n = X_n I(|X_n| \le \text{inv}H(\delta_n n)), Y_n' = Y_n - EY_n, T_n = \sum_{k=1}^{n} Y_k'$. By (10.3.5) we have

$$P\{X_n \ne Y_n, i.o.\} = 0. \tag{10.3.6}$$

Without loss of generality we can choose $\delta_n$ such that $n^{-\varepsilon} = o(\delta_n)$ for any $\varepsilon > 0$. From this and using the conditions $EX_j = 0$, (10.3.2) and (10.3.3),

we have for all large $n$,

$$\begin{aligned}\sum_{j=n+1}^{n+k}|EY_j| &\le c\sum_{j=n+1}^{n+k}\int_{\{H(|X_j|)>\delta_j j\}}(H(|X_j|))^{\frac{1}{2+\gamma}}dP\\ &\le c\sum_{j=n+1}^{n+k}(\delta_j j)^{-\frac{2+\gamma}{4+\gamma}}E(H(|X_j|))^{\frac{1}{2+\gamma}+\frac{2+\gamma}{4+\gamma}}\\ &\le c\sum_{j=n+1}^{n+k} j^{-\frac{3+\gamma}{6+\gamma}}E(H(|X_j|))^{\frac{\gamma^2+5\gamma+8}{\gamma^2+6\gamma+8}}\\ &\le ck^{-\frac{3+\gamma}{6+\gamma}}\Big\{\sum_{j=n+1}^{n+k}E(H(|X_j|))^{\frac{\gamma^2+5.5\gamma+8}{\gamma^2+6\gamma+8}}\Big\}^{\frac{\gamma^2+5\gamma+8}{\gamma^2+5.5\gamma+8}}\\ &\le ck^{\frac{3}{6+\gamma}}. \end{aligned} \tag{10.3.7}$$

Consequently (10.2.8) is also equivalent to (10.2.12). Put $h = 1/(l+1)$, $p_n = [n^h], q_n = [\alpha_n p_n]$, where $\alpha_n$ are positive numbers tending to zero slowly enough. Using conditions (iii)$'$ and (a)$'$, we obtain

$$\begin{aligned}&\varphi_{q_{a_N}}p_{a_N}(\text{inv}H(\delta_N N))^2\\ &\quad\le c\alpha_{a_N}^{-l}a_N^{-h(l-1)}(\text{inv}H(\delta_N N))^2\\ &\quad\le c\alpha_{a_N}^{-l}(\text{inv}H(\delta_N N)/\text{inv}H(N))^2.\end{aligned}$$

By condition (10.3.4) one can get $\text{inv}H(\delta_N N)/\text{inv}H(N)\to 0$ (see Lin and Lu, 1992, (2.3.12)). Hence ,we have

$$\varphi_{q_{a_N}}p_{a_N}(H(\delta_N N))^2\to 0,\quad \text{as } N\to\infty.$$

provided that $\alpha_N$ tends to zero slowly enough. Moreover the other conditions in Lemma 10.1.2 are also satisfied. Then we have (10.2.13) as well. The following proof is similar to that of Theorom 10.2.1, hence, is omitted.

**Remark 10.3.1.** When $l$ in condition (iii)' is large enough, i.e., $\varphi_n$ tends to zero at a great rate, $a_n$ can be $O((\text{inv}H(n))^{2+\varepsilon})$ for any given $\varepsilon > 0$. For an independent sequence, it is required that $a_n \ge c(\text{inv}H(n))^2/\log n$ (see Lin 1987).

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# Chapter 11 Strong Approximations for Mixing Random Fields

Let $\{X_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ be a stationary mixing random field with $EX_{\mathbf{j}} = 0$. The various definitions of mixing have been given in Chapter 6. We shall also employ the notations $\mathcal{A} \subset \mathcal{E} \subset \mathcal{B}^d, C_{\mathbf{j}}, \mathcal{A}_\delta, A(\delta)$, etc, which have been posed there. We define the partial-sum process by

$$S_n(A) = \sum_{\mathbf{j}} |nA \cap C_{\mathbf{j}}| X_{\mathbf{j}} \quad \text{for any } A \in \mathcal{A} \tag{11.0.1}$$

where $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}^d, \mathbf{n}A = \{(n_1 x_1, \cdots, n_d x_d),\ \mathbf{x} = (x_1, \cdots, x_d) \in A\}$.

The main subject of strong approximations of set-indexed processes is to establish a new probability space, on which there exists an independent identically distributed centered Gaussian field $\{Y_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ with covariance $\sigma^2 = EX_{\mathbf{1}}^2 + 2\sum_{\mathbf{j}} EX_{\mathbf{j}+\mathbf{1}} X_{\mathbf{1}}$, and without changing its joint distribution, the field $\{X_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ can be redefined on this new probability space, such that

$$\begin{aligned} D_n &= \sup_{A \in \mathcal{A}} \Big\{ \sum_{\mathbf{j} \in \mathbb{N}^d} |nA \cap C_{\mathbf{j}}| (X_{\mathbf{j}} - Y_{\mathbf{j}}) \Big\} \\ &= O(n^{d/2-\varepsilon}) \quad \text{or} \quad O(n^{d/2} (\log n)^{-a}) \qquad (a, \varepsilon > 0), \end{aligned} \tag{11.0.2}$$

under some conditions on $\mathcal{A}$ and $\{X_{\mathbf{j}}\}$.

Denote

$$\begin{aligned} &|\mathbf{n}| = n_1 n_2 \cdots n_d, \qquad \text{if} \quad \mathbf{n} = (n_1, \cdots, n_d), \\ &G_\beta = \Big\{ \mathbf{n} = (n_1, \cdots, n_d),\ n_i \geq \prod_{k \neq i} n_k^\beta,\ i = 1, 2, \cdots, d \Big\}, \end{aligned} \tag{11.0.3}$$

for $0 < \beta \leq 1$.

Berkes and Morrow (1981) first discussed the strong approximation for a weakly stationary $\alpha$-mixing random field $\{X_{\mathbf{j}},\ \mathbf{j} \in \mathbb{N}^d\}$. Suppose that

$EX_1 = 0, E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$ and $\alpha(n) = O(n^{-d(1+\varepsilon)(1+2/\delta)})$. They proved that for any $\mathbf{n} \in G_\beta$ and some $\lambda > 0$

$$\sup_{1\le \mathbf{j}\le \mathbf{n}} \Big|\sum_{\mathbf{i}\le \mathbf{j}}(X_\mathbf{i} - Y_\mathbf{i})\Big| = O(|\mathbf{n}|^{1/2-\lambda}) \quad \text{a.s.} \tag{11.0.4}$$

This result was improved by Strittmatter (1990), in which the condition "$\mathbf{n} \in G_\beta$" is left out and the following result is obtained: If

$$N(\varepsilon) := \operatorname{Card}(\mathcal{A}(\varepsilon)) \le c\varepsilon^{-u}, \qquad u < \frac{\delta s - 4 - 2\delta}{d(4+\delta)} - 2, \tag{11.0.5}$$

$$\begin{aligned} b(\varepsilon) := \sup\Big\{&n^{-d}|((\mathbf{n}A)^{n\varepsilon} \cap (\mathbf{n}A^c)^{n\varepsilon})| : \\ &A \in \bigcup_{\eta>0} A_\eta,\ n \ge 1/\varepsilon\Big\} \\ \le c\,\varepsilon^h &\qquad \text{for some } 0 < h \le 1, \end{aligned} \tag{11.0.6}$$

$$\alpha(n) = O(n^{-s}) \qquad \text{for some } s > 1 + (17/2)^{d-1}/(1\wedge\delta), \tag{11.0.7}$$

then for some $\gamma > 0$

$$\sup_{A\in\mathcal{A}} \sum_{\mathbf{j}\in\mathbb{N}^d} |nA \cap C_\mathbf{j}|(X_\mathbf{j} - Y_\mathbf{j}) = O(n^{d/2-\gamma}) \quad \text{a.s.} \tag{11.0.8}$$

The class $\mathcal{A}$ of subsets in the above results which satisfies $N(\varepsilon) \le c\varepsilon^{-u}$ is the smaller subset class. Su (1992) proved a strong approximation for a $\varphi$-mixing strictly stationary random field with $H(\varepsilon) \le c\varepsilon^{-r}$ for some $r > 0$, and $\mathbf{n} \in G_\beta$. We shall introduce Su's results in Section 11.1 and Strittmatter's results in Section 11.2.

## 11.1 Strong approximations of a $\varphi$-mixing random field

For simplicity, we only give the results for the case of $d = 2$.

**Theorem 11.1.1.** (Su 1992) *Let $\{X_\mathbf{j}, \mathbf{j} \in \mathbb{N}^2\}$ be a strictly stationary $\varphi$-mixing random field with $EX_\mathbf{j} = 0$ and $E|X_\mathbf{j}|^{2+\delta} < \infty$ for some $\delta > 0$. Suppose that the class $\mathcal{A}$ of subsets of $\mathcal{B}^2$ satisfies (11.0.6) and the entropy condition*

$$H(\varepsilon) \le c\varepsilon^{-r} \quad \textit{for some} \quad 0 < r < \delta/(4(1+\delta)), \tag{11.1.1}$$

*and suppose that*

$$\varphi(x) = O(x^{-q}) \quad \textit{for some} \quad q > \frac{4(2+\delta)}{\delta - r(4+\delta)} - 2. \tag{11.1.2}$$

*Then*

$$\sigma^2 := EX_{\mathbf{1}}^2 + 2\sum_{\mathbf{v}\neq\mathbf{1}} EX_{\mathbf{1}}X_{\mathbf{v}} = O(1), \tag{11.1.3}$$

*further, without changing the distribution of* $\{X_{\mathbf{j}},\ \mathbf{j}\in\mathbb{N}^2\}$, *we can redefine the field* $\{X_{\mathbf{j}},\ \mathbf{j}\in\mathbb{N}^2\}$ *on a richer probability space together with an independent normal random field* $\{Y_{\mathbf{j}},\ \mathbf{j}\in\mathbb{N}^2\}$ *with* $EY_{\mathbf{j}}=0, EY_{\mathbf{j}}^2=\sigma^2$, *such that*

$$\sup_{A\in\mathcal{A}}\sum_{\mathbf{j}\in\mathbb{N}^2}|nA\cap C_{\mathbf{j}}|(X_{\mathbf{j}}-Y_{\mathbf{j}}) = O(|n|(\log|n|)^{-\sigma_1}) \quad a.s. \tag{11.1.4}$$

*for every* $n\in G_\beta, \delta/(2(2+\delta))<\beta\le 1$, *where* $\sigma_1=\sigma_1(r,q,\delta)>0$.

First, we prove the *Bernstein inequality* for a $\varphi$-mixing random field.

**Lemma 11.1.1.** *Let* $\{X_{\mathbf{j}},\ \mathbf{j}\in\mathbb{N}^d\}$ *be a* $\varphi$*-mixing random field with* $EX_{\mathbf{j}}=0, |X_{\mathbf{j}}|\le\Delta_{\mathbf{n}}$ *a.s.* $\mathbf{1}\le\mathbf{j}\le\mathbf{n}$. *Denote* $\sigma_{\mathbf{n}}=\max_{\mathbf{1}\le\mathbf{j}\le\mathbf{n}}\|X_{\mathbf{j}}\|_2$. *If* $\sum_{n=1}^{\infty}\varphi^{1/2}(2^n)<\infty$, *then for any* $A\in\mathcal{B}^d$ *we have*

$$P\Big\{\Big|\sum_{\mathbf{j}}|\mathbf{n}A\cap C_{\mathbf{j}}|X_{\mathbf{j}}\Big|\ge x_n\Big\}$$
$$\le 2\exp\Big\{\frac{c|\mathbf{n}|\varphi(m)}{m^d}-\alpha x_n+c\alpha^2|\mathbf{n}A|\sigma_{\mathbf{n}}^2\Big\}, \tag{11.1.5}$$

*where* $x_n>0$, $m\le\min_{1\le i\le d}n_i$ *and* $2^d\alpha m^d\Delta_{\mathbf{n}}\le 1/2$.

*Particularly, if* $\alpha=1/(2^{d+1}m^d\Delta_{\mathbf{n}})$, *then*

$$P\Big\{\Big|\sum_{\mathbf{j}}|\mathbf{n}A\cap C_{\mathbf{j}}|X_{\mathbf{j}}\Big|\ge x_n\Big\}$$
$$\le 2\exp\Big\{\frac{c|\mathbf{n}|\varphi(m)}{m^d}-\frac{x_n}{2^{d+1}m^d\Delta_{\mathbf{n}}}+\frac{c|\mathbf{n}A|\sigma_{\mathbf{n}}^2}{(2^{d+1}m^d\Delta_{\mathbf{n}})^2}\Big\}.$$

**Proof.** Put $\mathbf{N}=(N_1,\cdots,N_d)$ such that $2m(N_i-1)\le n_i\le 2mN_i$, $i=1,\cdots,d$. Denote $V=\{(v_1,\cdots,v_d): v_i=1 \text{ or } 2,\ i=1,\cdots,d\}$. Define

$$I_{\mathbf{v},\mathbf{k}}=[l_1(1),l_2(1)]\times\cdots\times[l_1(d),l_2(d)],\quad \mathbf{v}\in V,\ \mathbf{1}\le\mathbf{k}\le\mathbf{N},$$

where

$$l_1(i)=\{(2k_i+v_i-3)m+1\}\wedge n_i,$$
$$l_2(i)=\{l_1(i)+m-1\}\wedge n_i,\ i=1,2,\cdots,d.$$

Moreover, denote

$$I_{\mathbf{v}} = \sum_{1 \leq \mathbf{k} \leq \mathbf{N}} I_{\mathbf{v},\mathbf{k}},$$
$$A_{\mathbf{v},\mathbf{k}} = \sum_{\mathbf{j}} |nA \cap I_{\mathbf{v},\mathbf{k}} \cap C_{\mathbf{j}}| X_{\mathbf{j}},$$
$$B_{\mathbf{v},\mathbf{N}} = \sum_{1 \leq \mathbf{k} \leq \mathbf{N}} A_{\mathbf{v},\mathbf{k}},$$
$$S = \sum_{\mathbf{v} \in V} B_{\mathbf{v},\,\mathbf{N}}.$$

Note that $e^x \leq 1 + x + x^2$ if $|x| \leq 1/2$ and $1 + x \leq e^x$ for any $x$. Therefore if $2^d \alpha\, m^d \Delta_{\mathbf{n}} \leq 1/2$, we have

$$E \exp(2^d \alpha A_{\mathbf{v},\mathbf{k}}) \leq \exp((2^d \alpha)^2 E A^2_{\mathbf{v},\mathbf{k}}).$$

By Lemmas 1.2.10 and 6.2.3 we have

$$\begin{aligned}
&E \exp(2^d \alpha B_{\mathbf{v},\mathbf{N}}) \\
&\quad = E \exp\Big(2^d \alpha \sum_{\mathbf{k} \neq \mathbf{N}} A_{\mathbf{v},\mathbf{k}}\Big) \exp(2^d \alpha A_{\mathbf{v},\mathbf{N}}) \\
&\quad \leq E \exp\Big(2^d \alpha \sum_{\mathbf{k} \neq \mathbf{N}} A_{\mathbf{v},\mathbf{k}}\Big) E \exp(2^d \alpha A_{\mathbf{v},\mathbf{N}}) \\
&\qquad + 2\varphi(m) \| \exp(2^d \alpha A_{\mathbf{v},\mathbf{N}}) \|_\infty E \exp\Big(2^d \alpha \sum_{\mathbf{k} \neq \mathbf{N}} A_{\mathbf{v},\mathbf{k}}\Big) \\
&\quad \leq (1 + 2e^{1/2} \varphi(m)) \exp(c(2^d \alpha)^2 |\, \mathbf{n}A \cap I_{\mathbf{v},\mathbf{N}} | \sigma_n^2) E \exp\Big(2^d \alpha \sum_{\mathbf{k} \neq \mathbf{N}} A_{\mathbf{v},\mathbf{k}}\Big) \\
&\quad \leq (1 + 2e^{1/2} \varphi(m))^{|\mathbf{N}|} \exp(c(2^d \alpha)^2 |\mathbf{n}A \cap I_{\mathbf{v}}| \sigma_n^2)
\end{aligned} \tag{11.1.6}$$

where $\|f(x)\|_\infty$ means the superior norm of $f(x)$. From the convexity of an exponential function and (11.1.6), it follows that

$$\begin{aligned}
E \exp(\alpha S) &\leq \frac{1}{2^d} \sum_{\mathbf{v} \in V} E \exp(2^d \alpha B_{\mathbf{v},\,\mathbf{N}}) \\
&\leq (1 + 2e^{1/2} \varphi(m))^{|\mathbf{N}|} \exp(c(2^d \alpha)^2 |\mathbf{n}A| \sigma_n^2).
\end{aligned} \tag{11.1.7}$$

Since $|\mathbf{N}| \leq (3/2)^d |\mathbf{n}| / m^d$ and (11.1.7) holds also true for $E \exp(-\alpha S)$, we obtain (11.1.5) by the Markov inequality.

**Proof of Theorem 11.1.1.**

First, under the conditions of Theorem 11.1.1, it is easy to see that (11.1.3) holds true and

$$\lim_{\substack{|\mathbf{n}|\to\infty \\ \mathbf{n}\in G_\beta}} E\Big(\sum_{1\le \mathbf{j}\le \mathbf{n}} X_{\mathbf{j}}\Big)^2/|\mathbf{n}| = \sigma^2 < \infty. \tag{11.1.8}$$

In order to prove (11.1.4) for any $\mathbf{n} \in G_\beta, \delta/(2+\delta) < \beta \le 1$, we shall use the truncation, nesting and blocking techniques in the following lemmas.

**1. Truncation** Take $\tau$ small enough (specified later on), let

$$X'_{\mathbf{j}} = X_{\mathbf{j}} I(|X_{\mathbf{j}}| \le |\mathbf{j}|^{(1+\tau)/(2+\delta)}),$$
$$S'_{\mathbf{n}}(A) = \sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}| X'_{\mathbf{j}}.$$

**Lemma 11.1.2.** *We have*

$$\sup_{A\in\mathcal{A}} |S_{\mathbf{n}}(A) - S'_{\mathbf{n}}(A)| = O(1) \quad a.s. \tag{11.1.9}$$

$$\sum_{\mathbf{j}\le n\mathbf{1}} E|X_{\mathbf{j}} - X'_{\mathbf{j}}| = O(n^{d/(2+\delta)}) \quad a.s. \tag{11.1.10}$$

**Proof.** Since

$$P\{X_{\mathbf{j}} \ne X'_{\mathbf{j}}\} = P\{|X_{\mathbf{j}}| \ge |\mathbf{j}|^{(1+\tau)/(2+\delta)}\} \le c|\mathbf{j}|^{-(1+\tau)},$$

we have $P\{X_{\mathbf{j}} - X'_{\mathbf{j}} \ne 0, \text{ i.o.}\} = 0$ by the Borel-Cantelli lemma. Thus for every $n \in \mathbb{N}$

$$\sup_A |S_{\mathbf{n}}(A) - S'_{\mathbf{n}}(A)| \le \sum_{\mathbf{j}\in n\mathbf{1}} |X_{\mathbf{j}} - X'_{\mathbf{j}}| \le c < \infty \quad \text{a.s.}$$

and

$$\sum_{\mathbf{j}\le n\mathbf{1}} E|X_{\mathbf{j}} - X'_{\mathbf{j}}| \le \sum_{\mathbf{j}\le n\mathbf{1}} \Big(|\mathbf{j}|^{-(1+\tau)(1+\delta)/(2+\delta)}\Big) = O(n^{d/(2+\delta)}).$$

**2. Nesting** For any given $\mathbf{n} = (n_1, n_2) \in G_\beta, \delta/(2(2+\delta)) < \beta \le 1$, let $a, b$ be least positive integers such that

$$2^a \ge (\log|\mathbf{n}|)^{1/r'}, \quad 2^b \ge |\mathbf{n}|^{\frac{1}{2}(\tau + \frac{4+\delta}{2+\delta})}, \tag{11.1.11}$$

where $r' > r$ is specified later on.

**Lemma 11.1.3.** *We have*

$$\sup_A |S'_{\mathbf{n}}(A) - S'_{\mathbf{n}}(A(2^{-b}))| = O(|n|^{\frac{1}{2}(1-\frac{\tau\delta}{2+\delta})}) \quad a.s.$$

**Proof.** For any $A \in \mathcal{A}$, by conditions (11.0.6), (11.1.1) and (11.1.11), we have

$$\begin{aligned}
&|S'_{\mathbf{n}}(A) - S'_{\mathbf{n}}(A(2^{-b}))| \\
&\quad \leq \sum_{\mathbf{j}} \Big| |\mathbf{n}A \cap C_{\mathbf{j}}| - |\mathbf{n}A(2^{-b}) \cap C_{\mathbf{j}}| \Big| |X'_{\mathbf{j}}| \\
&\quad \leq c|n||A \triangle A(2^{-b})| \cdot |n|^{\frac{1+\tau}{2+\delta}} = O(|n|^{\frac{1}{2}(1-\frac{\tau\delta}{2+\delta})}).
\end{aligned}$$

**Lemma 11.1.4.** *For every* $\mathbf{n} \in G_\beta$

$$\sup_{A \in \mathcal{A}} |S'_{\mathbf{n}}(A(2^{-b})) - S'_{\mathbf{n}}(A(2^{-a}))| = O(|\mathbf{n}|^{\frac{1}{2}}(\log |\mathbf{n}|)^{-\sigma_1}) \quad a.s.$$

**Proof.** Take $\tau, r'$ and $\tau'$ small enough such that

$$\Big(\frac{\delta - 2\tau}{2(2+\delta)} - \frac{1}{2}\tau'\Big(\tau + \frac{4+\delta}{2+\delta}\Big)\Big)(q+1) > 1, \quad \tau' - \tau > r' > r. \tag{11.1.12}$$

Denote $m_i = [|n|^{\frac{\delta-2\tau}{4(2+\delta)}} 2^{-\frac{i\tau'}{2}}]$. Since $n \in G_\beta$, it is clear that $m_i \leq n_1 \wedge n_2, i = 1, 2$. On the other hand we have

$$|X'_{\mathbf{j}} - EX'_{\mathbf{j}}| \leq 2|n|^{(1+\tau)/(2+\delta)}, \quad \mathbf{j} \leq \mathbf{n}.$$

Take $\Delta_{\mathbf{n}} = 2|\mathbf{n}|^{(1+\tau)/(2+\delta)}$ in Lemma 11.1.1, we obtain

$$\begin{aligned}
&P\{|S'_{\mathbf{n}}(A(2^{-(i+1)})) - S'_{\mathbf{n}}(A(2^{-i}))| \geq |n|^{1/2}2^{-i\tau}\} \\
&\quad \leq P\Big\{\Big|\sum_{\mathbf{j}} |n(A(2^{-(i+1)})\backslash A(2^{-i}))|(X'_{\mathbf{j}} - EX'_{\mathbf{j}})\Big| > \frac{1}{4}|n|^{1/2}2^{-i\tau}\Big\} \\
&\qquad + P\Big\{\Big|\sum_{\mathbf{j}} |n(A(2^{-i})\backslash A(2^{-(i+1)}))|(X'_{\mathbf{j}} - EX'_{\mathbf{j}})\Big| > \frac{1}{4}|n|^{1/2}2^{-i\tau}\Big\} \\
&\quad \leq c\exp(-c2^{i(\tau'-\tau)}).
\end{aligned} \tag{11.1.13}$$

Therefore from (11.1.1) it follows that

$$\sum_{\mathbf{n}\in G_\beta} P\Big\{\sup_{A\in\mathcal{A}} |S'_{\mathbf{n}}(A(2^{-b})) - S'_{\mathbf{n}}(A(2^{-a}))| \geq |n|^{1/2}(\log|n|)^{-\tau/\tau'}\Big\}$$

$$\leq \sum_{\mathbf{n}\in G_\beta}\sum_{i=a}^{b}\sum_{A\in\mathcal{A}} P\{|S'_{\mathbf{n}}(A(2^{-(i+1)})) - S'_{\mathbf{n}}(A(2^{-i}))| \geq |n|^{1/2}2^{-i\tau}\}$$

$$\leq c\sum_{\mathbf{n}\in G_\beta}\sum_{i=a}^{b}\exp\Big(H(2^{-i}) + H(2^{-(i+1)}) - c2^{i(\tau'-\tau)}\Big) < \infty,$$

which implies Lemma 11.1.4 by the Borel-Cantelli lemma.

Let $\mathbf{k} = (k(1), k(2)) \geq \mathbf{0}$, define $\mathbf{t_k} = (t_{\mathbf{k}}(1), t_{\mathbf{k}}(2))$ as follows:

$$t_{\mathbf{k}}(1) = [\exp(k(1))^{s/4}], \quad t_{\mathbf{k}}(2) = [\exp(k(2))^{s/4}], \quad 0 < s < 1/2.$$

For large $\mathbf{n}$, $\mathbf{n} \in G_\beta$ implies $\mathbf{t_k} \in G_\beta$ if $\mathbf{t_k} + \mathbf{1} \leq \mathbf{n} \leq \mathbf{t_{k+1}}$.

**Lemma 11.1.5.** *For $\mathbf{t_k} \in G_\beta$, we have*

$$\sup_{A\in\mathcal{A}}\max_{\mathbf{t_k}<\mathbf{n}\leq\mathbf{t_{k+1}}} |S'_{\mathbf{n}}(A(2^{-a})) - S'_{\mathbf{t_k}}(A(2^{-a}))|$$
$$= O(|\mathbf{t_k}|^{1/2}|\log|\mathbf{t_k}|)^{-1/4}) \quad a.s.$$

*as* $|\mathbf{k}| \to \infty$.

**Proof.** Note that if $\mathbf{t_k} < \mathbf{n} \leq \mathbf{t_{k+1}}, A \subset [0,1]^2$, we have

$$|\mathbf{n}A \,\triangle\, \mathbf{t_k}A| \leq c|\mathbf{n}|(\varepsilon_1 + \varepsilon_2),$$

where

$$\varepsilon_i = (t_{\mathbf{k+1}}(i) - t_{\mathbf{k}}(i))/n_i, \; i = 1, 2. \tag{11.1.14}$$

Put $m_k = \Big[|\mathbf{t_k}|^{\frac{\delta-2\tau}{4(2+\delta)}}|\mathbf{k}|^{-\frac{s}{4}}\Big]$. If $\mathbf{t_k} \in G_\beta$, we have $m_k \leq t_{\mathbf{k}}(1) \wedge t_{\mathbf{k}}(2)$. Let $\Delta_n = 2|\mathbf{t_k}|^{(1+\tau)/(2+\delta)}$ in Lemma 11.1.1, we obtain

$$P\{|S'_{\mathbf{n}}(A(2^{-a})) - S'_{t_{\mathbf{k}}}(A(2^{-a}))| \geq |\mathbf{t_k}|^{1/2}(k(1)^{-s/16} + k(2)^{-s/16})\}$$
$$\leq P\Big\{\Big|\sum_{\mathbf{j}} |(\mathbf{n}A(2^{-a})\backslash \mathbf{t_k}A(2^{-a}))\cap$$
$$\cdot C_{\mathbf{j}}|(X'_{\mathbf{j}} - EX'_{\mathbf{j}})\Big| \geq \frac{1}{4}|\mathbf{t_k}|^{\frac{1}{2}}(k(1)^{-\frac{s}{16}} + k(2)^{-\frac{s}{16}})\Big\}$$
$$+ P\Big\{\Big|\sum_{\mathbf{j}} |(\mathbf{t_k}A(2^{-a})\backslash \mathbf{n}A(2^{-a}))\cap$$
$$\cdot C_{\mathbf{j}}|(X'_{\mathbf{j}} - EX'_{\mathbf{j}})\Big| \geq \frac{1}{4}|\mathbf{t_k}|^{\frac{1}{2}}(k(1)^{-\frac{s}{16}} + k(2)^{-\frac{s}{16}})\Big\}$$
$$\leq c\exp(-c(k(1)^{s/2} + k(2)^{s/2})). \tag{11.1.15}$$

For $\mathbf{t_k} \in G_\beta$, we have

$$k(1)^{-s/16} + k(2)^{-s/16} \le c(\log|\mathbf{t_k}|)^{-1/4}.$$

It follows that

$$\begin{aligned}\sum_{\mathbf{t_k}\in G_\beta} P\Big\{\sup_{A\in\mathcal{A}} \sup_{\mathbf{t_k}<\mathbf{n}\le\mathbf{t_{k+1}}} & |S'_{\mathbf{n}}(A(2^{-a})) \\ & - S'_{\mathbf{t_k}}(A(2^{-a}))| \ge |\mathbf{t_k}|^{\frac{1}{2}}(\log|\mathbf{t_k}|)^{-\frac{1}{4}}\Big\} \\ \le c & \sum_{\mathbf{t_k}\in G_\beta} \exp\Big(2^{(\log|\mathbf{t_k}|)^{r/r'}}\Big)|\mathbf{t_{k+1}} - \mathbf{t_k}| \\ & \cdot \exp(-c(k(1)^{s/2} + k(2)^{s/2})) < \infty.\end{aligned}$$

Lemma 11.1.5 is proved by the Borel-Cantelli lemma.

Let $R_{\mathbf{l}} = [t_{\mathbf{l}}(1), t_{\mathbf{l+1}}(1)] \times [t_{\mathbf{l}}(2), t_{\mathbf{l+1}}(2)]$. For any $A \in \mathbf{n}\bigcup_{\delta>0}\mathcal{A}_\delta$, define

$$A^* = \bigcup\{R_{\mathbf{l}} : R_{\mathbf{l}} \cap A \ne \emptyset\}, \quad A_* = \bigcup\{R_{\mathbf{l}} : R_{\mathbf{l}} \subset A\}.$$

It is clear that $A_* \subset A \subset A^*$ and if $\mathbf{t_k} < \mathbf{n} \le \mathbf{t_{k+1}}$ we have $|A^*\backslash A_*| \le c|\mathbf{n}|(\varepsilon_1 + \varepsilon_2)$, where $\varepsilon_i$ are defined by (11.1.14). Similarly, we have

**Lemma 11.1.6.** *For $t_k \in G_\beta$, we have*

$$\begin{aligned}\sup_{A\in\mathcal{A}} \max_{\mathbf{t_k}<\mathbf{n}\le\mathbf{t_{k+1}}} & \Big|\sum_{\mathbf{j}} |(\mathbf{t_k}A(2^{-a})\backslash(\mathbf{t_k}A(2^{-a}))_*) \cap C_{\mathbf{j}}|X'_{\mathbf{j}}| \\ & = O(|\mathbf{t_k}|^{1/2}(\log|\mathbf{t_k}|)^{-1/4}) \quad a.s. \quad as\ |\mathbf{k}| \to \infty.\end{aligned}$$

**3. Blocking** To estimate

$$\sup_{A\in\mathcal{A}} \max_{\mathbf{t_k}<\mathbf{n}\le\mathbf{t_{k+1}}} \Big|\sum_{\mathbf{j}} |(\mathbf{t_k}A(2^{-a}))_* \cap C_{\mathbf{j}}|X_{\mathbf{j}}\Big|$$

is the key to the remainder of the proof. Let $\mathbf{u} = (u(1), u(2)) \ge \mathbf{0}$, $H_{\mathbf{u}} = \{\mathbf{v} \in \mathbb{N}^2, \mathbf{t_u} + \mathbf{1} \le \mathbf{v} \le \mathbf{t_{u+1}}\}, 0 < \rho < \beta$. Denote

$$L = \{\mathbf{u} : H_{\mathbf{u}} \subset G_\rho\}, \quad H = \bigcup_{\mathbf{u}\in L} H_{\mathbf{u}},$$

$$\begin{aligned}\Delta_{\mathbf{u}} = \bigcup_{i=1,2} \Big\{\mathbf{v} \in H_{\mathbf{u}} : & \ t_{\mathbf{u+1}}(i) \\ & - [\exp(\mathbf{u}(i)^{s/4}/4)] \le v_i \le t_{\mathbf{u+1}}(i)\Big\}.\end{aligned}$$

For $\mathbf{t_k} \in G_\beta$, define $\mathbf{t}_\mathbf{k}^{(p)} = (\mathbf{t}_\mathbf{k}^{(p)}(1), \mathbf{t}_\mathbf{k}^{(p)}(2)), p = 1, 2$, as follows:

$$\mathbf{t}_\mathbf{k}^{(p)}(m) = \delta_{m,p} \min_{\substack{\mathbf{v}\in H, \\ v_l = n_l, \text{if } l\neq m}} v_m + (1-\delta_{m,p})n_m, \quad m = 1, 2, \tag{11.1.16}$$

where $\delta_{m,p} = 1$ if $m = p$; $\delta_{m,p} = 0$ if $m \neq p$. Put

$$\begin{aligned} I_1 &= [\mathbf{0}, \mathbf{t}_\mathbf{k}^{(1)}] \cup [\mathbf{0}, \mathbf{t}_\mathbf{k}^{(2)}], \\ I_2 &= \bigcup_{(t_k^{(1)}(1), t_k^{(2)}(2)) \leq \mathbf{u} \leq \mathbf{k}} H_\mathbf{u} \backslash \Delta_\mathbf{u}, \\ I_3 &= \bigcup_{\mathbf{u}\in L, \mathbf{u} < \mathbf{k}} \Delta_\mathbf{u}. \end{aligned}$$

We have

$$\begin{aligned} &\Big|\sum_\mathbf{j} |(\mathbf{t_k}A(2^{-a}))_* \cap C_\mathbf{j}|(X_\mathbf{j} - Y_\mathbf{j})\Big| \\ &\quad \leq \Big|\sum_\mathbf{j} |(\mathbf{t_k}A(2^{-a}))_* \cap I_1 \cap C_\mathbf{j}|X_\mathbf{j}\Big| \\ &\qquad + \Big|\sum_\mathbf{j} |(\mathbf{t_k}A(2^{-a}))_* \cap I_1 \cap C_\mathbf{j}|Y_\mathbf{j}\Big| \\ &\qquad + \Big|\sum_\mathbf{j} |(\mathbf{t_k}A(2^{-a}))_* \cap I_2 \cap C_\mathbf{j}|(X_\mathbf{j} - Y_\mathbf{j})\Big| \\ &\qquad + \Big|\sum_\mathbf{j} |(\mathbf{t_k}A(2^{-a}))_* \cap I_3 \cap C_\mathbf{j}|X_\mathbf{j}\Big| \\ &\qquad + \Big|\sum_\mathbf{j} |(\mathbf{t_k}A(2^{-a}))_* \cap I_3 \cap C_\mathbf{j}|Y_\mathbf{j}\Big| \end{aligned}$$

where $Y_\mathbf{j}$ will be specified below.

**Lemma 11.1.7.** *For $t_k \in G_\beta$, we have*

$$\begin{aligned} &\sup_{A\in\mathcal{A}} \max_{\mathbf{t_k}<\mathbf{n}\leq\mathbf{t_{k+1}}} \Big|\sum_\mathbf{j} |(\mathbf{t_k}A(2^{-a}))_* \cap I_1 \cap C_\mathbf{j}|X_\mathbf{j}| \\ &\quad = O(|\mathbf{t_k}|^{\frac{1}{2}}(\log|\mathbf{t_k}|)^{-\sigma_1}) \quad a.s. \quad as\ |\mathbf{k}| \to \infty. \end{aligned}$$

**Proof.** It is easy to see that

$$\begin{aligned} &\sup_{A\in\mathcal{A}} \max_{\mathbf{t_k}<\mathbf{n}\leq\mathbf{t_{k+1}}} \Big|\sum_\mathbf{j} |(\mathbf{t_k}A(2^{-a}))_* \cap I_1 \cap C_\mathbf{j}|X_\mathbf{j}\Big| \\ &\quad \leq \sum_{\mathbf{l}<\mathbf{k}} \Big|\sum_\mathbf{j} |R_\mathbf{l} \cap I_1 \cap C_\mathbf{j}|X_\mathbf{j}\Big|. \end{aligned}$$

Since $t^{(1)}_{\mathbf{k}}(1) \le ct_{\mathbf{k}}(2)^{\rho}$, $t^{(2)}_{\mathbf{k}}(2) \le ct_{\mathbf{k}}(1)^{\rho}$, we have

$$|t^{(1)}_{\mathbf{k}}|/|t_{\mathbf{k}}| \le c|t_{\mathbf{k}}|^{-\beta(\beta-\rho)/2} \quad \text{and} \quad |t^{(2)}_{\mathbf{k}}|/|t_{\mathbf{k}}| \le c|t_{\mathbf{k}}|^{-\beta(\beta-\rho)/2}$$

for $t_{\mathbf{k}} \in G_\beta$. By the Markov inequality and Lemma 6.2.3 it follows that

$$\begin{aligned} P\Big\{ \sum_{\mathbf{l}<\mathbf{k}} \Big| \sum_{\mathbf{j}} |R_{\mathbf{l}} \cap I_1 \cap C_{\mathbf{j}}| X_{\mathbf{j}} \Big| \ge |\mathbf{t}_{\mathbf{k}}|^{1/2} (\log |\mathbf{t}_{\mathbf{k}}|)^{-\sigma_1} \Big\} \\ \le \sum_{\mathbf{l}<\mathbf{k}} P\Big\{ \Big| \sum_{\mathbf{j}} |R_{\mathbf{l}} \cap I_1 \cap C_{\mathbf{j}}| X_{\mathbf{j}} \Big| \ge |\mathbf{t}_{\mathbf{k}}|^{1/2} (\log |\mathbf{t}_{\mathbf{k}}|)^{-\sigma_1} |\mathbf{k}|^{-1} \Big\} \\ \le c \sum_{\mathbf{l}<\mathbf{k}} |R_{\mathbf{l}} \cap I_1|^{1+\delta/2} \Big/ \Big( |\mathbf{t}_{\mathbf{k}}|^{1+\delta/2} ((\log |\mathbf{t}_{\mathbf{k}}|)^{-\sigma_1} |\mathbf{k}|^{-1})^{2+\delta} \Big) \\ \le c |\mathbf{t}_{\mathbf{k}}|^{-\frac{\beta(\beta-\rho)}{2}(1+\frac{\delta}{2})} ((\log |\mathbf{t}_{\mathbf{k}}|)^{\sigma_1} |\mathbf{k}|)^{2+\delta}. \end{aligned} \tag{11.1.17}$$

Thus Lemma 11.1.7 holds true by the Borel-Cantelli lemma.

**Lemma 11.1.8.** *For $t_k \in G_\beta$, we have*

$$\begin{aligned} \sup_{A\in\mathcal{A}} \max_{\mathbf{t}_{\mathbf{k}}<\mathbf{n}\le\mathbf{t}_{\mathbf{k}+1}} \Big| \sum_{\mathbf{j}} |(\mathbf{t}_{\mathbf{k}} A(2^{-a}))_* \cap I_3 \cap C_{\mathbf{j}}| X_{\mathbf{j}} \Big| \\ = O(|\mathbf{t}_{\mathbf{k}}|^{1/2} (\log |\mathbf{t}_{\mathbf{k}}|)^{-\sigma_1}) \quad a.s. \quad as\ |\mathbf{k}| \to \infty. \end{aligned} \tag{11.1.18}$$

**Proof.** As in the proof of Lemma 11.1.7, we have

$$\begin{aligned} \sup_{A\in\mathcal{A}} \max_{\mathbf{t}_{\mathbf{k}}<\mathbf{n}\le\mathbf{t}_{\mathbf{k}+1}} \Big| \sum_{\mathbf{j}} |(\mathbf{t}_{\mathbf{k}} A(2^{-a}))_* \cap I_3 \cap C_{\mathbf{j}}| X_{\mathbf{j}} \Big| \\ \le \sum_{\mathbf{l}<\mathbf{k}} \Big| \sum_{\mathbf{j}} |R_{\mathbf{l}} \cap I_3 \cap C_{\mathbf{j}}| X_{\mathbf{j}} \Big|. \end{aligned}$$

Denote $\Delta_{\mathbf{u}} = \Delta_{\mathbf{u}}(1) \bigcup \Delta_{\mathbf{u}}(2)$, where $\Delta_{\mathbf{u}}(1)$ and $\Delta_{\mathbf{u}}(2)$ are disjoint rectangles, such that

$$\begin{aligned} |\Delta_{\mathbf{u}}(1)| =& [\exp(u(1)^{s/4}/4)] \Big( [\exp((u(2)+1)^{s/4})] \\ & - [\exp((u(2))^{s/4})] \Big), \\ |\Delta_{\mathbf{u}}(2)| =& [\exp(u(2)^{s/4}/4)] \Big( [\exp((u(1)+1)^{s/4})] \\ & - [\exp((u(1))^{s/4})] \Big). \end{aligned}$$

We have

$$\sum_{\substack{\mathbf{u}\in L\\ \mathbf{u}<\mathbf{k}}} |\Delta_{\mathbf{u}}(1)| \leq c\mathbf{k}(1)^{1-s/4}\mathbf{t}_{\mathbf{k}}(1)^{1/4}\mathbf{t}_{\mathbf{k}}(2),$$

$$\sum_{\substack{\mathbf{u}\in L\\ \mathbf{u}<\mathbf{k}}} |\Delta_{\mathbf{u}}(2)| \leq c\mathbf{k}(2)^{1-s/4}\mathbf{t}_{\mathbf{k}}(2)^{1/4}\mathbf{t}_{\mathbf{k}}(1).$$

Then, similarly to (11.1.17) in Lemma 11.1.7 we have

$$\begin{aligned} P\Big\{ \sum_{\mathbf{l}<\mathbf{k}} \Big| \sum_{\mathbf{j}} |R_{\mathbf{l}} \cap I_3 \cap C_{\mathbf{j}}| X_{\mathbf{j}} \Big| \geq |\mathbf{t}_{\mathbf{k}}|^{\frac{1}{2}} (\log |\mathbf{t}_{\mathbf{k}}|)^{-\sigma_1} \Big\} \\ \leq c \sum_{\mathbf{l}<\mathbf{k}} |R_{\mathbf{l}} \cap I_3|^{1+\delta/2} \Big/ \Big( |\mathbf{t}_{\mathbf{k}}|^{1+\delta/2} ((\log |\mathbf{t}_{\mathbf{k}}|)^{-\sigma_1} |\mathbf{k}|^{-1})^{2+\delta} \Big) \\ \leq c |\mathbf{t}_{\mathbf{k}}|^{-(3\beta/8)(1+\delta/2)} ((\log |\mathbf{t}_{\mathbf{k}}|)^{\sigma_1} |\mathbf{k}|)^{2+\delta} \\ \cdot \Big( \mathbf{k}(1)^{1-s/4} + \mathbf{k}(2)^{1-s/4} \Big)^{1+\delta/2}. \end{aligned} \tag{11.1.19}$$

Lemma 11.1.8 holds true by the Borel-Cantelli lemma .

**4. Constructions of $\{X_{\mathbf{j}}\}$ and $\{Y_{\mathbf{j}}\}$.**

In order to construct random fields $\{X_{\mathbf{j}}\}$ and $\{Y_{\mathbf{j}}\}$, we quote the following results without any proofs. Denote

$$h_u = \mathrm{Card}((H_{\mathbf{u}} \backslash \Delta_{\mathbf{u}}) \cap \mathbb{N}^2), \quad \overline{X}_{\mathbf{u}} = \sum_{\mathbf{j}\in H_{\mathbf{u}}\backslash\Delta_{\mathbf{u}}} X_{\mathbf{j}} / h_{\mathbf{u}}^{1/2}.$$

**Proposition 11.1.1.** (Berkes, Morrow 1981) Under the conditions of Theorem 11.1.1, there exists a $t$, $0 < t < 1$ such that for any $|x| \leq h_{\mathbf{u}}^t$ we have

$$|E\exp(ix\overline{X}_{\mathbf{u}}) - \exp(\frac{1}{2}\sigma^2 x^2)| \leq c h_{\mathbf{u}}^{-t}. \tag{11.1.20}$$

**Proposition 11.1.2.** (Berkes, Philipp 1979) Let $\{X_k, k \geq 1\}$ be a sequence of random variables with values in $R^{d_k}$, and let $\{\mathcal{F}_k, k \geq 1\}$ be a non-decreasing sequence of $\sigma$-fields such that $X_k$ is $\mathcal{F}_k$-measurable . Finally, let $\{G_k, k \geq 1\}$ be a sequence of probability distributions on $R^{d_k}$ with characteristic functions $g_k(\mathbf{u}), \mathbf{u} \in R^{d_k}$, respectively. Suppose that for some non-negative numbers $\lambda_k, \delta_k$ and $T_k \geq 10^8 d_k$

$$E|E\{\exp(i\langle\mathbf{u}, X_k\rangle)|\mathcal{F}_k\} - g_k(\mathbf{u})| \leq \lambda_k \tag{11.1.21}$$

for all $\mathbf{u}$ with $|\mathbf{u}| \le T_k$ and

$$G_k\{\mathbf{u} : |\mathbf{u}| > \frac{1}{4}T_k\} \le \delta_k. \tag{11.1.22}$$

Then without changing its distribution we can redefine the sequence $\{X_k, k \ge 1\}$ on a richer probability space together with a sequence $\{Y_k, k \ge 1\}$ of independent random variables such that $Y_k$ has distribution $G_k$ and

$$P\{|X_k - Y_k| \ge \alpha_k\} \le \alpha_k \qquad k = 1, 2, \cdots \tag{11.1.23}$$

where $\alpha_1 = 1$ and

$$\alpha_k = 16 d_k T_k^{-1} \log T_k + 4\lambda_k^{1/2} T_k^{d_k} + \delta_k. \tag{11.1.24}$$

**Proposition 11.1.3.** (Berkes, Philipp 1979) Let $S_i, i = 1, 2, 3$ be separable Banach spaces. Let $F$ be a distribution on $S_1 \times S_2$ and let $G$ be a distribution on $S_2 \times S_3$ such that the second marginal of $F$ equals to the first marginal of $G$. Then there exist a probability space and three random variables $Z_i, i = 1, 2, 3$ defined on it such that the joint distribution of $Z_1$ and $Z_2$ is $F$ and the joint distribution of $Z_2$ and $Z_3$ is $G$.

Let $(t_{k_0}^{(1)}(1), t_{k_0}^{(2)}(2))$ be large enough such that for any $\mathbf{u} \in L_0 := \{\mathbf{u} \in L, \mathbf{u} \ge (t_{k_0}^{(1)}(1), t_{k_0}^{(2)}(2))\}$, there exists a $\rho' > 0$ satisfying

$$\begin{aligned} t_{\mathbf{u+1}}(1) - t_{\mathbf{u}}(1) - [\exp(\mathbf{u}(1)^{s/4}/4)] \\ \ge \left(t_{\mathbf{u+1}}(2) - t_{\mathbf{u}}(2) - [\exp(\mathbf{u}(2)^{s/4}/4)]\right)^{\rho'}, \\ t_{\mathbf{u+1}}(2) - t_{\mathbf{u}}(2) - [\exp(\mathbf{u}(2)^{s/4}/4)] \\ \ge \left(t_{\mathbf{u+1}}(1) - t_{\mathbf{u}}(1) - [\exp(\mathbf{u}(1)^{s/4}/4)]\right)^{\rho'}. \end{aligned}$$

Put $\psi$ is a one to one correspondence from $\{1, 2, \cdots\}$ to $L_0$, denote $\psi(l) = (\psi_1(l), \psi_2(l))$. By Proposition 11.1.1, there exists a $t$, $0 < t < 1$ such that for any $|x| \le h_{\psi(l)}^t$ we have

$$|E\exp(ix\overline{X}_{\psi(l)}) - \exp(-\sigma^2 x^2/2)| \le c h_{\psi(l)}^{-t}. \tag{11.1.25}$$

On the other hand, since $h_{\psi(l)} = O(|t_{\psi(l)}|)$, for any $|x| \le h_{\psi(l)}^t$ we have

$$\begin{aligned} \lambda_l(x) :=& E|E\{\exp(ix\overline{X}_{\psi(l)})|\overline{X}_{\psi(1)}, \cdots, \overline{X}_{\psi(l-1)}\} - \exp(\sigma^2 x^2/2)| \\ \le& E|E\{\exp(ix\overline{X}_{\psi(l)})|\overline{X}_{\psi(1)}, \cdots, \overline{X}_{\psi(l-1)}\} - E\exp(ix\overline{X}_{\psi(l)})| \\ &+ |E\exp(ix\overline{X}_{\psi(l)}) - \exp(\sigma^2 x^2/2)| \\ \le& c h_{\psi(l)}^{-t} + 2\pi\varphi([\exp(\psi_1(l)^{s/4}/4)] \wedge [\exp(\psi_2(l)^{s/4}/4)]) \\ \le& c|t_{\psi(l)}|^{-t_0} \end{aligned}$$

by the property of $\varphi$-mixing, where $t_0 = \min(t, \rho q/8)$.

Put $t' < t_0/2$, $T_l = |t_{\psi(l)}|^{t'}$, $\lambda_l = |t_{\psi(l)}|^{-t_0}$, $\delta_l = N(0, \sigma^2)\{x : |x| > T_l/4\}$, $\alpha_l = 16T_l^{-1}\log T_l + 4\lambda_l^{1/2}T_l + \delta_l$. By Proposition 11.1.2, without changing its distribution we can redefine the sequence $\{\overline{X}_{\psi(l)}\}$ on a richer probability space together with a sequence $\{\overline{Y}_{\psi(l)}\}$ of independent normal random variables such that $E\overline{Y}^2_{\psi(l)} = \sigma^2$ and

$$P\{|\overline{X}_{\psi(l)} - \overline{Y}_{\psi(l)}\}| \geq \alpha_l\} \leq \alpha_l. \tag{11.1.26}$$

Moreover, by Proposition 11.1.3, there exist two random fields $\{X_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^2\}$ and $\{Y_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^2\}$ on a richer probability space such that the distribution of $\{X_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^2\}$ is not changed and $\{Y_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^2\}$ are independent centered normal random variables with $EY_{\mathbf{j}}^2 = \sigma^2$, and further

$$P\Big\{h_{\psi(l)}^{-1/2}\Big|\sum_{\mathbf{j}\in H_{\psi(l)}\setminus\Delta_{\psi(l)}}(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big| \geq \alpha_l\Big\} \leq \alpha_l. \tag{11.1.27}$$

Obviously we have $\sum_l \alpha_l < \infty$, which implies

$$\Big|\sum_{\mathbf{j}\in H_{\psi(l)}\setminus\Delta_{\psi(l)}}(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big| = O(\alpha_l h_{\psi(l)}^{1/2}) \quad \text{a.s.} \tag{11.1.28}$$

Note that $I_2 = \bigcup_{(t_k^{(1)}(1), t_k^{(2)}(2)) \leq \mathbf{u} < \mathbf{k}} H_{\mathbf{u}}\setminus\Delta_{\mathbf{u}}$. Take $\mathbf{t_k} \in G_\beta$ such that $t_{\mathbf{k}}^{(1)}(1) \geq t_{\mathbf{k}_0}^{(1)}(1), t_{\mathbf{k}}^{(2)}(2) \geq t_{\mathbf{k}_0}^{(2)}(2)$. Then for every $A \in \mathcal{A}$ and some $t'' > 0$ we have

$$\begin{aligned}
&\Big|\sum_{\mathbf{j}}|(\mathbf{t_k}A(2^{-a}))_* \cap I_2 \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big| \\
&\leq \sum_{(t_k^{(1)}(1), t_k^{(2)}(2)) \leq \mathbf{u} < \mathbf{k}} \Big|\sum_{\mathbf{j}\in H_{\mathbf{u}}\setminus\Delta_{\mathbf{u}}}(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big| \\
&\leq \sum_{\psi(l) < \mathbf{k}, \psi_i(l) \geq t_{\mathbf{k}}^{(i)}(i), i=1,2} \alpha_l h_{\psi(l)}^{1/2} \\
&\leq c|\mathbf{t_k}|^{1/2 - t''}. \quad \text{a.s.}
\end{aligned} \tag{11.1.29}$$

For the random field $\{Y_{\mathbf{j}}\}$ which defined in (11.1.27), we define $Y_{\mathbf{j}}'$ like

$X'_{\mathbf{j}}$, then for $t_{\mathbf{k}} < n \le t_{\mathbf{k+1}}$, $A \in \mathcal{A}$ we have

$$\begin{aligned}
&\Big|\sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - Y_{\mathbf{j}})| \\
&\quad \le \Big|\sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - X'_{\mathbf{j}})\Big| \\
&\qquad + \Big|\sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}|(X'_{\mathbf{j}} - Y'_{\mathbf{j}})\Big| + \Big|\sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}|(Y'_{\mathbf{j}} - Y_{\mathbf{j}})\Big|,
\end{aligned}$$

where

$$\begin{aligned}
&\Big|\sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}|(X'_{\mathbf{j}} - Y'_{\mathbf{j}})\Big| \\
&\quad \le \Big|\sum_{\mathbf{j}} \big|\,|\mathbf{n}A \cap C_{\mathbf{j}}| - |\mathbf{n}A(2^{-b}) \cap C_{\mathbf{j}}|\,\big|(X'_{\mathbf{j}} - Y'_{\mathbf{j}})\Big| \\
&\qquad + \Big|\sum_{\mathbf{j}} \big|\,|\mathbf{n}A(2^{-b}) \cap C_{\mathbf{j}}| - |\mathbf{n}A(2^{-a}) \cap C_{\mathbf{j}}|\,\big|(X'_{\mathbf{j}} - Y'_{\mathbf{j}})\Big| \\
&\qquad + \Big|\sum_{\mathbf{j}} \big|\,|\mathbf{n}A(2^{-a}) \cap C_{\mathbf{j}}| - |t_{\mathbf{k}}A(2^{-a}) \cap C_{\mathbf{j}}|\,\big|(X'_{\mathbf{j}} - Y'_{\mathbf{j}})\Big| \\
&\qquad + \Big|\sum_{\mathbf{j}} |(t_{\mathbf{k}}A(2^{-a}) \backslash (t_{\mathbf{k}}A(2^{-a}))_* \cap C_{\mathbf{j}}|(X'_{\mathbf{j}} - Y'_{\mathbf{j}})\Big| \\
&\qquad + \Big|\sum_{\mathbf{j}} |(t_{\mathbf{k}}A(2^{-a}))_* \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big| \\
&\qquad + \sum_{\mathbf{j}} |X_{\mathbf{j}} - X'_{\mathbf{j}}| + \sum_{\mathbf{j}} |Y_{\mathbf{j}} - Y'_{\mathbf{j}}|
\end{aligned} \tag{11.1.30}$$

and

$$\begin{aligned}
&\Big|\sum_{\mathbf{j}} |(t_{\mathbf{k}}A(2^{-a}))_* \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big| \\
&\quad \le \Big|\sum_{\mathbf{j}} |(t_{\mathbf{k}}A(2^{-a}))_* \cap I_1 \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big| \\
&\qquad + \Big|\sum_{\mathbf{j}} |(t_{\mathbf{k}}A(2^{-a}))_* \cap I_2 \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big| \\
&\qquad + \Big|\sum_{\mathbf{j}} |(t_{\mathbf{k}}A(2^{-a}))_* \cap I_3 \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - Y_{\mathbf{j}})\Big|.
\end{aligned} \tag{11.1.31}$$

It is clear that Lemmas 11.1.2-11.1.8 also hold true for the independent normal random field $\{Y_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^2\}$. Then Theorem 11.1.1 is proved from these lemmas and (11.1.29)–(11.1.31).

## 11.2 Strong approximations for $\alpha$-mixing random fields

The strong approximation results for $\alpha$-mixing random fields were given by Strittmatter (1990) for a smaller class of sets with $N(\varepsilon) \le c\varepsilon^{-u}$ for some $u > 0$ and the condition "$\mathbf{n} \in G_\beta$" was left out.

**Theorem 11.2.1.** *Let $\{X_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ be a weakly stationary $\alpha$-mixing random field of $R^q$-valued random vectors with $EX_{\mathbf{j}} = 0$. Put $\|X_{\mathbf{j}}\|^2 = (X_{\mathbf{j}1}^2 + \cdots + X_{\mathbf{j}q}^2)$ and suppose that there are positive constants $C_i > 0, i = 1, \cdots, 4$, such that*

$$E\|X_{\mathbf{j}}\|^{2+\delta} \le C_1, \quad \text{for every } \mathbf{j} \in \mathbb{N}^d \text{ and some } \delta > 0, \tag{11.2.1}$$

$$\alpha(t) \le C_2 t^{-s} \quad \text{for some } s > 1 + 2(17/2)^{d-1}/(1 \wedge \delta), \tag{11.2.2}$$

$$N(\varepsilon) \le C_3 \varepsilon^{-u} \quad \text{for some } u < \frac{\delta s - 4 - 2\delta}{d(4+\delta)} - 2, \tag{11.2.3}$$

$$\begin{aligned} b(\varepsilon) &:= \sup\{n^{-d}|((\mathbf{n}A)^{n\varepsilon} \cap (\mathbf{n}A^c)^{n\varepsilon})| : A \in \bigcup_{\eta>0} A_\eta,\ n \ge 1/\varepsilon\} \\ &\le C_4\, \varepsilon^h \qquad \text{for some } 0 < h \le 1. \end{aligned} \tag{11.2.4}$$

*Put*

$$r(\mathbf{j}) = \operatorname{Cov}(X_{\mathbf{j}}, X_{\mathbf{j}}).$$

*Then the series*

$$T = \sum_{\mathbf{j} \in \mathbb{Z}^d} r(\mathbf{j})$$

*converges absolutely and $T$ is a non-negative definite matrix. Furthermore, without changing its joint distribution, we can redefine the field $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ on a richer probability space together with a field $\{Y_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ of i.i.d. centered Gaussian random vectors with covariance matrix $T$ such that for some $\gamma = \gamma(u, s, d, h, \delta, N(\varepsilon))$ we have*

$$\sup_{A \in \mathcal{A}} \Big\| \sum_{\mathbf{j} \in \mathbb{N}^d} |\mathbf{n}A \cap C_{\mathbf{j}}|(X_{\mathbf{j}} - Y_{\mathbf{j}}) \Big\| = O(n^{d/2-\gamma}) \quad a.s. \tag{11.2.5}$$

The proof of Theorem 11.2.1 will need the following lemmas.

**Lemma 11.2.1.** *Let* $\{X_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ *be an* $\alpha$*-mixing random field with* $EX_{\mathbf{j}} = 0$, $E\|X_{\mathbf{j}}\|^{2+\delta} \le C_1$ *and*

$$C_0 = \sum_{r=1}^{\infty} r^{d-1}\alpha(r)^{\delta/(2+\delta)} < \infty. \tag{11.2.6}$$

*Then for* $0 \le d_{\mathbf{j}} \le 1$, $\mathbf{j} \in \mathbb{N}^d$, *where only finitely many* $\mathbf{j}$ *such that* $d_{\mathbf{j}} \ne 0$, *we have*

$$E\Big\|\sum_{\mathbf{j}\in\mathbb{N}^d} d_{\mathbf{j}}X_{\mathbf{j}}\Big\|^2 \le C\sum_{\mathbf{j}\in\mathbb{N}^d} d_{\mathbf{j}},$$

*where* $C = (1 + 15d3^dC_0)C_1^{2/(2+\delta)}$.

**Proof.** By Lemma 1.2.4 we have

$$\begin{aligned}\|EX_{\mathbf{j}}X_{\mathbf{k}}\| &\le 10\alpha(d(\mathbf{j},\mathbf{k}))^{\delta/(2+\delta)}\|X_{\mathbf{j}}\|_{2+\delta}\|X_{\mathbf{k}}\|_{2+\delta} \\ &= 10\alpha(d(\mathbf{j},\mathbf{k}))^{\delta/(2+\delta)}C_1^{2/(2+\delta)},\end{aligned}$$

where $d(\mathbf{j},\mathbf{k}) = \max_{1\le i\le d}|j_i - k_i|$. Lemma 11.2.1 is proved.

**Lemma 11.2.2.** *Let* $\{X_{\mathbf{j}},\mathbf{j} \in \mathbb{N}^d\}$ *be a weakly stationary* $\alpha$*-mixing random field with* $EX_{\mathbf{j}} = 0$, $\|X_{\mathbf{j}}\| \le M < \infty$ *and satisfying (11.2.6). Let* $D_1, D_2, \cdots, D_q, D_k = (\mathbf{n}_k - 2p\mathbf{1}, \mathbf{n}_k]$, *be mutually disjoint cubes with* $\mathbf{n}_k \in 2p\mathbb{N}^d$ *for some fixed* $p \in \mathbb{N}$. *Let* $0 \le d_{\mathbf{j}} \le 1$, $\mathbf{j} \in \mathbb{N}^d$. *Put*

$$D = \bigcup_{k=1}^{q} D_k \quad F_k = \sum_{\mathbf{j}\in D_k\cap\mathbb{N}^d} d_{\mathbf{j}}, \quad F = \sum_{\mathbf{j}\in D} d_{\mathbf{j}}.$$

*Then for every* $K > 0$ *we have*

$$\begin{aligned}&P\Big\{\Big\|\sum_{\mathbf{j}\in D} d_{\mathbf{j}}X_{\mathbf{j}}\Big\| > 2^{d+1}K\Big\} \\ &\quad\le c(K^{-2}M^2F^2\alpha(p) + p^{2d}C^{-2}M^4\alpha(p) \\ &\qquad + \exp(-K^2/(8CF)) + \exp(-2^{-d-2}K/Mp^d),\end{aligned}$$

*where* $C$ *is defined as in Lemma 11.2.1.*

This lemma is a generalization of Theorem 4 in Philipp (1984). The proof follows closely by his lines, and hence is not given here.

**Proof of Theorem 11.2.1.**

From (11.2.3) it follows that there exist positive numbers $\tau$ and $\zeta$ such that $\zeta < \frac{1}{2} - \frac{1+\tau}{2+\delta}$ and

$$u < \frac{\zeta s - 1}{d(\frac{1}{2} + \frac{1+\tau}{2+\delta})} - 2. \tag{11.2.7}$$

Define for $\mathbf{j} \in \mathbb{N}^d$

$$X'_{\mathbf{j}} = X_{\mathbf{j}} I(\|X_{\mathbf{j}}\| \le |\mathbf{j}|^{(1+\tau)/(2+\delta)}),$$
$$\overline{X}_{\mathbf{j}} = X'_{\mathbf{j}} - EX'_{\mathbf{j}}, \qquad X''_{\mathbf{j}} = X_{\mathbf{j}} - X'_{\mathbf{j}}.$$

For a constant $\beta$ specified later define

$$\psi(m) = \sum_{k=1}^{m} k^{\beta} \quad m \in \mathbb{N}. \tag{11.2.8}$$

Assume that $m$ and $n$ are linked by

$$\psi(m) < n \le \psi(m+1). \tag{11.2.9}$$

For $\mathbf{r} = (r_1, \cdots, r_d) \in \mathbb{N}^d$, write

$$R_r = \{(v_1, \cdots, v_d) \in R_+^d : \psi(r_i) < v_i \le \psi(r_i + 1), i \le d\}.$$

For $A \subset R_+^d \cap \mathcal{B}^d$ let

$$A_* = \bigcup_{R_r \subset A} R_r.$$

and for $A \in \mathcal{B}^d \cap [0,1]^d$

$$S_{\mathbf{n}}(A) = \sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}| X_{\mathbf{j}},$$
$$\overline{S}_{\mathbf{n}}(A) = \sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}| \overline{X}_{\mathbf{j}},$$
$$\overline{V}_n(A) = \sum_{\mathbf{j}} |(\mathbf{n}A \backslash (\mathbf{n}A)_*) \cap C_{\mathbf{j}}| \overline{X}_{\mathbf{j}}.$$

We introduce the following result of Berkes and Morrow (1981) without proof.

**Proposition 11.2.1.** Let $\{\xi_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ be a weakly stationary $\alpha$-mixing random field with $E\xi_{\mathbf{j}} = 0$, $E\|\xi_{\mathbf{j}}\|^{2+\delta} < \infty$, $\alpha(t) = O(t^{-d(1+\varepsilon)(1+2/\delta)})$. Denote

$$G_\theta = \bigcap_{k=1}^{d} \{\mathbf{j} \in \mathbb{N}^d : j_k \ge \prod_{l \ne k} j_l^{\theta}\}, \quad 0 < \theta < 1.$$

Then the series

$$\sigma^2 = E\xi_{\mathbf{1}}^2 + 2\sum_{\mathbf{j}} E\xi_{\mathbf{1}}\xi_{\mathbf{j}}$$

converges absolutely. Without loss of generality, assume that $\sigma^2 = 1$. Furthermore without changing the distribution of $\{\xi_{\mathbf{j}}\}$, we can redefine the sequence $\{\xi_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ on a richer probability space together with a random field $\{\eta_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ of i.i.d. centered Gaussian random variables with $E\eta_{\mathbf{j}}^2 = 1$ such that for any $\mathbf{n} \in G_\theta$ we have

$$\sup_{1 \le \mathbf{m} \le \mathbf{n}} \Big\| \sum_{\mathbf{k} \le \mathbf{m}} \xi_{\mathbf{k}} - \sum_{\mathbf{k} \le \mathbf{m}} \eta_{\mathbf{k}} \Big\| = O(|\mathbf{n}|^{1/2-\lambda}) \quad \text{a.s.} \tag{11.2.10}$$

where $\lambda > 0$ depends only on the field $\{\xi_{\mathbf{j}}\}$.

Now we take $\theta = 1/(8d-1)$ and denote

$$G = G_{1/(8d-1)}, \quad L = \{\mathbf{r} \in \mathbb{N}^d : (\psi(r_1), \cdots, \psi(r_d)) \in G\}.$$

By Proposition 11.2.1, we have a random field $\{Y_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$ of i.i.d. centered Gaussian random variables with $EY_{\mathbf{j}}^2 = 1$ such that

$$\sum_{\substack{\mathbf{r} \in L \\ \forall i, \psi(r_i) \le n}} \Big\| \sum_{\mathbf{j} \in R_{\mathbf{r}}} (X_{\mathbf{j}} - Y_{\mathbf{j}}) \Big\| = O(n^{d/2-\gamma_1}) \quad \text{a.s.} \tag{11.2.11}$$

for some $\gamma_1 > 0$. Denote

$$\begin{aligned}
Y_{\mathbf{j}}' &= Y_{\mathbf{j}} I(\|Y_{\mathbf{j}}\| \le |\mathbf{j}|^{(1+\tau)/(2+\delta)}), \\
T_{\mathbf{n}}(A) &= \sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}| Y_{\mathbf{j}}, \\
T_{\mathbf{n}}'(A) &= \sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}| Y_{\mathbf{j}}', \\
U_n'(A) &= \sum_{\mathbf{j}} |(\mathbf{n}A \backslash (\mathbf{n}A)_*) \cap C_{\mathbf{j}}| Y_{\mathbf{j}}'.
\end{aligned}$$

Then for $\kappa > 0$(specified later on) we have

$$\begin{aligned}
&\Big\| \sum_{\mathbf{j}} |\mathbf{n}A \cap C_{\mathbf{j}}| (X_{\mathbf{j}} - Y_{\mathbf{j}}) \Big\| \\
&\quad = \|S_{\mathbf{n}}(A) - T_{\mathbf{n}}(A)\| \\
&\quad \le \|S_{\mathbf{n}}(A) - \overline{S}_{\mathbf{n}}(A)\| + \|\overline{S}_{\mathbf{n}}(A) - \overline{S}_{\mathbf{n}}(A(n^{-\kappa}))\| \\
&\qquad + \|\overline{V}_n(A(n^{-\kappa}))\| + \|T_{\mathbf{n}}(A) - T'_{\mathbf{n}}(A)\| \\
&\qquad + \|T'_{\mathbf{n}}(A) - T'_{\mathbf{n}}(A(n^{-\kappa}))\| + \|U_n'(A(n^{-\kappa}))\| \\
&\qquad + \sum_{1 \le \mathbf{j} \le n\mathbf{1}} (\|X_{\mathbf{j}} - \overline{X}_{\mathbf{j}}\| + \|Y_{\mathbf{j}} - Y_{\mathbf{j}}'\|) \\
&\qquad + \Big\| \sum_{\mathbf{j}} |(\mathbf{n}A(n^{-\kappa}))_* \cap C_{\mathbf{j}}| (X_{\mathbf{j}} - Y_{\mathbf{j}}) \Big\| \\
&\quad =: I_{11} + I_{12} + I_{13} + I_{21} + I_{22} + I_{23} + I_3 + I_4.
\end{aligned} \tag{11.2.12}$$

From Lemma 11.1.2 it follows that

$$I_{11} \vee I_{21} \vee I_3 = O(n^{d/(2+\delta)}). \tag{11.2.13}$$

From Lemma 11.1.3 it follows that

$$I_{12} \vee I_{22} = O(n^{d/2-\gamma_2}) \tag{11.2.14}$$

where $\gamma_2 = \kappa - d(1+\tau)/(2+\delta) - d/2 > 0$.

In order to estimate $I_{13}, I_{23}$ and $I_4$, we need the following lemmas.

**Lemma 11.2.3.** *There exist* $d\Big(\frac{1}{2} + \frac{1+\tau}{2+\delta}\Big) < \kappa \le d$, $\gamma'$, $\gamma'' > 0$ *such that for any* $n \in \mathbb{N}$ *we have*

$$P\{\sup(\|\overline{V}_n(A(n^{-\kappa}))\| :\ A(n^{-\kappa}) \in \mathcal{A}(n^{-\kappa})) > cn^{d/2-\gamma'}\} \le cn^{-1-\gamma''}.$$

**Proof.** Put $p = [n^\zeta]$, for fixed $A \in \mathcal{B}^d \cap [0,1]^d$ cover $(\mathbf{n}A)\backslash(\mathbf{n}A)_*$ with cubes in $R_+^d$ with length of edges equal to $2p$ and of the form explained in Lemma 11.2.2; as there let $D$ be the union of these cubes. For $\mathbf{j} \in \mathbb{N}^d$ let $d_{\,\mathbf{j}} = |((\mathbf{n}A)\backslash(\mathbf{n}A)_*) \cap C_{\,\mathbf{j}}|$. Then by (11.2.4) we have

$$\begin{aligned} F = \sum_{\mathbf{j}\in D} d_{\,\mathbf{j}} &= |(\mathbf{n}A)\backslash(\mathbf{n}A)_*| \\ &\le n^d b((\psi(m+1) - \psi(m))/n) \le cn^{d-h}(\psi(m+1) - \psi(m))^h \\ &\le cn^{d-h} n^{h\beta/(\beta+1)} = cn^{d-\gamma}, \end{aligned} \tag{11.2.15}$$

where $\gamma = h/(\beta+1)$. There is a $\gamma' > 0$ such that $\gamma' < \gamma/2$ and

$$\gamma' < d\Big(\frac{1}{2} - \frac{1+\tau}{2+\delta} - \zeta\Big). \tag{11.2.16}$$

Then by Lemma 11.2.1, (11.2.15) and Lemma 11.2.2, we have

$$\begin{aligned} &P\{\|\overline{V_n}(A)\| > cn^{d/2-\gamma'}\} \\ &\quad\le c\Big(n^{-d+2\gamma'} M^2 F^2 \alpha(p) + p^{2d} M^4 \alpha(p) \\ &\qquad + \exp\Big(-\frac{n^{d-2\gamma'}}{cF}\Big) + \exp\Big(-\frac{cn^{d/2-\gamma'}}{Mp^d}\Big)\Big). \end{aligned}$$

Since $M = 2n^{d(1+\tau)/(2+\delta)}$, $p \sim n^{\zeta}$, we have

$$\begin{aligned}
&P\{\|\overline{V}_n(A)\| > cn^{d/2-\gamma'}\} \\
&\le c\Big(n^{-d+2\gamma'+2d\frac{1+\tau}{2+\delta}+2d-2\gamma-\zeta s} \\
&\quad + n^{2d\zeta+4d\frac{1+\tau}{2+\delta}-\zeta s} \\
&\quad + \exp\Big(-cn^{-2\gamma'+\gamma}\Big) + \exp\Big(-cn^{d/2-\gamma'-d\frac{1+\tau}{2+\delta}-d\zeta}\Big)\Big) \\
&\le cn^{-\zeta s+2\left(1+d\frac{1+\tau}{2+\delta}\right)}.
\end{aligned} \tag{11.2.17}$$

There exist $\theta$ and $\kappa$ with $0 < \theta < \zeta s - 1 - d(1 + \dfrac{2(1+\tau)}{2+\delta})$, $d(\frac{1}{2} + \frac{1+\tau}{2+\delta}) < \kappa \le d$ and $\theta/\kappa = u$. We multiply both sides of (11.2.17) by $N(n^{-\kappa}) \le cn^{\kappa u} = cn^{\theta}$ and obtain

$$\begin{aligned}
&P\Big\{\sup_{A(n^{-\kappa})} \|\overline{V_n}(A(n^{-\kappa}))\| > cn^{d/2-\gamma'}\Big\} \\
&\le cn^{\theta-\zeta s+2\left(1+d\frac{1+\tau}{2+\delta}\right)} \\
&= cn^{-1-\gamma''}.
\end{aligned}$$

The proof of Lemma 11.2.3 is completed.

**Lemma 11.2.4.** *If in (11.2.8) $\beta \ge 6d$ then*

$$P\Big\{\sum_{\mathbf{r}\notin L,\psi(r_i)\le n} \Big\|\sum_{\mathbf{j}\in R_{\mathbf{r}}} X_{\mathbf{j}}\Big\| > n^{d/2-1/16}\Big\} \le cm^{-2}, \tag{11.2.18}$$

*where $n$ and $m$ are linked by (11.2.9), hence by the Borel-Cantelli lemma*

$$\sum_{\mathbf{r}\notin L,\psi(r_i)\le n} \Big|\sum_{\mathbf{j}\in R_{\mathbf{r}}} X_{\mathbf{j}}\Big| = O(n^{d/2-1/16}) \quad a.s. \tag{11.2.19}$$

**Proof.** For $r \notin L$ with $\psi(r_i) \le n$, $i = 1, \cdots, d$ we have by definition of $L$ for some $i \le d$, $\psi(r_i)^{8d} < \prod_{i=1}^{d} \psi(r_i) \le n^d$, hence $\psi(r_i) \le n^{1/8}$. And therefore

$$\mathrm{Card}(R_{\mathbf{r}} \cap \mathbb{N}^d) \le n^{1/8} \cdot n^{d-1} = n^{d-7/8}.$$

Furthermore we have

$$\begin{aligned}
&\mathrm{Card}\{t : t \in \mathbb{N}, \psi(t) \le n\} \le \mathrm{Card}\{t : t \in \mathbb{N}, ct^{\beta+1} \le n\} \le cn^{1/(\beta+1)}, \\
&\mathrm{Card}\{\mathbf{r} : \mathbf{r} \in \mathbb{N}^d, \psi(r_i) \le n, i = 1, \cdots, d\} \le cn^{d/(\beta+1)}.
\end{aligned}$$

By Lemma 11.2.1 and the Chebyshev inequality this implies

$$
\begin{aligned}
&P\Big\{ \sum_{\mathbf{r}\notin L, \psi(r_i)\le n} \Big\| \sum_{\mathbf{j}\in R_{\mathbf{r}}} X_{\mathbf{j}} \Big\| > n^{d/2-1/16} \Big\} \\
&\le cn^{d/(\beta+1)} \max_{\mathbf{r}\notin L, \psi(r_i)\le n} P\Big\{ \Big\| \sum_{\mathbf{j}\in R_{\mathbf{r}}} X_{\mathbf{j}} \Big\| \ge cn^{\frac{d}{2}-\frac{1}{16}-\frac{d}{\beta+1}} \Big\} \\
&\le cn^{\frac{d}{\beta+1}} \cdot n^{d-\frac{7}{8}-d+\frac{1}{8}+\frac{2d}{\beta+1}} \\
&= cn^{-\frac{3}{4}+\frac{3d}{\beta+1}} \\
&\le cm^{-2}.
\end{aligned}
$$

Lemma 11.2.4 is proved.

From Lemma 11.2.3 it follows that $I_{13} \vee I_{23} = O(n^{d/2-\gamma'})$. Finally since

$$
\begin{aligned}
I_4 \le &\sum_{\mathbf{r}\in L, \psi(r_i)\le n} \Big\| \sum_{\mathbf{j}\in R_{\mathbf{r}}} (X_{\mathbf{j}} - Y_{\mathbf{j}}) \Big\| \\
&+ \sum_{\mathbf{r}\notin L, \psi(r_i)\le n} \Big\| \sum_{\mathbf{j}\in R_{\mathbf{r}}} (X_{\mathbf{j}} - Y_{\mathbf{j}}) \Big\|,
\end{aligned}
$$

we have $I_4 = O(n^{d/2-\gamma'})$ a.s. by Lemma 11.2.4 and (11.2.11). Therefore the proof of Theorem 11.2.1 is completed.

# Part IV Statistics of a Dependent Sample

The limit behavior of various kinds of statistics with a dependent random sample have been studied by many authors since the sixties. We shall introduce some of them in this part. We first introduce weak convergence and strong approximations of an empirical process with a mixing dependent sample in Chapter 12. The limit behavior of U-statistics, estimations of error variance in a linear model and estimations of density function with a mixing dependent sample are discussed in Chapter 13. We investigate in Chapter 14 the asymptotic fluctuation behavior of sums of other kinds of dependent random variables, such as lacunary trigonometric series, a Gaussian sequence and the additive functional of a Markov process.

# Chapter 12 Empirical Processes

Let $\{X_n, n \geq 1\}$ be a sequence of random variables with a common distribution function $F$. Then the empirical distribution function $F_n$ of $X_1, \cdots, X_n$ is defined by $F_n(t) = n^{-1} \sum_{i=1}^{n} I(X_i \leq t)$, $-\infty < x < \infty$. The $n$th empirical process $\beta_n$ is defined by

$$\beta_n(t) = \sqrt{n}(F_n(t) - F(t)), \quad -\infty < t < \infty. \tag{12.0.1}$$

If $F$ is continuous, then $U_n := F(X_n)$ for all $n \geq 1$ are uniformly distributed over [0,1]. The corresponding empirical process is

$$\alpha_n(t) = \sqrt{n}(E_n(t) - t), \quad 0 \leq t \leq 1, \tag{12.0.2}$$

where the empirical distribution function

$$E_n(t) = F_n(\mathrm{inv}F(t)) = \frac{1}{n} \sum_{i=1}^{n} I(U_i \leq t), \quad 0 \leq t \leq 1. \tag{12.0.3}$$

Thus, in term of $U_i = F(X_i)$, $i = 1, \cdots, n$, we have for any continuous $F$

$$\{\beta_n(\mathrm{inv}F(t)), 0 \leq t \leq 1\} = \{\alpha_n(t), 0 \leq t \leq 1\}, \quad n = 1, 2, \cdots. \tag{12.0.4}$$

This implies that all theorems proved for $\alpha_n$ will hold automatically for $\beta_n$ as well, simply by letting $y = F(x)$ in (12.0.4). So we shall mainly concern with uniform empirical process $\alpha_n$ in this Chapter.

In the independent case, it is well-known that

$$\alpha_n \Longrightarrow B \quad \text{as } n \to \infty \tag{12.0.5}$$

where $B$ is a Brownian bridge. The strong approximations of $\{\alpha_n(\cdot)\}$ by a sequence of Brownian bridges were discussed by Komlós-Major-Tusnády (1975). They showed that without changing the distribution of $\{\alpha_n(t), n \geq 1\}$, one can redefine the $\{\alpha_n(t)\}$ on a richer probability space together with a sequence $\{B_n\}$ of independent Brownian bridges such that

$$\sup_{0 \leq t \leq 1} |\alpha_n(t) - B_n(t)| = O(n^{-1/2} \log n) \quad \text{a.s..} \tag{12.0.6}$$

For a strictly stationary sequence $\{U_n, n \geq 1\}$, the conditions, under which (12 0.5) and (12.0.6) also hold true, have been given by many mathematicians. This chapter is organized as follows. We shall introduce the best results of weak convergence of empirical processes when the sample is mixing dependent in Section 12.1. The weighted weak convergence for empirical processes of $\alpha$-mixing and $\rho$-mixing sequences will be established in Section 12.2. The strong approximations for empirical processes with a mixing sample by Gaussian processes will be introduced in Section 12.3 and the moduli of continuity of empirical processes when the sample is mixing dependent will be studied in Section 12.4.

## 12.1 Weak convergence

Let $\{U_n, n \geq 1\}$ be a sequence of strictly stationary random variables uniformly distributed over $[0, 1]$, denote

$$R(s,t) = s \wedge t - st + \sum_{k=2}^{\infty} E\{[I(U_k \leq s) - s][I(U_1 \leq t) - t] + [I(U_1 \leq s) - s][I(U_k \leq t) - t]\}. \tag{12.1.1}$$

When $\{U_n\}$ is $\alpha$-mixing, the series in (12.1.1) converges absolutely if $\sum_n \alpha(n) < \infty$ by Lemma 1.2.1.

The weak convergence of empirical processes $\{\alpha_n(t), 0 \leq t \leq 1\}$ of $\{U_n\}$ has been discussed by some scientists. For an insightful view of this subject we refer to Billingsley (1968), Sen (1971, 1974), Yoshihara (1975, 1978) and Shao (1986), etc, until now the best results are due to Shao (1986).

### *12.1.1 Weak convergence for a $\varphi$-mixing sample*

**Theorem 12.1.1.** *Let $\{U_n, n \geq 1\}$ be a sequence of strictly stationary $\varphi$-mixing random variables uniformly distributed over [0,1], and let*

$$\alpha_n(t) = \sqrt{n}(E_n(t) - t). \tag{12.1.2}$$

*If the series of (12.1.1) converges absolutely and*

$$\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty, \tag{12.1.3}$$

*then we have*

$$\alpha_n \Longrightarrow Y \quad \text{in } D[0, 1],$$

*where* $Y = \{Y(t), 0 \le t \le 1\}$ *is a Gaussian process with* $EY(t) = 0$, $EY(s)Y(t) = R(s,t)$.

**Proof.** For any given $t \in [0,1]$,

$$\alpha_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (I(X_k \le t) - t).$$

By Corollary 4.1.1, $\alpha_n(t)$ converge to $Y(t)$ in distribution and it is easy to see that the finite dimensional distributions of $\alpha_n$ converge to the corresponding finite dimensional distributions of $Y$.

In order to prove the tightness of $\{\alpha_n\}$, we need only to show that for any $\varepsilon > 0$, $\eta > 0$ there exists a $\delta, 0 < \delta < 1$, such that for any $s, 0 \le s \le 1-\delta$, and large $n$

$$P\{\sup_{s \le t \le s+\delta} |\alpha_n(t) - \alpha_n(s)| \ge 4\varepsilon\} < \eta\delta. \tag{12.1.4}$$

For $0 \le s < t \le 1$ denote $\xi_i = (I(U_i \le t) - t) - (I(U_i \le s) - s) = I(s < U_i \le t) - (t-s)$. We have

$$E\xi_i = 0,\ |\xi_i| \le 1,\ E\xi_i^2 = t - s - (t-s)^2 \le t - s.$$

From Lemma 2.2.2 and Lemma 2.2.8 it follows that

$$\begin{aligned} E|\alpha_n(t) - \alpha_n(s)|^4 &\le \frac{1}{n^2} E\Big(\sum_{j=1}^{n} \xi_j\Big)^4 \\ &\le C(t-s)^2 \end{aligned} \tag{12.1.5}$$

for some $C > 0$. Let $p$ be a positive number such that $\varepsilon/n \le p$. Consider the random variables $\alpha_n(s+ip) - \alpha_n(s+(i-1)p)$, $i = 1, 2, \cdots, m$. From Theorem 12.2 of Billingsley (1968) we have

$$P\Big\{\max_{0 \le i \le m} |\alpha_n(s+ip) - \alpha_n(s)| \ge \lambda\Big\} \le \frac{K}{\varepsilon\lambda^4} m^2 p^2 \tag{12.1.6}$$

where constant $K$ depends only on $\varphi(\cdot)$.

For $s \le t \le s+p$, we have

$$|\alpha_n(t) - \alpha_n(s)| \le |\alpha_n(s+p) - \alpha_n(s)| + p\sqrt{n}, \tag{12.1.7}$$

which implies

$$\begin{aligned} &\sup_{s \le t \le s+pm} |\alpha_n(t) - \alpha_n(s)| \\ &\quad \le \sup_{s \le t \le s+pm} \{|\alpha_n(t) - \alpha_n(s+ip)| + |\alpha_n(s+ip) - \alpha_n(s)|\} \\ &\quad \le 3\max_{i \le m} |\alpha_n(s+ip) - \alpha_n(s)| + p\sqrt{n}. \end{aligned} \tag{12.1.8}$$

If $\varepsilon/n \le p < \varepsilon/\sqrt{n}$, from (12.1.6) and (12.1.8) it follows that

$$P\Big\{\sup_{s\le t\le s+pm} |\alpha_n(t) - \alpha_n(s)| \ge 4\varepsilon\Big\} \le \frac{K}{\varepsilon^5} m^2 p^2. \tag{12.1.9}$$

Take $\delta$ such that $K\delta/\varepsilon^5 < \eta$. For large $n$ there exists an $m$ such that $(\delta/\varepsilon)\sqrt{n} < m \le (\delta/\varepsilon)n$ and $mp = \delta$. It follows from (12.1.8) and (12.1.9) that (12.1.4) holds true. The proof of Theorem 12.1.1 is completed.

**Corollary 12.1.1.** *Let $\{U_n, n \ge 1\}$ be a sequence of strictly stationary $\rho$-mixing random variables uniformly distributed over $[0,1]$. If the series in (12.1.1) converges absolutely and $\sum_{n=1}^{\infty} \rho(2^n) < \infty$, then $\alpha_n \Longrightarrow Y$.*

**Remark 12.1.1.** In the case of a $p$-dimensional sample ($p \ge 2$), the result of Theorem 12.1.1 is also true.

**Remark 12.1.2.** The Glivenko-Cantelli theorem, i.e. almost sure convergence of empirical processes of mixing sequences, is an immediate consequence of the results in Sections 8.3–8.5. For example, from Corollary 8.3.1 we have

**Proposition 12.1.1.** Let $\{X_n, \ge 1\}$ be a sequence of $\varphi$-mixing random variables with a common continuous distribution $F(x)$. Then for any $\theta > 0$, $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{|F_n(x) - F(x)| \ge \varepsilon\, n^{-1/2+\theta}\} < \infty,$$

that is to say the rate of convergence in the Glivenko-Cantelli theorem is given as follows:

$$|F_n(x) - F(x)| = o(n^{-1/2+\theta}) \qquad \text{a.s.}$$

When the sample is $\rho$-mixing or $\alpha$-mixng, there is the similar results under certain conditions.

### *12.1.2 Weak convergence for an $\alpha$-mixing sample*

**Theorem 12.1.2.** *Let $\{U_n, n \ge 1\}$ be a sequence of strictly stationary $\alpha$-mixing random variables uniformly distributed over $[0,1]$ and*

$$\alpha(n) = O(n^{-r}) \quad r > 2. \tag{12.1.10}$$

*Then $\alpha_n \Longrightarrow Y$.*

By the following lemmas, from the proof of Theoem 12.1.1 it follows that Theorem 12.1.2 holds true.

**Lemma 12.1.1.** *Let $\{\xi_n, n \geq 1\}$ be a sequence of strictly stationary $\alpha$-mixing random variables with $|\xi_1| \leq 1$ a.s., $E\xi_1 = 0$, $E\xi_1^2 = \tau$, $E|\xi_1| = 2\tau$ and satisfying (12.1.10). Put $S_n = \sum_{j=1}^n \xi_j$. Suppose that $r > 2$ is a non-integer and $m = 2k$, where $k$ is an integer, satisfies $r - 1 < m < r + 1$. Then we have*

$$E|S_n|^m \leq c \sum_{i=0}^{(m-2)/2} \left(n^{i+1}\tau^{i\theta_1+\theta_{m-1-2i}}\right) \tag{12.1.11}$$

*where $[r] - 1 < r\theta_1 < r - 1$, $\theta_k = \theta_1 - (k-1)/r$, $k = 1, \cdots, [r]$.*

**Proof.** By (12.1.10) and the definition of $\theta_k$, we have

$$\sum_{i=0}^{\infty}(i+1)^{k-1}\alpha^{1-\theta_k}(i) < \infty, \tag{12.1.12}$$

for $k = 1, \cdots, [r]$. Let $\sum_{n(k)}$ be a summation for $i_1, \cdots, i_k \geq 0$, $i_1 + \cdots + i_k \leq n$, and $\sum_{n(k)}^{(l)}$ be a summation for $i_1, \cdots, i_k \geq 0$, $i_1 + \cdots + i_k \leq n$ and $i_l = \max_{1 \leq j \leq k} i_j$, where $l = 1, \cdots, k$; $k = 1, \cdots, [r]$. Denote

$$\sum\nolimits_{n(k)}^{(l)} = \sum\nolimits_{n(k)}^{(l)} |E\xi_0\xi_{i_1}\xi_{i_1+i_2}\cdots\xi_{i_1+\cdots+i_k}|.$$

Note that for any even $d$, by the stationarity, we have

$$E|S_n|^d \leq d!n\sum\nolimits_{n(d-1)}|E\xi_0\xi_{i_1}\cdots\xi_{i_1+\cdots+i_{d-1}}|. \tag{12.1.13}$$

From Lemma 1.2.5 and (12.1.12) we have

$$\sum\nolimits_{n(1)}|E\xi_0\xi_{i_1}| \leq 6\sum_{i_1=0}^{n}\alpha^{1-\theta_1}(i_1)\tau^{\theta_1} \leq c\,\tau^{\theta_1}, \tag{12.1.14}$$

$$\begin{aligned}\sum\nolimits_{n(2)}|E\xi_0\xi_{i_1}\xi_{i_1+i_2}| &\leq \left\{\sum\nolimits_{n(2)}^{(1)} + \sum\nolimits_{n(2)}^{(2)}\right\}|E\xi_0\xi_{i_1}\xi_{i_1+i_2}| \\ &\leq 6\sum\nolimits_{n(2)}^{(1)}\alpha^{1-\theta_2}(i_1)\tau^{\theta_2} + 6\sum\nolimits_{n(2)}^{(2)}\alpha^{1-\theta_2}(i_2)\tau^{\theta_2} \\ &\leq 6\sum_{i_1=0}^{n}(i_1+1)\alpha^{1-\theta_2}(i_1)\tau^{\theta_2} + 6\sum_{i_2=0}^{n}(i_2+1)\alpha^{1-\theta_2}(i_2)\tau^{\theta_2} \\ &\leq c\tau^{\theta_2}.\end{aligned} \tag{12.1.15}$$

Next we prove

$$\sum\nolimits_{n(k)} \leq c\sum_{i=0}^{[(k-1)/2]} n^i\tau^{i\theta_1+\theta_{k-2i}} \tag{12.1.16}$$

for $k = 1, 2, \cdots, [r]$ by the induction. From (12.1.14) and (12.1.15) it follows that (12.1.16) holds true for $k = 1, 2$. Assume that (12.1.16) holds true for $k-1$, $2 \le k-1 < [r]$, we show that (12.1.16) also holds true for $k$. Note that

$$\sum\nolimits_{n(k)} \le \sum\nolimits_{n(k)}^{(1)} + \cdots + \sum\nolimits_{n(k)}^{(k)},$$
$$\sum\nolimits_{n(k)}^{(1)} \le 6\sum\nolimits_{n(k)}^{(1)} \alpha^{1-\theta_k}(i_1)\tau^{\theta_k}$$
$$\le 6\sum_{i_1=0}^{n}(i_1+1)^{k-1}\alpha^{1-\theta_k}(i_1)\tau^{\theta_k} \le c\tau^{\theta_k}.$$

Similarly, $\sum_{n(k)}^{(k)} \le c\tau^{\theta_k}$. For $2 \le l \le k-1$,

$$\sum\nolimits_{n(k)}^{(l)} \le 6\sum\nolimits_{n(k)}^{(l)} \alpha^{1-\theta_k}(i_l)\tau^{\theta_k}$$
$$+ \sum\nolimits_{n(k)}^{(l)} |E\xi_0\xi_{i_1}\cdots\xi_{i_1+\cdots+i_{l-1}}| \cdot |E\xi_{i_1+\cdots+i_l}\cdots\xi_{i_1+\cdots+i_k}|$$
$$=: I_1 + I_2.$$

We also have $I_1 \le c\tau^{\theta_k}$. By the induction and stationarity we have

$$I_2 \le cn\Big(\sum_{i=0}^{[(l-2)/2]} n^i\tau^{i\theta_1+\theta_{l-1-2i}}\Big)\Big(\sum_{j=0}^{[(k-l-1)/2]} n^j\tau^{j\theta_1+\theta_{k-l-2j}}\Big)$$
$$\le c \sum_{0\le i\le[\frac{l-2}{2}],\ 0\le j\le[\frac{k-l-1}{2}]} n^{i+j+1}\tau^{(i+j+1)\theta_1+\theta_{k-2(i+j+1)}}$$
$$\le c\sum_{i=1}^{[(k-1)/2]} n^i\tau^{i\theta_1+\theta_{k-2i}}.$$

Thus (12.1.16) holds true for $k = 1, 2, \cdots, [r]$. From (12.1.13), (12.1.16) and noting $m-1 \le [r]$ we complete the proof of Lemma 12.1.1.

**Lemma 12.1.2.** *If the conditions of Lemma 12.1.1 are satisfied with* $2 < r < 3$, *then*

$$ES_n^4 \le c(n^{4-r} + n^2\tau^{2\theta_1}). \tag{12.1.17}$$

**Proof.** From Lemma 1.2.5 and (12.1.14) we have

$$\sum\nolimits_{n(3)}^{(1)} \le 6\sum\nolimits_{n(3)}^{(1)}\alpha(i_1) \le 6\sum_{i_1=0}^{n}(i_1+1)^2\alpha(i_1) \le cn^{3-r}.$$

Similarly $\sum_{n(3)}^{(3)} \le cn^{3-r}$. And by (12.1.16) we have

$$\begin{aligned}\sum\nolimits_{n(3)}^{(2)} &\le 6\sum\nolimits_{n(3)}^{(2)}\alpha(i_2) + \sum\nolimits_{n(3)}^{(2)}|E\xi_0\xi_{i_1}| \cdot |E\xi_{i_1+i_2}\xi_{i_1+i_2+i_3}| \\ &\le c(n^{3-r} + n\tau^{2\theta_1}).\end{aligned}$$

Combining these inequalities with (12.1.13) yields (12.1.17).

Now Theorem 12.1.2 can be proved along the similar lines to that of Theorem 12.1.1 by applying Lemma 12.1.2 instead of Lemma 2.2.2 and Lemma 2.2.8.

**Remark 12.1.3.** In the case of a $p$-dimensional sample ($p \ge 2$), by using Lemma 12.1.1, the result of Theorem 12.1.2 is also true if

$r > p + 1$ when $p$ is even;

$r > p^2/(p-1)$ when $p$ is odd.

**Remark 12.1.4.** The weak convergence of partial-sum processes and empirical processes with random indexes has been discussed by some scientist. For this subject we refer to Rényi (1958, 1960), Billingsley (1968 §17), Aldous (1978) and Lu (1984), etc. We enumerate only a result without proof for the weak convergence of empirical processes with random indexes as follows:

Let $\{\alpha_n, n \ge 1\}$ be a sequence of empirical processes as in Theorem 12.1.2 and let $\{\tau_n, n \ge 1\}$ be a sequence of positive integer-valued random variables on the same probability space. Suppose that $\alpha_n \Longrightarrow Y$, where $Y$ is a Gaussian process and $\{\tau_n\}$ satisfies

$$\tau_n/n \xrightarrow{P} \tau,$$

where $\tau$ is a positive random variable. Then we have

$$\alpha_{\tau_n} \Longrightarrow Y.$$

## 12.2 Weighted weak convergence

Let $q$ be a positive weight function on $(0,1)$, i.e. $\inf_{\delta \le t \le 1-\delta} q(t) > 0$ for all $0 < \delta < 1/2$, and define the weighted uniform empirical processes as $\{\alpha_n(t)/q(t), 0 < t < 1\}$. When $\{U_n, n \ge 1\}$ is a sequence of independent random variables uniformly distributed on $[0,1]$, the weighted weak convergence of empirical processes has been intensively studied in recent years. For an insightful view of this subject we refer to M. Csörgő, S. Csörgő, Horváth and Mason (1986a), Shorack and Wellner (1986) and

Csörgő and Horváth (1993), etc. We restate a theorem of M. Csörgő, S. Csörgő, Horváth and Mason (1986a) as follows. A shorter and more direct proof was given by Csörgő and Horváth (1993 Chapter 4).

**Theorem 12.2.1.** *Assume that $q$ is positive and continuous on $(0,1)$, and is nondecreasing in a neighborhood of $0$ and nonincreasing in a neighborhood of $1$. Then*

$$I(q,\lambda) := \int_0^1 \frac{1}{t(1-t)} \exp(-\lambda q^2(t)/(t(1-t)))dt < \infty \tag{12.2.1}$$

*for all $\lambda > 0$ if and only if, as $n \to \infty$*

$$\alpha_n/q \Longrightarrow B/q \qquad \text{in } D[0,1]. \tag{12.2.2}$$

There are only a few studies concerned with the weighted weak convergence for empirical processes of a dependent sequence. In the latter case the limit process is changed from being a Brownian bridge, due to the appearence of covariances among observations (cf. (12.1.1)).

Shao and Yu (1995) studied the weighted weak convergence for empirical processes of strictly stationary observations under mixing and associated dependence assumptions. We only introduce the case of mixing dependence here. First we give the following basic theorem.

**Theorem 12.2.2.** *Let $\{U_n, n \geq 1\}$ be a strictly stationary sequence of uniform-[0, 1] random variables. Assume that for all $0 \leq s,t \leq 1$ and $n \geq 1$ we have*

*(A1)* $E|\alpha_n(t) - \alpha_n(s)|^p \leq C_1(|t-s|^{p_1} + n^{-p_2/2}|t-s|^{r_1})$ *for some* $C_1 > 0, p > 2$, $p_1 > 1$, $0 \leq r_1 \leq 1$ *and* $p_2 > 1 - r_1$*;*

*(A2)* $E(\alpha_n(t)-\alpha_n(s))^2 \leq C_2|t-s|^{r_2}$ *for some* $C_2 > 0$ *and* $0 < r_2 \leq 1$.
*If we have*

$$\alpha_n \Longrightarrow Y \qquad \text{in } D[0,1] \tag{12.2.3}$$

*with the Gaussian process $Y(\cdot)$ defined as in § 12.1, then*

$$\alpha_n/q \Longrightarrow Y/q \qquad \text{in } D[0,1], \tag{12.2.4}$$

*where $q$ is a weight function such that for some $C > 0$ and $\beta > 1/2$*

$$q(t) \geq C(t(1-t))^{\mu}(\log 1/(t(1-t)))^{\beta} \qquad \text{for all } 0 < t < 1 \tag{12.2.5}$$

*and*

$$\mu = \min\Big(\frac{p_1}{p}, \frac{r_1+p_2}{p+p_2}, \frac{r_2}{2}\Big). \tag{12.2.6}$$

**Remark 12.2.1.** By using a standard argument (cf. Theorem 12.2 and (22.18) in Billingsley 1968), one can easily verify that $\{\alpha_n(t), 0 \le t \le 1\}$ is tight by condition (A1). Hence to show (12.2.3), one needs only to prove that any finite dimensional distribution of $\{\alpha_n(t)\}$ converges to that of $\{Y(t)\}$ and the series in (12.1.1) converges absolutely.

**Remark 12.2.2.** The weight function $q$ used in Theorem 12.2.1 is usually called a Chibisov-O'Reilly weight function. If we write $q(t) = (t(1-t)\log\log(1/(t(1-t))))^{1/2}g(t)$, then, necessarily, $g(t) \to \infty$ as $t \to 0$ or $t \to 1$. Thus the weight function $q$ in (12.2.5) can be compared to a Chibisov-O'Reilly weight function by taking $\mu$ in (12.2.6) close to $1/2$ or exactly $1/2$ for properly chosen $p, p_1, p_2, r_1$, and $r_2$. In fact, Theorems 12.2.3 and 12.2.4 below show this possibility of taking $\mu = 1/(2+\varepsilon)$ for some $\varepsilon > 0$ in the case of mixing sequences. In particular our sharpest rate with $\mu = 1/2$ is obtained for $\rho$-mixing under a stronger mixing decay rate. In most cases, however, $\mu < 1/2$. We note in passing that if for a general weight function $q$ we have $\int_0^1 1/q^2(t)dt < \infty$, then we have (12.2.1) as well for all $\lambda > 0$, i.e., $q$ is then necessarily a Chibisov-O'Reilly weight function.

**Remark 12.2.3.** If $\mu = (r_1 + p_2)/(p + p_2) < \min(p_1/p, r_2/2)$ in (12.2.6), then from the proof of Theorem 12.2.2, one can relax the restriction on $\beta$ from $\beta > 1/2$ to $\beta > 1/(p+p_2) = (1-\mu)/(p-r_1)$. Moreover, in the case of $\mu \ge 1/(p+1-r_1)$, one can use a simple sufficient condition $\int_0^1 1/q^{1/\mu}(t)dt < \infty$ to replace (12.2.5).

A direct application of Theorem 12.2.2 is to obtain weak convergence for integral functionals of $\alpha_n$. For example, we consider the integral functional

$$\Delta_n(t) = \int_0^t \alpha_n(s)dQ(s) = \int_0^t \beta_n(\text{inv}F(s))dQ(s), \quad 0 \le t \le 1$$

and its approximating Gaussian counterpart

$$\Delta(t) = \int_0^t Y(s)dQ(s), \quad 0 \le t \le 1, \tag{12.2.7}$$

where $Q(s) = \text{inv}F(s)$ is the quantile function of distribution function $F$ of $X$ (recalling (12.0.4)). The function $\Delta(t)$ plays a central role in weak approximation theory for empirical total time on test, mean residual life, empirical Lorenz and Goldie concentration processes which are of interest in reliability and economic concentration theories. (cf. e.g., M. Csörgő, S. Csörgő, Horváth and Mason 1986b).

**Corollary 12.2.1.** *Under the conditions of Theorem 12.2.2, if*

$$\int_0^1 (t(1-t))^{\mu}(\log 1/(t(1-t)))^{\beta} dQ(t) < \infty, \tag{12.2.8}$$

*then*

$$\Delta_n \Longrightarrow \Delta \qquad \text{in } D[0,1].$$

**Remark 12.2.4.** Let $F$ be the distribution function of a random variable $X$. Then condition (12.2.8) is sightly stronger than the existence of the $(1/\mu)$-th moment of $X$. This is not necessarily true conversely, but $E|X|^{1/\mu}(\log(1+|X|))^{(1+\beta)/\mu+\delta} < \infty$, with any $\delta > 0$, implies (12.2.8).

Theorem 12.2.2 enables us to establish weighted weak convergence for empirical processes of a stationary mixing sequence.

**Theorem 12.2.3.** *Let $\{U_n, n \geq 1\}$ be a strictly stationary $\alpha$-mixing sequence of uniform-[0, 1] random variables. If*

$$\alpha(n) = O(n^{-\theta-\varepsilon}) \tag{12.2.9}$$

*for some $\theta \geq 1+\sqrt{2}$ and $\varepsilon > 0$, then we have*

$$\alpha_n/q \Longrightarrow Y/q \quad \text{in } D[0,1]$$

*for $q$ satisfying $q(t) \geq C(t(1-t))^{(1-1/\theta)/2}$ for some $C > 0$.*

**Theorem 12.2.4.** *Let $\{U_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequence of uniform-[0, 1] random variables. Suppose that the series in (12.1.1) converges absolutely. If*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty, \tag{12.2.10}$$

*then for any $\varepsilon > 0$ we have*

$$\alpha_n/q \Longrightarrow Y/q \quad \text{in } D[0,1]$$

*for $q$ satisfying $q(t) \geq C(t(1-t))^{1/(2+\varepsilon)}$ for some $C > 0$.*

*If, in addition,*

$$\sum_{n=1}^{\infty} \rho^{2/p}(2^n) < \infty \tag{12.2.11}$$

*for some $p > 2$, then we have*

$$\alpha_n/q \Longrightarrow Y/q \quad \text{in } D[0,1]$$

*for $q$ satisfying $q(t) \geq C(t(1-t))^{1/2}(\log 1/(t(1-t)))^{\beta}$ for some $C > 0$ and $\beta > 1/2$.*

**Corollary 12.2.2.** *Under the conditions of Theorem 12.2.3, if*

$$\int_{-\infty}^{\infty} |x|^{2/(1-1/\theta)} dF(x) < \infty,$$

*then*

$$\Delta_n \Longrightarrow \Delta \quad \text{in } D[0,1].$$

**Corollary 12.2.3.** *Let $\{U_n, n \geq 1\}$ be a strictly stationary $\rho$-mixing sequences of uniform-[0, 1] random variables. If (12.2.10) holds and for any $\varepsilon > 0$ we have*

$$\int_{-\infty}^{\infty} |x|^{2+\varepsilon} dF(x) < \infty,$$

*then*

$$\Delta_n \Longrightarrow \Delta.$$

*If, in addition, (12.2.11) holds and*

$$\int_0^1 (t(1-t))^{1/2}(\log 1/(t(1-t)))^{\beta} dQ(t) < \infty,$$

*then we have*

$$\Delta_n \Longrightarrow \Delta \quad \textit{in } D[0,1].$$

In order to prove Theorems, we need the following lemmas.

**Lemma 12.2.1.** *Let $\{\xi_i, i \geq 1\}$ be a sequence of random variables and let $\mathcal{F}_i = \sigma(\xi_j, j \leq i)$. Then, for any $p \geq 2$, there exists a constant $D = D(p)$ such that*

$$\begin{aligned} E\Big|\sum_{i=1}^n \xi_i\Big|^p \leq & D\Big(\Big(\sum_{i=1}^n E\xi_i^2\Big)^{p/2} + \sum_{i=1}^n E|\xi_i|^p + n^{p-1}\sum_{i=1}^n E|E(\xi_i|\mathcal{F}_{i-1})|^p \\ & + n^{p/2-1}\sum_{i=1}^n E|E(\xi_i^2|\mathcal{F}_{i-1}) - E\xi_i^2|^{p/2}\Big). \end{aligned} \tag{12.2.12}$$

**Proof.** Let $\eta_i = \xi_i - E(\xi_i|\mathcal{F}_{i-1})$ for $1 \leq i \leq n$. Then, $\{\eta_i, \mathcal{F}_{i-1}, 1 \leq i \leq n\}$ is a martingale difference sequence. By the well-known Burkholder

(1973) inequality, there is a $D = D(p) < \infty$ such that

$$\begin{aligned} E\Big|\sum_{i=1}^n \eta_i\Big|^p &\le D\Big(E\Big(\sum_{i=1}^n E(\eta_i^2|\mathcal{F}_{i-1})\Big)^{p/2} + \sum_{i=1}^n E|\eta_i|^p\Big) \\ &\le 2^p D\Big(\Big(\sum_{i=1}^n E\xi_i^2\Big)^{p/2} + \sum_{i=1}^n E|\xi_i|^p \\ &\quad + E\Big(\sum_{i=1}^n |E(\xi_i^2|\mathcal{F}_{i-1}) - E\xi_i^2|\Big)^{p/2}\Big) \\ &\le 2^{2p} D\Big(\Big(\sum_{i=1}^n E\xi_i^2\Big)^{p/2} + \sum_{i=1}^n E|\xi_i|^p \\ &\quad + n^{p/2-1}\sum_{i=1}^n E|E(\xi_i^2|\mathcal{F}_{i-1}) - E\xi_i^2|^{p/2}\Big). \end{aligned} \tag{12.2.13}$$

On the other hand, it is easy to see that

$$E\Big|\sum_{i=1}^n \xi_i\Big|^p \le 2^p\Big(E\Big|\sum_{i=1}^n \eta_i\Big|^p + n^{p-1}\sum_{i=1}^n E|E(\xi_i|\mathcal{F}_{i-1})|^p\Big).$$

This proves (12.2.12) by the inequalities above.

We now develop a Rosenthal-type inequality for an $\alpha$-mixing sequence which is of its own interest.

**Lemma 12.2.2.** *Let $2 < p < r \le \infty$, $2 < v \le r$ and $\{X_n, n \ge 1\}$ be an $\alpha$-mixing sequence of random variables with $EX_n = 0$ and $\|X_n\|_r < \infty$. Assume that*

$$\alpha(n) \le Cn^{-\theta} \tag{12.2.14}$$

*for some $C > 0$ and $\theta > 0$. Then, for any $\varepsilon > 0$, there exists a $K = K(\varepsilon, r, p, v, \theta, C) < \infty$ such that*

$$E|S_n|^p \le K\Big((nC_n)^{p/2}\max_{i\le n}\|X_i\|_v^p + n^{(p-(r-p)\theta/r)\vee(1+\varepsilon)}\max_{i\le n}\|X_i\|_r^p\Big), \tag{12.2.15}$$

*where $C_n = \Big(\sum_{i=o}^n (i+1)^{2/(v-2)}\alpha(i)\Big)^{(v-2)/v}$. In particular, for any $\varepsilon > 0$,*

$$E|S_n|^p \le K\Big(n^{p/2}\max_{i\le n}\|X_i\|_v^p + n^{1+\varepsilon}\max_{i\le n}\|X_i\|_r^p\Big), \tag{12.2.16}$$

*if $\theta > v/(v-2)$ and $\theta \ge (p-1)r/(r-p)$, and*

$$E|S_n|^p \le Kn^{p/2}\max_{i\le n}\|X_i\|_r^p, \tag{12.2.17}$$

*if* $\theta \geq pr/(2(r-p))$.

**Proof.** For the sake of convenience of statement, we assume that $\{X, X_n, n \geq 1\}$ is a strictly stationary $\alpha$-mixing sequence. By a result of Rio (1993), there is $D_1 = D_1(v)$ such that

$$ES_n^2 \leq D_1 n C_n \|X\|_v^2. \tag{12.2.18}$$

We shall prove (12.2.15) by induction on $n$. Suppose that for each $1 \leq k < n$,

$$E|S_k|^p \leq K\Big((kC_k)^{p/2}\|X\|_v^p + k^{(p-(r-p)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big). \tag{12.2.19}$$

We now prove that (12.2.19) is still true for $k = n$. Let $0 < a < 1/2$ that will be specified later and let $m = [an] + 1$. Define

$$\xi_i = \sum_{j=2(i-1)m+1}^{n\wedge(2i-1)m} X_j \quad \text{and} \quad \eta_i = \sum_{j=(2i-1)m+1}^{n\wedge 2im} X_j$$

for $1 \leq i \leq k_n := [n/(2m)] + 1$. Clearly, we have

$$E|S_n|^p \leq 2^{p-1}\Big(E\Big|\sum_{i=1}^{k_n}\xi_i\Big|^p + E\Big|\sum_{i=1}^{k_n}\eta_i\Big|^p\Big) =: 2^{p-1}(I_1 + I_2).$$

Let $\mathcal{F}_i = \sigma(\xi_j, j \leq i)$. It follows from Lemma 12.2.1 that there is $D_2$ such that $D_2 \geq (2D_1)^{p/2}$ and

$$\begin{aligned} I_1 &\leq D_2\Big(\sum_{i=1}^{k_n} E|\xi_i|^p + \Big(\sum_{i=1}^{k_n} E\xi_i^2\Big)^{p/2} + k_n^{p-1}\sum_{i=1}^{k_n} E|E(\xi_i|\mathcal{F}_{i-1})|^p \\ &\quad + k_n^{p/2-1}\sum_{i=1}^{k_n} E|E(\xi_i^2|\mathcal{F}_{i-1}) - E\xi_i^2|^{p/2}\Big) \\ &=: D_2\Big(\sum_{i=1}^{k_n} E|\xi_i|^p + I_{11} + I_{12} + I_{13}\Big). \end{aligned} \tag{12.2.20}$$

In terms of (12.2.18), we have

$$\begin{aligned} I_{11} &\leq (D_1 k_n m C_m \|X\|_v^2)^{p/2} \\ &\leq (2D_1 n C_n)^{p/2}\|X\|_v^p \leq D_2(nC_n)^{p/2}\|X\|_v^p. \end{aligned} \tag{12.2.21}$$

To estimate $I_{13}$, we write

$$Y_i = E(\xi_i^2|\mathcal{F}_{i-1}) - E\xi_i^2.$$

Then, by Lemma 1.2.4

$$\begin{aligned}
E|Y_i|^{p/2} &= E|Y_i|^{p/2-1}\mathrm{sgn}(Y_i)Y_i = E(|Y_i|^{p/2-1}\mathrm{sgn}(Y_i)(\xi_i^2 - E\xi_i^2)) \\
&= \sum_{2(i-1)m<j,l\le n\wedge(2i-1)m} E|Y_i|^{p/2-1}\mathrm{sgn}(Y_i)(X_jX_l - EX_jX_l) \\
&\le 12 \sum_{2(i-1)m<j,l\le n\wedge(2i-1)m} \alpha^{1-\frac{p-2}{p}-\frac{2}{r}}(m) \\
&\quad \cdot (E|Y_i|^{p/2})^{(p-2)/p}\|X_jX_l\|_{r/2} \\
&\le 12m^2\alpha^{\frac{2}{p}-\frac{2}{r}}(m)(E|Y_i|^{p/2})^{(p-2)/p}\|X\|_r^2,
\end{aligned}$$

and hence

$$E|Y_i|^{p/2} \le 12^{p/2}m^p\alpha^{1-p/r}(m)\|X\|_r^p, \tag{12.2.22}$$

which, together with (12.2.14), yields

$$\begin{aligned}
I_{13} &\le k_n^{p/2}12^pm^p\alpha^{1-p/r}(m)\|X\|_r^p \\
&\le C24^pn^pm^{(p-r)\theta/r}\|X\|_r^p \\
&\le C24^pa^{(p-r)\theta/r}n^{p+(p-r)\theta/r}\|X\|_r^p \\
&\le C24^pa^{(p-r)\theta/r}n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p.
\end{aligned} \tag{12.2.23}$$

Similarly to (12.2.22), we have

$$E|E(\xi_i|\mathcal{F}_{i-1})|^p \le 12^pm^p\alpha^{1-p/r}(m)\|X\|_r^p. \tag{12.2.24}$$

Therefore

$$\begin{aligned}
I_{12} &\le k_n^p12^pm^p\alpha^{1-p/r}(m)\|X\|_r^p \\
&\le C24^pa^{(p-r)\theta/r}n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p.
\end{aligned} \tag{12.2.25}$$

Putting the inequalities above together yields

$$\begin{aligned}
I_1 \le D_2\Big(&\sum_{i=1}^{k_n} E|\xi_i|^p + D_2(nC_n)^{p/2}\|X\|_v^p \\
&+ 2C24^pa^{(p-r)\theta/r}n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 \le D_2\Big(&\sum_{i=1}^{k_n} E|\eta_i|^p + D_2(nC_n)^{p/2}\|X\|_v^p \\
&+ 2C24^pa^{(p-r)\theta/r}n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big).
\end{aligned}$$

Consequently, we have

$$E|S_n|^p \le 2^{p-1}D_2\Big(\sum_{i=1}^{k_n}(E|\xi_i|^p + E|\eta_i|^p) + 2D_2(nC_n)^{p/2}\|X\|_v^p + 4C24^p a^{(p-r)\theta/r} n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big). \tag{12.2.26}$$

Now we let

$$a = (2^{p+4}D_2)^{-1/\varepsilon - p/(p-2)},$$
$$K = 2^{p+1}D_2(D_2 + 2C24^p a^{(p-r)\theta/r}).$$

By (12.2.26) and induction hypothesis (12.2.19), we get

$$\begin{aligned}
E|S_n|^p &\le 2^p D_2\Big(k_n K\Big((mC_m)^{p/2}\|X\|_v^p + m^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big)\\
&\quad + D_2(nC_n)^{p/2}\|X\|_v^p + 2C24^p a^{(p-r)\theta/r} n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big)\\
&\le 2^p D_2(n/m)K\Big((mC_n)^{p/2}\|X\|_v^p + m^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big)\\
&\quad + 2^p D_2(D_2 + 2C24^p a^{(p-r)\theta/r})\\
&\quad \cdot\Big((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big)\\
&\le 2^p D_2 K\Big(a^{(p-2)/2}(nC_n)^{p/2}\|X\|_v^p + a^{\varepsilon} n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big)\\
&\quad + (K/2)\Big((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big)\\
&\le (K/2)\Big((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big)\\
&\quad + (K/2)\Big((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big)\\
&= K\Big((nC_n)^{p/2}\|X\|_v^p + n^{(p+(p-r)\theta/r)\vee(1+\varepsilon)}\|X\|_r^p\Big).
\end{aligned}$$

This proves that (12.2.19) remains valid for $k = n$, as desired.

**Proof of Theorem 12.2.2.** By Theorem 4.2 in Billingsley (1968), it is sufficient to prove that for any $\varepsilon > 0$

$$\lim_{\theta\to 0}\limsup_{n\to\infty} P\Big\{\sup_{0<t\le\theta}|\alpha_n(t)/q(t)| \ge \varepsilon\Big\} = 0, \tag{12.2.27}$$

$$\lim_{\theta\to 0}\limsup_{n\to\infty} P\Big\{\sup_{1-\theta\le t<1}|\alpha_n(t)/q(t)| \ge \varepsilon\Big\} = 0, \tag{12.2.28}$$

$$\lim_{\theta\to 0} P\Big\{\sup_{0<t\le\theta}|Y(t)/q(t)| \ge \varepsilon\Big\} = 0, \tag{12.2.29}$$

$$\lim_{\theta\to 0} P\Big\{\sup_{1-\theta\le t<1}|Y(t)/q(t)| \ge \varepsilon\Big\} = 0. \tag{12.2.30}$$

Note that

$$
\begin{aligned}
P\Big\{\sup_{0<t\le\theta}|\alpha_n(t)/q(t)|\ge\varepsilon\Big\}
&\le \sum_{j=1}^{\infty} P\Big\{\sup_{\theta 2^{-j}<t\le\theta 2^{-j+1}}|\alpha_n(t)/q(t)|\ge\varepsilon\Big\}\\
&\le \sum_{j=1}^{\infty} P\Big\{\sup_{\theta 2^{-j}<t\le\theta 2^{-j+1}}|\alpha_n(t)|\ge\varepsilon q(\theta 2^{-j})\Big\}.
\end{aligned}
$$

Hence, (12.2.27) can be rewritten as

$$
\begin{aligned}
\lim_{\theta\to 0}\limsup_{n\to\infty} P\Big\{\sup_{0<t\le\theta}|\alpha_n(t)/q(t)|\ge\varepsilon\Big\}
&\le \limsup_{\theta\to 0}\limsup_{n\to\infty}\sum_{j=1}^{\infty} B_{j,n}, \qquad (12.2.31)
\end{aligned}
$$

where $B_{j,n}=P\{\sup_{0\le t\le\theta 2^{-j+1}}|\alpha_n(t)|\ge\varepsilon q(\theta 2^{-j})\}$.

Put

$$
\varepsilon_j=\varepsilon q(\theta 2^{-j}),\qquad G_n=\{j: n^{1/2}\theta 2^{-j+1}\le\varepsilon_j/2\},
$$
$$
H_n=\{j: n^{1/2}\theta 2^{-j+1}>\varepsilon_j/2\}.
$$

It is easy to check that for any $0\le s\le t\le s+h\le 1$

$$
|\alpha_n(t)-\alpha_n(s)|\le|\alpha_n(s+h)-\alpha_n(s)|+n^{1/2}h. \qquad (12.2.32)
$$

Hence we have for $j\in G_n$

$$
\begin{aligned}
B_{j,n}&\le P\{|\alpha_n(\theta 2^{-j+1})|+n^{1/2}\theta 2^{-j+1}\ge\varepsilon_j\}\\
&\le P\{|\alpha_n(\theta 2^{-j+1})|\ge\varepsilon_j/2\}\\
&\le c\varepsilon_j^{-2}(\theta 2^{-j})^{r_2}\\
&\le c\varepsilon^{-2}\Big(\log(2^j/\theta)\Big)^{-2\beta} \qquad (12.2.33)
\end{aligned}
$$

by (A2), (12.2.5) and (12.2.6). Hence (12.2.33) implies that

$$
\limsup_{\theta\to 0}\limsup_{n\to\infty}\sum_{j\in G_n} B_{j,n}=0. \qquad (12.2.34)
$$

Note that in the case of $\mu<r_2/2$, (12.2.34) holds also for $\beta=0$.

Write

$$
\Delta:=\Delta_{j,n}=\frac{1}{4}\frac{\varepsilon_j}{n^{1/2}}=\frac{\varepsilon}{4}\frac{q(\theta 2^{-j})}{n^{1/2}}. \qquad (12.2.35)
$$

When $j \in H_n$, using (12.2.32) again, we obtain

$$\begin{aligned}
B_{j,n} &\le P\Big\{\max_{1\le i\le \theta 2^{-j+1}/\Delta} |\alpha_n(i\Delta)| \ge \varepsilon_j/2\Big\} \\
&\quad + P\Big\{\max_{0\le i\le \theta 2^{-j+1}/\Delta} \sup_{i\Delta<t\le (i+1)\Delta} |\alpha_n(t)-\alpha_n(i\Delta)| \ge \varepsilon_j/2\Big\} \\
&\le P\Big\{\max_{1\le i\le \theta 2^{-j+1}/\Delta} |\alpha_n(i\Delta)| \ge \varepsilon_j/2\Big\} \\
&\quad + P\Big\{\max_{0\le i\le \theta 2^{-j+1}/\Delta} |\alpha_n((i+1)\Delta)-\alpha_n(i\Delta)| + \Delta n^{1/2} \ge \varepsilon_j/2\Big\} \\
&\le P\Big\{\max_{1\le i\le \theta 2^{-j+1}/\Delta} |\alpha_n(i\Delta)| \ge \varepsilon_j/2\Big\} \\
&\quad + P\Big\{\max_{0\le i\le \theta 2^{-j+1}/\Delta} |\alpha_n((i+1)\Delta)-\alpha_n(i\Delta)| \ge \varepsilon_j/4\Big\} \\
&\le 3P\Big\{\max_{1\le i\le \theta 2^{-j+2}/\Delta} |\alpha_n(i\Delta)| \ge \varepsilon_j/8\Big\}. \qquad (12.2.36)
\end{aligned}$$

From (A1) it follows that for all $0 \le i < k \le \theta 2^{-j+2}/\Delta$

$$\begin{aligned}
E|\alpha_n(k\Delta)-\alpha_n(i\Delta)|^p &\le C_1\Big(((k-i)\Delta)^{p_1} + n^{-p_2/2}((k-i)\Delta)^{r_1}\Big) \\
&\le C_1\Big(((k-i)\Delta)^{p_1} + n^{-p_2/2}(k-i)\Delta^{r_1}\Big).
\end{aligned}$$

Thus, by Móricz's theorem (Lemma 4.1.2), there is a constant $C$, depending only on $C_1$ and $p_1$, such that

$$\begin{aligned}
&E \max_{0\le i\le \theta 2^{-j+2}/\Delta} |\alpha_n(i\Delta)|^p \\
&\le C\Big((\theta 2^{-j+2}/\Delta)^{p_1}\Delta^{p_1} + n^{-p_2/2}(\theta 2^{-j+2}/\Delta)\Delta^{r_1}\log^p(\theta 2^{-j+2}/\Delta)\Big) \\
&\le C4^p\Big((\theta 2^{-j})^{p_1} + n^{-p_2/2}\theta 2^{-j}\Delta^{r_1-1}\log^p(\theta 2^{-j+2}/\Delta)\Big). \qquad (12.2.37)
\end{aligned}$$

Since $p_2 > 1 - r_1$ in (A1), $l(x) = (\log x)^p/x^{-1+r_1+p_2} \le c$ for all $x \ge 1$. Thus from (12.2.5), (12.2.6), (12.2.35), (12.2.37) and the fact that $\theta 2^{-j+4}n^{1/2}/\varepsilon_j \ge 8$, we conclude that for $j \in H_n$

$$\begin{aligned}
&P\Big\{\max_{0\le i\le \theta 2^{-j+2}/\Delta} |\alpha_n(i\Delta)| \ge \varepsilon_j/8\Big\} \\
&\le c\varepsilon_j^{-p}\Big((\theta 2^{-j})^{p_1} + n^{-p_2/2}\theta 2^{-j}\Delta^{r_1-1}\log^p(\theta 2^{-j+2}/\Delta)\Big) \\
&\le c\varepsilon_j^{-p}\Big((\theta 2^{-j})^{p_1} + \varepsilon_j^{r_1-1}(\theta 2^{-j})n^{(1-r_1-p_2)/2}\log^p(\theta 2^{-j+4}n^{1/2}/\varepsilon_j)\Big) \\
&\le c\varepsilon_j^{-p}\Big((\theta 2^{-j})^{p_1} + \varepsilon_j^{-p_2}(\theta 2^{-j})^{r_1+p_2}l(\theta 2^{-j+4}n^{1/2}/\varepsilon_j)\Big) \\
&\le c\Big(\varepsilon_j^{-p}(\theta 2^{-j})^{p_1} + \varepsilon_j^{-p-p_2}(\theta 2^{-j})^{r_1+p_2}\Big) \\
&\le c\varepsilon^{-p-p_2}\Big(\log(2^j/\theta)\Big)^{-p\beta}.
\end{aligned}$$

This, together with (12.2.36), proves that

$$\limsup_{\theta\to 0}\limsup_{n\to\infty}\sum_{j\in H_n} B_{j,n}=0. \tag{12.2.38}$$

Note that in the case of $\mu < p_1/p$, (12.2.38) is true for $\beta > 1/(p+p_2)$. The proof of (12.2.27) is now completed by (12.2.31), (12.2.34) and (12.2.38). Similarly one can prove (12.2.28).

By (12.2.3) we have for any $0 \le s,t \le 1$

$$\alpha_n(t)-\alpha_n(s)\Longrightarrow Y(t)-Y(s).$$

Then by (A2) and Theorem 5.3 in Billingsley (1968)

$$E(Y(t)-Y(s))^2 \le \liminf_{n\to\infty} E(\alpha_n(t)-\alpha_n(s))^2 \le C_2|t-s|^{r_2}. \tag{12.2.39}$$

Thus, based on the fact that $\{Y(t), 0\le t\le 1\}$ is a Gaussian process, we have

$$E(Y(t)-Y(s))^4 \le cE^2(Y(t)-Y(s))^2 \le c|t-s|^{2r_2}$$

for all $0 \le s,t \le 1$. Applying Theorem 12.2 in Billingsley (1968), we can immediately get (12.2.29) and (12.2.30). This completes the proof of Theorem 12.2.2.

**Proof of Corollary 12.2.1.** First we verify that $\{\Delta_n(t), n\ge 1\}$ and $\Delta(t)$ are well defined on [0, 1]. By Schwarz's inequality, (A2), (12.2.6) and (12.2.8), for $0\le t\le 1$,

$$\begin{aligned} E\Delta_n^2(t) &= \int_0^t\int_0^t E\alpha_n(u)\alpha_n(v)dQ(u)dQ(v) \\ &\le \Big(\int_0^1 E^{1/2}(\alpha_n(t))^2 dQ(t)\Big)^2 \\ &\le 2^{r_2}C_2\Big(\int_0^1 (t(1-t))^{r_2/2}dQ(t)\Big)^2 < \infty, \end{aligned}$$

where the last inequality follows from $\alpha_n(0)=\alpha_n(1)=0$, $E(\alpha_n(t))^2 \le C_2 t^{r_2}$ for $0\le t\le 1/2$, and $E(\alpha_n(t))^2 \le C_2(1-t)^{r_2}$ for $1/2\le t\le 1$. Similarly, using (12.2.39) in conjunction with (A2), we have

$$\begin{aligned} E\Delta^2(t) &= \int_0^1\int_0^1 EY(u)Y(v)dQ(u)dQ(v) \\ &\le \Big(\int_0^1 E^{1/2}(Y(t))^2 dQ(t)\Big)^2 \\ &\le 2^{r_2}C_2\Big(\int_0^1 (t(1-t))^{r_2/2}dQ(t)\Big)^2 < \infty. \end{aligned}$$

This shows that $\{\Delta_n(t), 0 \le t \le 1; n \ge 1\}$ and $\{\Delta(t), 0 \le t \le 1\}$ are square integrable processes. Now we have for any $\theta > 0$

$$\sup_{0<t\le\theta} |\Delta_n(t)| \le \sup_{0<t\le\theta} |\alpha_n(t)/q^*(t)| \int_0^1 q^*(t)dQ(t)$$

and

$$\sup_{1-\theta\le t<1} |\Delta_n(1) - \Delta_n(t)| \le \sup_{1-\theta\le t<1} |\alpha_n(t)/q^*(t)| \int_0^1 q^*(t)dQ(t),$$

where $q^*(t) = (t(1-t))^\mu(\log 1/(t(1-t)))^\beta$. Thus (12.2.27), (12.2.28) and (12.2.8) imply that for any $\varepsilon > 0$

$$\begin{aligned}&\lim_{\theta\to0}\limsup_{n\to\infty} P\Big\{\sup_{0<t\le\theta} |\Delta_n(t)| \ge \varepsilon\Big\}\\ &\quad= \lim_{\theta\to0}\limsup_{n\to\infty} P\Big\{\sup_{1-0\le t<1} |\Delta_n(1) - \Delta_n(t)| \ge \varepsilon\Big\} = 0. \quad (12.2.40)\end{aligned}$$

Similarly, (12.2.29), (12.2.30) and (12.2.8) imply that for any $\varepsilon > 0$

$$\begin{aligned}&\lim_{\theta\to0} P\Big\{\sup_{0<t\le\theta} |\Delta(t)| \ge \varepsilon\Big\}\\ &\quad= \lim_{\theta\to0} P\Big\{\sup_{1-\theta\le t<1} |\Delta(1) - \Delta(t)| \ge \varepsilon\Big\} = 0. \quad (12.2.41)\end{aligned}$$

Hence Corollary 12.2.1 follows from Theorem 12.2.2 and Theorem 4.2 in Billingsley (1968).

**Proof of Theorem 12.2.3.** Let $\theta$ and $\varepsilon$ be as in (12.2.9). Since $\theta \ge 1 + \sqrt{2}$, we can take $r = \infty, v$ and $p$ in Lemma 12.2.2 such that

$$\frac{2(\theta+\varepsilon)}{\theta+\varepsilon-1} < v < \frac{2\theta}{\theta-1} \quad \text{and} \quad v < \theta + 1 < p < \theta + 1 + \varepsilon. \qquad (12.2.42)$$

Therefore, by (12.2.16) and (12.2.18), for any $0 < \eta < (p - 1 - \theta)/\theta$, there is a $K < \infty$ such that for any $0 \le s, t \le 1$

$$\begin{aligned}&E\Big|\sum_{i=1}^n (I(U_i \le t) - I(U_i \le s) - (t-s))\Big|^p\\ &\qquad\qquad \le K(n^{p/2}|t-s|^{p/v} + n^{1+\eta/2})\end{aligned}$$

and

$$E\Big|\sum_{i=1}^n (I(U_i \le t) - I(U_i \le s) - (t-s))\Big|^2 \le Kn|t-s|^{2/v},$$

which imply that

$$E|\alpha_n(t) - \alpha_n(s)|^p \le K(|t-s|^{p/v} + n^{-(p-2-\eta)/2})$$

and

$$E|\alpha_n(t) - \alpha_n(s)|^2 \le K|t-s|^{2/v}.$$

Hence (A1) and (A2) hold for $p_1 = p/v > 1$, $p_2 = p - 2 - \eta > 1$, $r_1 = 0$ and $r_2 = 2/v$. Note that $0 < \eta < (p-1-\theta)/\theta$, it is easy to see from (12.2.42) that

$$\begin{aligned} \min\Big(\frac{p_1}{p}, \frac{r_1+p_2}{p+p_2}, \frac{r_2}{2}\Big) & \\ &\ge \min\Big(\frac{1}{v}, \frac{p-2-\eta}{2p-2}\Big) \\ &> \frac{1}{2}\Big(1 - \frac{1}{\theta}\Big). \end{aligned} \tag{12.2.43}$$

By Theorem 12.1.2, (12.2.3) holds. This proves Theorem 12.2.3 by Theorem 12.2.2.

**Proof of Theorem 12.2.4.** From Corollary 12.1.1 it follows that (12.2.3) holds. By Theorem 1.1 of Shao (1995), we have for $p \ge 2$

$$\begin{aligned} E\Big|\sum_{i=1}^n (I(U_i \le t) - I(U_i \le s) - (t-s))\Big|^p & \\ &\le c\Big(n^{p/2}\exp\Big(c\sum_{i=0}^{[\log n]} \rho(2^i)\Big)(E(I(U_1 \le t) \\ &\quad - I(U_1 \le s) - (t-s))^2)^{p/2} \\ &\quad + n\exp\Big(K\sum_{i=0}^{[\log n]} \rho^{2/p}(2^i)\Big)E|I(U_1 \le t) \\ &\quad - I(U_1 \le s) - (t-s)|^p\Big) \\ &\le c\Big(n^{p/2}\exp\Big(c\sum_{i=0}^{[\log n]} \rho(2^i)\Big)|t-s|^{p/2} \\ &\quad + n\exp\Big(c\sum_{i=0}^{[\log n]} \rho^{2/p}(2^i)\Big)|t-s|\Big). \end{aligned}$$

Clearly, under condition (12.2.10), we have for $p \ge 2$ and any $0 < \delta <$

$\min\{\varepsilon(p-1)/(2(2+\varepsilon)),(p-2)/2\}$,

$$\begin{aligned}
E\Big|\sum_{i=1}^{n}(I(U_i\le t)-I(U_i\le s)-(t-s))\Big|^p
&\le c\Big(n^{p/2}|t-s|^{p/2}+n\exp\Big(c\sum_{i=0}^{[\log n]}\rho^{2/p}(2^i)\Big)|t-s|\Big)\\
&\le c(n^{p/2}|t-s|^{p/2}+n^{1+\delta}|t-s|),
\end{aligned}$$

since $\exp(c\sum_{i=0}^{[\log n]}\rho^{2/p}(2^i))$ is a slowly varying function. This, in turns, gives us for $p>2$

$$E|\alpha_n(t)-\alpha_n(s)|^p\le c(|t-s|^{p/2}+n^{-(p-2-2\delta)/2}|t-s|),$$

and for $p=2$

$$E|\alpha_n(t)-\alpha_n(s)|^2\le c|t-s|.$$

Thus (A1) and (A2) are satisfied for $p_1=p/2$, $p_2=p-2-2\delta>0$ and $r_1=r_2=1$. Hence by (12.2.6)

$$\mu=(1+p-2-2\delta)/(p+p-2-2\delta)>1/(2+\varepsilon). \tag{12.2.44}$$

On the other hand, under condition (12.2.11), we have for $p\ge 2$

$$E\Big|\sum_{i=1}^{n}(I(U_i\le t)-I(U_i\le s)-(t-s))|^p \le c(n^{p/2}|t-s|^{p/2}+n|t-s|),$$

which implies that for $p\ge 2$

$$E|\alpha_n(t)-\alpha_n(s)|^p\le c(|t-s|^{p/2}+n^{-(p-2)/2}|t-s|).$$

Thus (A1) and (A2) are satisfied for $p_1=p/2$, $p_2=p-2$ and $r_1=r_2=1$. Obviously $\mu=1/2$. Now the proof of Theorem 12.2.4 is complete.

## 12.3 Strong approximations

In this section, we introduce the strong approximations for empirical processes with a dependent sample. For simplicity, we only give the results for an $\alpha$-mixing sample due to Philipp (1982). Strittmatter (1990) has given a further result for an absolutely regular sample.

In order to discuss the strong approximations of empirical processes with a dependent sample, we first introduce a result of strong approximations for random elements with values in a Banach space. Let $\{x_j, j \geq 1\}$ be a sequence of strictly stationary $\alpha$-mixing random elements on probability space $(\Omega, \Sigma, P)$ and with values in a measurable space $(A, \mathcal{A})$. Let $(S, \|\cdot\|)$ be a Banach space and let $h : A \to S$ be a mapping. We call $\{X_j := h(x_j), j \geq 1\}$ a sequence of strictly stationary $\alpha$-mixing S-valued random elements.

**Theorem 12.3.1.** *Let $\{X_j, j \geq 1\}$ be a sequence of strictly stationary $\alpha$-mixing S-valued random elements as above, such that $\|X_1\| \leq \xi$ for some $\xi$ with $E\xi^{2+\delta} < \infty$, $0 < \delta \leq 1$ and*

$$\alpha(n) \leq cn^{-(2+\rho)(1+2/\delta)} \tag{12.3.1}$$

*for some $\rho > 0$. Suppose that for each $m \geq 1$ there is a linear mapping $\Lambda_m : S \to S$ with the following properties:*

$$\sup_{m\geq 1} \|\Lambda_m\| < \infty, \tag{12.3.2}$$

$$\dim \Lambda_m S \leq C\,m, \qquad \textit{for some } C > 0. \tag{12.3.3}$$

*And suppose that there is a $\theta$ with the following property: for each $m \geq 1$ there is an $n_0(m) \leq c\exp(m^{1-\theta})$ and a non-increasing function $g(m) \geq m^{-1/2}$ such that for all $n \geq n_0$*

$$P^*\Big\{n^{-1/2}\|\sum_{j\leq n}(X_j - \Lambda_m X_j)\| \geq g(m)\Big\} \leq m^{-1-\theta}, \tag{12.3.4}$$

*where $P^*$ is the outer measure of $P$. Moreover assume that for each $m \geq 1$ the mapping $\Lambda_m \circ h$ is a measurable function from $(A, \mathcal{A})$ into the linear span $L_m S$ of $\Lambda_m S$, and that*

$$E\Lambda_m X_1 = 0, \quad E\|\Lambda_m X_1\|^{2+\delta} < \infty. \tag{12.3.5}$$

*Let $T$ be the completion of the linear span of $\cup_{m\geq 1}\Lambda_m S$, so $T$ is a separable Banach space. Then there exists a sequence $\{Y_j, j \geq 1\}$ of i.i.d. $T$-valued Gaussian random variables defined on $(\Omega, \Sigma, P)$ such that $EY_1 = 0$. Moreover, for each $s, t \in T'$, the space of bounded linear functional on $T$, the following limits exist and the series converge absolutely and the covariance*

*structure of* $Y_1$ *is given by*

$$\begin{aligned} Es(Y_1)t(Y_1) = \lim_{m\to\infty} E\{s(\Lambda_m X_1)t(\Lambda_m X_1)\} \\ + \lim_{m\to\infty} \sum_{j\geq 2} E\{s(\Lambda_m X_1)t(\Lambda_m X_j)\} \\ + \lim_{m\to\infty} \sum_{j\geq 2} E\{s(\Lambda_m X_j)t(\Lambda_m X_1)\} \qquad (12.3.6) \end{aligned}$$

*and*

$$\begin{aligned} \|\sum_{j\leq n}(X_j - Y_j)\| \leq V_n \\ = O\Big(n^{1/2}\Big\{(\log n)^{-(1-\beta)/4} \\ + (\log n)^{(1-\beta)/\beta} g\Big(a(\log n)^{1-\theta/2}\Big)\Big\}\Big) \quad a.s. (12.3.7) \end{aligned}$$

*for some measurable* $V_n$ *and some positive constant* $a$ *and any* $\beta$ *with* $1 - \theta/4 \leq \beta < 1$.

The proof of Theorem 12.3.1 will not be presented here.

**Theorem 12.3.2.** *Let* $\mathcal{C} \subset \mathcal{A}$ *be a class of measurable sets with*

$$N(\delta, \mathcal{C}) \leq c\,\delta^{-\tau}, \quad \text{for some } \tau > 1/6. \qquad (12.3.8)$$

*Let* $\{x_n, n \geq 1\}$ *be a sequence of strictly stationary A-valued* $\alpha$*-mixing random elements with*

$$\alpha(n) \leq c\,n^{-2\tau-4}. \qquad (12.3.9)$$

*Put* $g_n(C) = I(x_n \in C) - P(C), C \in \mathcal{C}$. *Then there exists a sequence* $\{Y_j, j \geq 1\}$ *of i.i.d. Gaussian processes, defined on the same probability space* $(\Omega, \Sigma, P)$, *indexed by* $C \in \mathcal{C}$ *with* $EY_1(C) = 0$,

$$\begin{aligned} E(Y_1(C)Y_1(D)) \\ = Eg_1(C)g_1(D) + \sum_{n\geq 2} Eg_1(C)g_n(D) \\ + \sum_{n\geq 2} Eg_n(C)g_1(D), \quad C, D \in \mathcal{C} \qquad (12.3.10) \end{aligned}$$

*such that with probability* 1

$$\begin{aligned} \sup_{C\in\mathcal{C}}\Big|\sum_{j\leq n}(I(x_j \in C) - P(C) - Y_j(C))\Big| \\ \leq V_n = O(n^{1/2}(\log n)^{-\lambda}) \qquad (12.3.11) \end{aligned}$$

*for some measurable* $V_n$ *and some* $\lambda > 0$.

The proof of Theorem 12.3.2 will need the following lemmas.

Let $\{\xi_n\}$ be a sequence of strictly stationary $\alpha$-mixing real random variables with $E\xi_1 = 0$, $|\xi_1| \le 1$ and $\alpha(n) \le cn^{-\rho}$, where $c > 0$, $\rho > 2$. We need an exponential bound for the partial sums of $\{\xi_i, i \ge 1\}$. Put $p = [n^{1/2-\kappa}]$, $q = [n/(2p_n)] + 1$ for some $0 < \kappa < 1/2$. Define blocks of consecutive integers $H_j$ and $I_j$, $1 \le j < q$ of length $p$ each and $H_q$ consisting of $n - 2p(q-1)$ integers. Then $\text{Card}H_q \le 2p$. The order of the blocks is $H_1, I_1, \cdots, H_{q-1}, I_{q-1}, H_q$. We leave no gaps between blocks. Put

$$y_j = \sum_{i \in H_j} \xi_i, \quad z_j = \sum_{i \in I_j} \xi_i.$$

It is easy to see that

$$\sigma^2 := E\xi_1^2 + 2\sum_{n \ge 2} E\xi_1\xi_n < \infty.$$

Note that for sufficiently large $n$

$$Ey_j^2 < \begin{cases} 2\sigma^2 p, & 1 \le j < q, \\ 5\sigma^2 p, & j = q. \end{cases} \tag{12.3.12}$$

Denote $\mathcal{L}_j = \sigma(y_1, \cdots, y_j)$. We write

$$y_j = Y_j + v_j, \quad 1 \le j \le q,$$

where $v_j = E(y_j|\mathcal{L}_{j-1}), Y_j = y_j - E(y_j|\mathcal{L}_{j-1})$. It is clear that $(Y_j, \mathcal{L}_j, 1 \le j \le q)$ is a martingale difference sequence and from Lemma 1.2.1, it follows that

$$\|v_j\|_2 \le 2\|y_j\|_\infty \alpha^{1/2}(p) \le c\, p^{1-\rho/2}. \tag{12.3.13}$$

**Lemma 12.3.1.** *Let* $A \ge \sigma^2$. *We have*

$$P\Big\{\sum_{j \le q} E(Y_j^2|\mathcal{L}_{j-1}) \ge 4An\Big\} \le cA^{-2}p^{2-\rho}.$$

**Proof.** By the Hölder inequality we have

$$E(Y_j^2|\mathcal{L}_{j-1}) \le E(y_j^2|\mathcal{L}_{j-1}). \tag{12.3.14}$$

By Lemma 1.2.1, we have

$$\|E(y_j^2|\mathcal{L}_{j-1}) - Ey_j^2\|_2 \le 4\|y_j\|_\infty^2 \alpha^{1/2}(p) \le 4p^{2-\rho/2}.$$

Thus by the Minkowski and Chebyshev inequalities we have

$$\begin{aligned}P\Big\{\sum_{j\le q}(E(y_j^2|\mathcal{L}_{j-1}) - Ey_j^2) \ge An\Big\}\\ \le 16n^{-2}A^{-2}(qp^{2-\rho/2})^2 \le cA^{-2}p^{2-\rho}.\end{aligned}$$

Hence by (12.3.12) and (12.3.14), the probability in question does not exceed

$$P\Big\{\sum_{j\le q} E(y_j^2|\mathcal{L}_{j-1}) \ge \sum_{j\le q} Ey_j^2 + An\Big\} \le cA^{-2}p^{2-\rho}.$$

Lemma 12.3.1 is proved.

**Lemma 12.3.2.** *For all $R > 0$ we have*

$$P\Big\{\Big|\sum_{j\le q} Y_j\Big| > 5Rn^{1/2}\Big\} \le c\{\exp(-R^2/A) + A^{-2}p^{2-\rho}\}.$$

**Proof.** Let $M$ be the index $j$ of $H_j$ or $I_j$ containing $n$. Define

$$U_k = \begin{cases}\sum_{j\le k} Y_j, & \text{if } k \le M,\\ U_M, & \text{if } k > M;\end{cases}$$

$$s_k^2 = \begin{cases}\sum_{j\le k} E(Y_j^2|\mathcal{L}_{j-1}), & \text{if } k \le M,\\ s_M^2, & \text{if } k > M.\end{cases}$$

Obviously $U_j - U_{j-1} = Y_j \le 4p =: C$. If $R \le An^{1/2}/p$, we set $\lambda = R/(4An^{1/2})$, $K = 20An$, so that $\lambda C \le 1$. Put

$$T_k = \exp(\lambda U_k - \frac{1}{2}\lambda^2(1 + \frac{1}{2}\lambda C)s_k^2).$$

By Lemma 5.4.1 and Corollary 5.4.1 of Stout (1974) and Lemma 12.3.1 we have

$$\begin{aligned}&P\Big\{\sup_{k\ge 0} U_k > 5Rn^{1/2}\Big\}\\ &\le P\Big\{\sup_{k\ge 0} T_k > \exp\Big(\lambda^2 K - \frac{1}{2}\lambda^2\Big(1 + \frac{1}{2}\lambda C\Big)s_q^2\Big)\Big\}\\ &\le c\Big\{\exp(-R^2/A) + A^{-2}p^{2-\rho}\Big\}.\end{aligned}$$

But if $R > An^{1/2}/p$ we set $\lambda = 1/(4p)$ and $K = 20Rpn^{1/2}$. Then $\lambda C = 1$ and by the same calculation we obtain

$$
\begin{aligned}
&P\Big\{\sup_{k\ge 0} U_k > 5Rn^{1/2}\Big\} \\
&\le c\exp\Big(\frac{-20Rpn^{1/2}+3An}{16p^2}\Big) + A^{-2}p^{2-\rho} \\
&\le cA^{-2}p^{2-\rho}.
\end{aligned}
$$

Lemma 12.3.2 is proved.

**Lemma 12.3.3.** *We have*

$$
P\Big\{\Big|\sum_{j\le q} y_j\Big| \ge 7Rn^{1/2}\Big\} \le c\Big(\exp(-R^2/A) + n^{1+\rho\kappa-\rho/2}(A^{-2}+R^{-2})\Big).
$$

**Proof.** We have

$$
\Big|\sum_{j\le q} y_j\Big| \le \Big|\sum_{j\le q} Y_j\Big| + \Big|\sum_{j\le q} v_j\Big|.
$$

By the Chebyshev and Minkowski inequalities we obtain from (12.3.13) and expressing $p$ and $q$ in terms of $n$

$$
P\Big\{\Big|\sum_{j\le q} v_j\Big| \ge Rn^{1/2}\Big\} \le cR^{-2}n^{-1}(qp^{1-\rho/2})^2 \le cR^{-2}np^{-\rho}.
$$

Combining it with Lemma 12.3.2 implies Lemma 12.3.3.

Lemma 12.3.3 remains valid if we replace $y_j$ by $z_j$,$1 \le j < q$ and set $z_q = 0$. Hence we have an exponential bound for the partial sums of $\{\xi_i, i \ge 1\}$ as follows

$$
\begin{aligned}
&P\Big\{\Big|\sum_{k\le n}\xi_k\Big| > 14Rn^{1/2}\Big\} \\
&\le c\Big(\exp(-R^2/A) + n^{1+\rho\kappa-\rho/2}(A^{-2}+R^{-2})\Big). \qquad (12.3.15)
\end{aligned}
$$

**Lemma 12.3.4.** *If the hypotheses of Theorem 12.3.2 are satisfied, then for any given $\varepsilon > 0$, there exists a $\delta > 0$, $\delta \le c\varepsilon^6$ and $n_0 \le c\exp\big(-1/(4\varepsilon)\big)$ such that for all $n \ge n_0$*

$$
P^*\{\sup(|\nu_n(C)-\nu_n(D)| : C, D \in \mathcal{C}, P_x(C \,\triangle\, D) < \delta) > \varepsilon\} \le c\exp\Big(-\frac{1}{8\varepsilon}\Big),
$$

*where*

$$\nu_n = n^{1/2}(P_n - P_x),$$
$$P_n(B) = n^{-1}\sum_{j=1}^{n} I(x_j \in B),$$
$$P_x(B) = P(x_1 \in B),\ B \in \mathcal{A}.$$

**Proof.** Let $r$ be so large that

$$2^r > \varepsilon^{-6}. \tag{12.3.16}$$

Put $\delta_k = 2^{-r-k}(k = 0, 1, 2, \cdots)$, $m_k = N(\delta_k, \mathcal{C}, P_x)$, $d_i = (i+1)^{-2}\varepsilon/32$. Take sets $A_{k1}, \cdots, A_{km(k)}$ as in the definition of $N(\delta_k, \mathcal{C}, P_x)$, so that for every $C \in \mathcal{C}$ and $k = 0, 1, \cdots$ there exist $r(k) = r(k, C)$ and $s(k) = s(k, C)$ such that $A_{kr(k)} \subset C \subset A_{ks(k)}$ and $P_x(A_{ks(k)}\backslash A_{kr(k)}) < \delta_k$. Denote

$$B_k := B_k(C) = A_{ks(k)}\backslash A_{k+1,s(k+1)}, \quad D_k := D_k(C) = A_{k+1,s(k+1)}\backslash A_{ks(k)}.$$

Then $P_x(B_k) < \delta_k$ and $P_x(D_k) < \delta_{k+1} < \delta_k$.

Put $n_0 := n_0(\varepsilon) = \varepsilon^2/(256\delta_0^2)$. For every $n > n_0$ there is a unique $k = k(n)$ such that

$$1/2 < 8\delta_k n^{1/2}/\varepsilon \le 1. \tag{12.3.17}$$

Then for $n > n_0$, $k = k(n)$, $\delta = \delta_k$ and $C \in \mathcal{C}$, $r = r(k, C)$, $s = s(k, C)$ we have

$$\nu_n(A_{kr}) - \varepsilon/8 \le \nu_n(A_{kr}) - \delta n^{1/2} \le \nu_n(C) \le \nu_n(A_{ks}) + \varepsilon/8. \tag{12.3.18}$$

From this we obtain

$$\begin{aligned} &|\nu_n(A_{ks(k)}) - \nu_n(A_{0s(0)})| \\ &\le \sum_{i=0}^{k-1} |\nu_n(A_{is(i)}) - \nu_n(A_{i+1,s(i+1)})| \\ &\le \sum_{i=0}^{k-1}(|\nu_n(B_i)| + |\nu_n(D_i)|). \end{aligned} \tag{12.3.19}$$

Let $\mathcal{B}_i$ be a collection of sets $B = A_{is}\backslash A_{i+1,t}$ or $B = A_{i+1,t}\backslash A_{i,s}$ with $P_x(B) < \delta_i$. Then for every $C \in \mathcal{C}$, $B_i(C)$ and $D_i(C) \in \mathcal{B}_i$. The number of sets in $\mathcal{B}_i$

$$\mathrm{Card}(\mathcal{B}_i) \le 2m(i)m(i+1).$$

Now let us estimate $P\{|\nu_n(B)| > d_i\}$, $B \in \mathcal{B}_i$. Put $\xi_n = I(x_n \in B) - P_x(B)$. Since

$$|E\xi_1\xi_n| \le \|\xi_1\|_4\|\xi_n\|_4(\alpha(n-1))^{1/2} \le P_x^{1/2}(B)n^{-2-\tau},$$

we have

$$\sigma^2 = E\xi_1^2 + 2\sum_{n=2}^{\infty} E\xi_1\xi_n \le cP_x^{1/2}(B).$$

Hence we can take $\delta_i^{1/2}$ such that $\sigma^2 \le \delta_i^{1/2}$, it follows from (12.3.15) that

$$P\{|\nu_n(B)| > d_i\} \le c\Big(\exp\Big\{-\frac{d_i^2}{196}\delta_i^{-1/2}\Big\} + n^{-\tau-1+\kappa(2\tau+4)}(\delta_i^{-1} + i^2)\Big).$$

Then from (12.3.8) and taking $\kappa = 1/(8\tau+16)$ we have

$$\begin{aligned} p_i &:= P\{|\nu_n(B)| > d_i \text{ for some } B \in \mathcal{B}_i\} \\ &\le c\exp\Big((-\varepsilon^2/(\delta_i^{1/2}196\cdot 32^2(i+1)^4))\cdot\delta_i^{-2\tau} + n^{-\tau-3/4}\delta_i^{-1-2\tau}\Big). \end{aligned}$$

Therefore by (12.3.16) and (12.3.17) and taking $r$ large enough, we obtain

$$\begin{aligned} \sum_{i=0}^{k} p_i &\le c\Big(\exp\Big(-\frac{1}{\varepsilon}\Big) + n^{-1/4}\varepsilon^{-(1+2\tau)}\Big) \\ &\le c\exp\Big(-\frac{1}{8\varepsilon}\Big) \end{aligned}$$

for $n \ge n_0$. Put $Q_n = P(V_n > \varepsilon/8)$, where

$$\begin{aligned} V_n = \sup\{|\nu_n(A_{kr}) - \nu_n(A_{ks})| : A_{kr} \subset A_{ks}, \\ P(A_{ks}\backslash A_{kr}) < \delta_k,\ r, s = 1, \cdots, m_k\}. \end{aligned}$$

Then from (12.3.15) and (12.3.17) it follows that for $n \ge n_0$

$$\begin{aligned} Q_n &\le c\delta_k^{-2\tau}\exp\Big(\frac{-\varepsilon^2}{64}\cdot\delta_k^{-1/2}\Big) + n^{-\tau-3/4}\delta_k^{-1-2\tau} \\ &\le c\exp\Big(-\frac{1}{8\varepsilon}\Big). \end{aligned}$$

By using (12.3.16) again we have

$$\begin{aligned} p_0 &:= P\{\sup(|\nu_n(A_{0i}) - \nu_n(A_{0j})| : P(A_{0i}\triangle A_{0j}) < 3\delta_0) > \varepsilon/4\} \\ &\le c\delta_0^{-2\tau}\exp\Big(-\frac{\varepsilon^2}{16}\delta_0^{-1/2}\Big) + n^{-1-\tau+\rho\kappa}\delta_0^{-1-2\tau} \\ &\le c\exp\Big(-\frac{1}{8\varepsilon}\Big). \end{aligned}$$

Lemma 12.3.4 is proved.

**Proof of Theorem 12.3.2.**

Let $S$ be the space of all bounded real-valued functions on $\mathcal{C}$. If $f \in S$ we set $\|f\| = \sup_{C\in\mathcal{C}} |f(C)|$. If $x \in A$ we set $h(x) = I(x \in C) - P(C)$. Thus $h : A \to S$. Let $m \geq 1$ and put $\varepsilon = m^{-1/(6\tau)}$. Find $\delta$ and $n_0$ according to Lemma 12.3.4. Then $\delta \leq c\ m^{-1/\tau}$ and $n_0 \leq c\exp\Big(m^{1/(6\tau)}/4\Big)$. Let $A_1, \cdots, A_d$, $d = N(\delta)$, be sets such that for all $C \in \mathcal{C}$ there exists $A_r$ with $P(A_r \triangle C) < \delta$ and $r$ minimal. Now $N(\delta) \leq c\delta^{-\tau} \leq cm$. We define $\Lambda_m : S \to S$ by $\Lambda_m h(x) = I(x \in A_r) - P(A_r)$ if $h(x) = I(x \in C) - P(C)$ with $P(A_r \triangle C) < \delta$. Then $\dim \Lambda_m S \leq cm$. From Lemma 12.3.4 (with $D = A_r$) we conclude that (12.3.4) is satisfied with $g(m) = m^{-1/(6\tau)}$ and $\theta = 1 - 1/(6\tau)$. The result follows now from Theorem 12.3.1 choosing $\beta$ close to 1, but subject to $1 - \theta/4 \leq \beta < 1$. Theorem 12.3.2 is proved.

From Theorem 12.3.2 we have the following theorem immediately.

**Theorem 12.3.3.** *Let $\{X_j, j \geq 1\}$ be a sequence of strictly stationary $\alpha$-mixing d-dimensional random vectors with a distribution $F(x)$ and*

$$\alpha(n) = O(n^{-4-2d}). \tag{12.3.20}$$

*Denote $g_n(s) = I(X_n \leq s) - F(s)$, for $s \in R^d$. So the series below defining the covariance function*

$$\Gamma(s, s') = Eg_1(s)g_1(s') + \sum_{n=2}^{\infty} E(g_1(s)g_n(s') + g_n(s)g_1(s'))$$

*converges absolutely for $s, s' \in R^d$. Then there exists a sequence $\{Y_j, j \geq 1\}$ of i.i.d. Gaussian processes, defined on the same probability space $(\Omega, \Sigma, P)$, indexed by $s \in R^d$ with*

$$EY_1(s) = 0, \quad EY_1(s)Y_1(s') = \Gamma(s, s'), \quad s, s' \in R^d$$

*and a positive constant $\lambda$ depending on d only such that with probability* 1

$$\sup_{s\in R^d}\Big|\sum_{j=1}^{n}(I(X_j \leq s) - F(s) - Y_j(s))\Big| = O(n^{1/2}(\log n)^{-\lambda}).$$

**Proof.** Let $P$ be the probability measure induced by $F$. Let $F_i(s_i)$, $1 \leq i \leq d$ be the $i$-th marginal of $F(s)$, $s = (s_1, \cdots, s_d)$. Let $r \geq 1$ be given. We define

$$s_{ij} = \mathrm{inv}F_i(j2^{-r}), \quad 1 \leq i \leq d, \quad 0 \leq j \leq 2^r.$$

Let $\mathcal{C}$ be the collection of all intervals $C = (-\infty, s]$, $s \in R^d$. For any $C \in \mathcal{C}$ there exist $A_p$ and $A_q$ both of the form $(-\infty, (s_{1j_1}, \cdots, s_{dj_d})]$ for some $(s_{1j_1}, \cdots, s_{dj_d})$ such that

$$A_p \subset C \subset A_q \quad \text{and} \quad P(A_q \backslash A_p) \le d\,2^{-r}.$$

The collection of all such sets $A_p$ has cardinality $\le 2^{dr}$, i.e. $N(d2^{-r}, \mathcal{C}, P) \le 2^{dr}$. Hence by interpolation (12.3.8) is satisfied with $\tau = d$ and $c = (2d)^d$ and (12.3.10) holds because of (12.3.20).

**Remark 12.3.1.** Philipp and Pinzur (1980) gave an almost sure approximation of the multivariate empirical process by a Kiefer process. Let $\{X_n, n \ge 1\}$ be a strictly stationary $\alpha$-mixing sequence of random vectors in $R^q$ with continuous distribution $F$ and

$$\alpha(n) = O(n^{-4-q(1+\varepsilon)})$$

for some $0 \le \varepsilon \le 1/4$. The empirical process of $\{X_n, n \ge 1\}$ is defined as

$$R(s,t) = [t](F_{[t]}(s) - F(s)), \quad t \ge 0,\ s \in R^q.$$

Let

$$\Gamma(s,s') = E\{g_1(s)g_1(s')\} + \sum_{n=2}^{\infty} E\{g_1(s)g_n(s') + g_n(s)g_1(s')\}, \quad s,\ s' \in R^q$$

where $g_n(s) = I(X_n \le s) - F(s)$. Then without changing its distribution we can redefine the empirical process $\{R(s,t), s \in R^q, t \ge 0\}$ on a richer probability space on which there exists a Kiefer process $\{K(s,t), s \in R^q, t \ge 0\}$ with covariance function $(t \wedge t')\Gamma(s,s')$ and a constant $\lambda = \lambda(q,\varepsilon)$ such that

$$\sup_{t \le T} \sup_{s \in R^q} |R(s,t) - K(s,t)| = O(T^{1/2}(\log T)^{-\lambda}) \quad \text{a.s.} \tag{12.3.21}$$

When $q = 1$ and $\{X_n, n \ge 1\}$ are uniformly distributed over $[0,1]$, (12.3.21) coincides with the result by Yoshihara (1979), but with less mixing rate. By Theorem 1.15.1 in Csörgő and Révész (1981), (12.3.21) implies that the Strassen-type law of iterated logarithm holds true for

$$\{\eta_s(t) = R(s,t)/\sqrt{2t \log\log t}, 0 \le s \le 1\}.$$

## 12.4 Moduli of continuity of empirical processes

Let $\{X_n, n \geq 1\}$ be a sequence of random variables with a common distribution $F(x)$, $\{\beta_n(t), -\infty < t < \infty\}, n = 1, 2, \cdots$, a sequence of its empirical processes. The moduli of continuity of empirical processes are defined as follows:

$$w_n(a_n) = \sup_{\substack{|t-s|\leq a_n \\ -\infty<s<t<\infty}} |\beta_n(t) - \beta_n(s)|.$$

Stute (1982) proved for the i.i.d. case the following theorem.

**Theorem 12.4.1.** *Suppose that*
*(i)* $0 < a_n < 1$, $a_n \downarrow$ *and* $na_n \uparrow \infty$,
*(ii)* $na_n / \log n \to \infty$,
*(iii)* $\log a_n^{-1} / \log\log n \longrightarrow 0$.
*Then*

$$\lim_{n\to\infty} (a_n \log a_n^{-1})^{-1/2} w_n(a_n) = 1 \quad a.s. \tag{12.4.1}$$

**Definition 12.4.1.** Let $0 < \lambda \leq 1$, $A \subset R$. The function $g(x)$ is said to satisfy the uniformly local $\lambda$-order Lipschtz ($\lambda$-ulL) condition on $A$, if there exist $\delta > 0$, $M < \infty$ such that

$$\sup_{x\in A} |g(x+z) - g(x)| \leq M|z|^\lambda, \quad |z| \leq \delta. \tag{12.4.2}$$

Zhou (1994) discussed the moduli of continuity of empirical processes when the sample is $\varphi$-mixing or $\alpha$-mixing, and proved the following theorems.

**Theorem 12.4.2.** *Let $\{X_n, n \geq 1\}$ be a sequence of strictly stationary $\varphi$-mixing random variables with a common distribution $F(x)$. Suppose that $F(x)$ satisfies the 1-ulL condition and $\sum_{i=1}^\infty \varphi^{1/2}(2^i) < \infty$. If there is a sequence of positive integers $\{m_n\}$ such that*

$$1 \leq m_n \leq n, \quad \frac{n}{m_n}\varphi(m_n) \leq A \quad \text{and} \quad \left(\frac{\log n}{na_n}\right)^{1/2} m_n \leq C,$$

*where $A$ and $C$ are two constants, then*

$$w_n(a_n) = O((a_n \log n)^{1/2}) \quad a.s. \tag{12.4.3}$$

**Theorem 12.4.3.** *Let $\{X_n, n \geq 1\}$ be a sequence of strictly stationary $\alpha$-mixing random variables with a common distribution $F(x)$. Suppose that $F(x)$ satisfies the 1-ulL condition and $\alpha(n) = O(\rho^n)$ for some $0 < \rho < 1$ and $a_n \to 0 (n \to \infty)$. Then for any $0 < \theta < 1$ we have*

$$w_n(a_n) = O\Big(a_n^{\frac{1-\theta}{2}} \log^2 n\Big) \quad a.s. \tag{12.4.4}$$

**Remark 12.4.1.** If $F(x)$ satisfies the $\lambda$-ulL condition $(0 < \lambda < 1)$ on $A(\subset R)$, then (12.4.3) and (12.4.4) are rewritten as

$$w_n(a_n, A) := \sup_{t,s \in A, |t-s| \leq a_n} |\beta_n(t) - \beta_n(s)| = O((a_n^\lambda \log n)^{1/2}) \quad \text{a.s.} \tag{12.4.5}$$

and

$$w_n(a_n, A) = O\Big(a_n^{\frac{1-\theta}{2}\lambda} \log^2 n\Big) \quad \text{a.s.} \tag{12.4.6}$$

**Remark 12.4.2.** Because of Lemma 12.4.1 below, the condition $\sum_{n=1}^\infty \varphi(n) < \infty$, which is required in Zhou (1994), is weakened to $\sum_{i=1}^\infty \cdot \varphi^{1/2}(2^i) < \infty$ in Theorem 12.4.2.

The proof of Theorems will need the following Bernstein type inequalities.

**Lemma 12.4.1.** *Let $\{X_n, n \geq 1\}$ be a $\varphi$-mixing sequence with $EX_n = 0$, $|X_n| \leq d$, $EX_n^2 \leq D$ and $\sum_{i=1}^\infty \varphi^{1/2}(2^i) < \infty$. Then there is $C_1 = C_1(\varphi(\cdot)) > 0$ such that*

$$P\Big\{\Big|\sum_{i=1}^n X_i\Big| > \varepsilon\Big\} \leq \exp\Big\{3\sqrt{e}n\frac{\varphi(m)}{m} - \alpha\varepsilon + C_1\alpha^2 Dn\Big\}$$

*where $\alpha$ is a real number, $m$ is a positive integer satisfying $m \leq n, \alpha \leq n$ and $\alpha m d \leq 1/4$.*

Lemma 12.4.1 is an improvement of Lemma 1 of Collomb (1984). Its proof follows Collomb's lines provided one uses Lemma 2.2.2 instead of Lemma 1.2.10 which is employed by Collomb (cf. Lemma 11.1.1).

**Lemma 12.4.2.** (Doukhan, Leon and Portal 1984) *Let $\{X_i, i \geq 1\}$ be an $\alpha$-mixing sequence with $EX_i = 0, |X_i| \leq 1$ and $\alpha(n) \leq C\rho^n$. Denote*

$\sigma = \sup \|X_i\|_\gamma$, *where* $\gamma = 2/(1-\theta), 0 < \theta < 1$. *Then there exist* $C_1, C_2$ *which depend only on* $\alpha(\cdot)$ *such that*

$$P\Big\{\Big|\sum_{i=1}^{n} X_i\Big| > \varepsilon\Big\} \le C_1\theta^{-1}\exp\{-C_{2,n}\varepsilon^{1/2}/(n^{1/4}\sigma^{1/2})\}$$

*where*

$$C_{2,n} = \begin{cases} C_2 & \text{if } \; n^{1/2}\sigma \le 1, \\ C_2 n^{1/4}\sigma^{1/2} & \text{if } \; n^{1/2}\sigma > 1. \end{cases}$$

**Proof of Theorem 12.4.2.**

As we have mentioned in the beginning of this chapter, we need only to consider uniform empirical processes. And by the 1-ulL condition, it is enough to show

$$\sup_{0\le t,s\le 1}\sup_{|t-s|\le Ma_n} |\alpha_n(t) - \alpha_n(s)| = O((a_n \log n)^{1/2}) \quad \text{a.s.} \tag{12.4.7}$$

where $\alpha_n(\cdot)$ is a uniform empirical process. Without loss of generality, we can assume that $M = 1$ in (12.4.2).

Divide the interval $[0,1]$ into $K_n$ subintervals by points $t_0 = 0, t_j = j/K_n$, $j = 1, \cdots, K_n$, where $K_n = [a_n^{-1}\log n]$. Denote

$$V_n(t) = \sup_{|t-s|\le a_n} |\alpha_n(t) - \alpha_n(s)|.$$

For any fixed $t, s \in [0,1]$ with $|t-s| \le a_n$, there are two cases:

(i) $s$ and $t$ both fall into the same subinterval, i.e. there is a $j$, $0 \le j \le K_n - 1$ such that $s, t \in [t_j, t_{j+1}]$. Then

$$\begin{aligned} |\alpha_n(t) - \alpha_n(s)| &\le |\alpha_n(t) - \alpha_n(t_j)| + |\alpha_n(t_j) - \alpha_n(s)| \\ &\le 2\max_{0\le j\le K_n} V_n(t_j). \end{aligned}$$

(ii) $s$ and $t$ fall into the different subintervals, i.e. there exist $j$ and $r, 1 \le j+1 \le r \le K_n$ such that $s \in [t_j, t_{j+1}]$, $t \in [t_r, t_{r+1}]$. Since $|s-t| \le a_n$, $t_r - t_{j+1} \le a_n$. Then

$$\begin{aligned} &|\alpha_n(t) - \alpha_n(s)| \\ &\quad\le |\alpha_n(t) - \alpha_n(t_r)| + |\alpha_n(t_r) - \alpha_n(t_{j+1})| + |\alpha_n(t_{j+1}) - \alpha_n(s)| \\ &\quad\le 3\max_{0\le j\le K_n} V_n(t_j). \end{aligned}$$

Hence, in any case, we have

$$\sup_{|t-s|\le a_n} |\alpha_n(t) - \alpha_n(s)| \le 3\max_{1\le j\le K_n} V_n(t_j). \tag{12.4.8}$$

For any fixed $j$, $1 \le j \le K_n$, divide the interval $[t_j - a_n, t_j + a_n]$ by points

$$\eta_{jr} = t_j + r\frac{a_n}{b_n}, \quad r = -b_n, -b_n + 1, \cdots, b_n - 1, b_n,$$

where $b_n = B[(na_n/\log n)^{1/2}]$, constant $B$ will be specified later on. Denote $\phi_{jr} = n^{-1/2}|\alpha_n(t_j) - \alpha_n(\eta_{jr})|$. For any given $s \in [t_j - a_n, t_j + a_n]$, there is an $r$, $-b_n \le r \le b_n$ such that $s \in [\eta_{jr}, \eta_{j,r+1}]$. By monotoncity of the empirical distribution $E_n(t)$, we have

$$\begin{aligned} n^{-1/2}V_n(t_j) &= \sup_{|t_j - s| \le a_n} |(E_n(t_j) - t_j) - (E_n(s) - s)| \\ &\le \max_{-b_n \le r \le b_n} \max_{\eta_{jr} \le s \le \eta_{j,r+1}} |(E_n(t_j) - t_j) - (E_n(s) - s)| \\ &\le \max_{-b_n \le r \le b_n} \{|E_n(t_j) - t_j - (E_n(\eta_{jr}) - \eta_{j,r+1})|, \\ &\qquad |E_n(t_j) - t_j - (E_n(\eta_{j,r+1}) - \eta_{j,r+1})|\} \\ &\le \max_{-b_n \le r \le b_n} \{\phi_{jr}, \phi_{j,r+1}\} + |\eta_{j,r+1} - \eta_{jr}| \\ &\le \max_{-b_n \le r \le b_n} \{\phi_{jr}\} + a_n/b_n. \end{aligned} \tag{12.4.9}$$

Write

$$\phi_{jr} = \Big|\frac{1}{n}\sum_{i=1}^{n}[I(\eta_{jr} < U_i \le t_j) - (t_j - \eta_{jr})]\Big| =: \Big|\sum_{i=1}^{n} Z_i\Big|. \tag{12.4.10}$$

Obviously $|Z_i| \le 2/n$, $EZ_i = 0$ and $EZ_i^2 \le a_n/n^2$. Take $\varepsilon = B(a_n \log n/n)^{1/2}$ and $\alpha = (B^{-1}na_n^{-1}\log n)^{1/2}$. By the assumption of the theorem, for large $B$ we have

$$\alpha m_n d = \Big(\frac{n\log n}{Ba_n}\Big)^{1/2} m_n\frac{2}{n} = 2\Big(\frac{\log n}{Bna_n}\Big)^{1/2} m_n \le \frac{1}{4}.$$

By Lemma 12.4.1 we have

$$\begin{aligned} &P\Big\{\Big|\sum_{i=1}^{n} Z_i\Big| > \varepsilon\Big\} \\ &\le C_1 \exp\Big\{-\alpha\varepsilon + C_1\alpha^2 a_n/n\Big\} \\ &\le C_1 \exp\Big\{-B^{1/2}\log n(1 - C_1/B^{3/2})\Big\}. \end{aligned} \tag{12.4.11}$$

where $C_1 = \exp(3\sqrt{e}A)$. Choosing $B$ large enough, we have

$$\begin{aligned} &P\Big\{\max_{0 \le j \le K_n} \max_{-b_n \le r \le b_n} |\phi_{jr}| > B(a_n \log n/n)^{1/2}\Big\} \\ &\le cK_n b_n n^{-B^{1/2}/2} \le cn^{-2}. \end{aligned}$$

Therefore, from the Borel-Cantelli lemma it follows that

$$\max_{0\le j\le K_n}\max_{-b_n\le r\le b_n}|\phi_{jr}|\le B(a_n\log n/n)^{1/2}\quad\text{a.s.}\tag{12.4.12}$$

Combining it with (12.4.8) and (12.4.9) yields (12.4.7). Theorem 12.4.2 is proved.

**Proof of Theorem 12.4.3.**

The proof is along the same lines of that of Theorem 12.4.2 with Lemma 12.4.2 instead of Lemma 12.4.1. Let $0<\theta<1$, $\gamma=2/(1-\theta)$. It is clear that $E|Z_i|^\gamma\le 2n^{-\gamma}a_n$, $i=1,\cdots,n$. Hence $\sigma:=\sup\{\|Z_i\|_\gamma : i=1,\cdots,n\}\le 2^{1/\gamma}n^{-1}a_n^{1/\gamma}$ and $n^{1/2}\sigma\le 2^{1/\gamma}n^{-1/2}a_n^{1/\gamma}\le 1$. By Lemma 12.4.2 with $\varepsilon=\varepsilon_n=B(n^{-1/2}a_n^{(1-\theta)/2}\log^2 n)$, we have

$$\begin{aligned}P\{\phi_{jr}\ge\varepsilon_n\}&\le C_1\exp\{-C_2B(n^{-1}a_n^{1-\theta}\log^4 n)^{1/4}/n^{-1/4}a_n^{1/(2\gamma)}\}\\&\le C_1\exp\{-C_2B\log n\}.\end{aligned}$$

Therefore for large $B$, we have

$$P\Big\{\max_{0\le j\le K_n}\max_{-b_n\le r\le b_n}\phi_{jr}>\varepsilon_n\Big\}\le cn^{-2}.$$

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# Chapter 13 Convergence of Some Statistics with a Mixing Sample

Large sample theory in statistics is an important subject. In general, the sample is assumed to be independent. But, in some practical cases, the observations are dependent. In this part, we shall give some large sample properties for several interesting and useful statistics, such as U-statistics, error variance estimations in linear models, density function estimations with a mixing sample.

## 13.1 $U$-statistics

Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence with a common distribution $F(\cdot)$, $h: R^m \to R$ be a symmetric function in its $m$ arguments. A $U$-**statistic** is given by

$$U_n = \binom{n}{m}^{-1} \sum_{1 \leq t_1 < \cdots < t_m \leq n} h(X_{t_1}, \cdots, X_{t_m}) \quad n \geq m.$$

Here $h$ is called a *kernel function* of $U_n$. This class of statistics was introduced by Hoeffding (1948) as a generalization of the sample mean. Many statistics of interest fall within this class or may be approximated by a member of this class.

**Remark 13.1.1.** $U$-statistics are closely connected with another class of statistics, the so-called *von-Mises statistics* (von Mises 1947) defined by

$$V_n = n^{-m} \sum_{t_1=1}^{n} \cdots \sum_{t_m=1}^{n} h(X_{t_1}, \cdots, X_{t_m}) \qquad n \geq 1.$$

These two kinds of statistics have similar limit behavior. So we discuss only $U$-statistic as a representative.

A kernel $h$ is called degenerate (for the distribution $F$) if for all choices of $a_i, 1 \le i \le m$ and every $j \in \{1, \cdots, m\}$,

$$Eh(a_1, \cdots, a_{j-1}, X_j, a_{j+1}, \cdots, a_m) = 0;$$

a $U$-statistic will be called degenerate if the corresponding kernel has this property.

First of all, we introduce an important tool in deriving the asymptotic theory of $U$-statistics, Hoeffding's projection method. Put

$$\theta = \int \cdots \int h(x_1, \cdots, x_m) \prod_{k=1}^{m} dF(x_k), \tag{13.1.1}$$

$$h_r(x_1, \cdots, x_r) = \int \cdots \int h(x_1, \cdots, x_m) \prod_{i=r+1}^{m} dF(x_i), \quad r = 1, \cdots, m-1,$$

and

$$\widetilde{h}_r(x_1, \cdots, x_r) = h_r(x_1, \cdots, x_r) - \theta, \quad r = 1, \cdots, m-1.$$

The projection of $U_n$ is defined as

$$\widehat{U}_n = \frac{m}{n} \sum_{i=1}^{n} \widetilde{h}_1(X_i) + \theta.$$

$U_n - \widehat{U}_n$ may itself be expressed as a $U$-statistic

$$\begin{aligned} U_n - \widehat{U}_n &= \binom{n}{m}^{-1} \sum_{1 \le t_1 < \cdots < t_m \le n} H(X_{t_1}, \cdots, X_{t_m}) \\ &=: R_n, \end{aligned} \tag{13.1.2}$$

where

$$H(x_1, \cdots, x_m) = h(x_1, \cdots, x_m) - \widetilde{h}_1(x_1) - \cdots - \widetilde{h}_1(x_m) - \theta$$

is a degenerate kernel. We call $R_n$ the remainder of $U_n$.

At first, modifying definition of a $\varphi$-mixing sequence, we call a sequence $\{X_n, n \ge 1\}$ $\varphi^*$-mixing or $\varphi$-mixing in both directions of time if the sequence itself and the time reversed sequence are $\varphi$-mixing, that is

$$\varphi^*(n) := \sup_{k \in \mathbb{N}} \sup_{\substack{A \in \mathcal{F}_1^k, \\ B \in \mathcal{F}_{k+n}^\infty}} \max\{|P(B|A) - P(A)|, |P(B|A) - P(B)|\} \to 0 \quad \text{as} \quad n \to \infty.$$

Obviously, $\varphi^*$-mixing implies $\varphi$-mixing.

In this section, we shall establish weak and strong convergence for a $\varphi^*$-mixing sequence.

Denker and Keller (1983) proved the CLT and their rate of convergence, functional CLT and a.s. approximation by a Wiener process. Combining the results of weak convergence and strong approximation for a $\varphi$-mixing sequence in Chapters 5 and 9, we can weaken the conditions on moments and/or $\varphi^*(n)$.

*13.1.1 Bounds for the remainder $R_n$*

Let $\{X_n, n \geq 1\}$ be a strictly stationary $\varphi^*$-mixing sequence. Assume

$$s^2 := \sup_{1 \leq t_1 < \cdots < t_m} E(h(X_{t_1}, \cdots, X_{t_m}))^2 < \infty. \tag{13.1.3}$$

First, we cite two lemmas given by Denker and Keller (1983). Lemma 13.1.1 is a conditional version of Lemma 1.2.8 with $p = q = 2$. Let $\mathcal{A}, \mathcal{B}, \mathcal{B}_1, \mathcal{B}_2$ be sub-$\sigma$-fields of $\mathcal{F}$. For probabilities $P$ and $Q$ on $\mathcal{F}$ define the distance of $P$ and $Q$ over $\mathcal{A}$ given $\mathcal{B}$ by

$$d(P, Q : \mathcal{A}|\mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A|B) - Q(A|B)|.$$

Moreover put

$$d(P : \mathcal{A}|\mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A|B) - P(A)|.$$

**Lemma 13.1.1.** *Let $f_1$ and $f_2$ be an $\mathcal{A} \vee \mathcal{B}_1$- and $\mathcal{A} \vee \mathcal{B}_2$-measurable function, and let $P, Q_1$ and $Q_2$ be probability measures coinciding on $\mathcal{A}$. Then*

$$\begin{aligned}
&|E_P(f_1 f_2) - E_P[E_{Q_1}(f_1|\mathcal{A}) \cdot E_{Q_2}(f_2|\mathcal{A})]| \\
&\quad \leq (4 + 2\sqrt{2}) \\
&\qquad \cdot \max\{d^{1/2}(P : \mathcal{B}_1|\mathcal{A} \vee \mathcal{B}_2), d^{1/2}(P, Q_1 : \mathcal{B}_1|\mathcal{A}), d^{1/2}(P, Q_2 : \mathcal{B}_2|\mathcal{A})\} \\
&\qquad \cdot \{E_P(f_1^2)^{1/2}, E_{Q_1}(f_1^2)^{1/2}\} \cdot \max\{E_P(f_2^2)^{1/2}, E_{Q_2}(f_2^2)^{1/2}\}.
\end{aligned}$$

**Lemma 13.1.2.** *Let $f$ be an $\mathcal{A} \vee \mathcal{B}$-measurable function, and let $P_n (n \geq 1)$ and $Q$ be probability measures coinciding on $\mathcal{A}$. If*

$$\lim_{n \to \infty} d(P_n, Q : \mathcal{B}|\mathcal{A}) = 0,$$

*then*

$$E_Q|f| \leq \liminf_{n\to\infty} E_{P_n}|f|.$$

**Lemma 13.1.3.** *For given $\varepsilon > 0$, there exists a $C = C_\varepsilon > 0$ such that*

$$ER_n^2 \leq Cn^{-2+\varepsilon}s^2 \quad (n \geq m).$$

**Proof.** By (13.1.2), it suffices to estimate the variance of a degenerate $U$-statistics. We shall show: if $h$ is degenerate, then

$$\binom{n}{m}^2 EU_n^2 \leq cn^{2(m-1)+\varepsilon}s^2. \tag{13.1.4}$$

For $\mathbf{a} = (a_1, \cdots, a_m), \mathbf{b} = (b_1, \cdots, b_m) \in \mathbb{N}^m$, we put

$$W(\mathbf{a},\ \mathbf{b}) = \sum h(X_{t_1}, \cdots, X_{t_m}), \tag{13.1.5}$$

where the summation extends over all indices $t_1, \cdots, t_m$ satisfying $a_i \leq t_i \leq b_i$ and $t_i \neq t_j$ for $1 \leq i \neq j \leq m$. Putting $\mathbf{1} = (1, \cdots, 1) \in \mathbb{N}^m$, we thus have

$$\binom{n}{m} U_n = \frac{1}{m!} W(\mathbf{1}, n\mathbf{1}). \tag{13.1.6}$$

For the estimation of $E(W(\mathbf{1}, n\mathbf{1}))^2$ we proceed recursively, decomposing $W(\mathbf{1}, n\mathbf{1})$ into sums over smaller index-blocks (This is inspired by the proof of Theorem 2.1.2).

We need some preparations: let $(X_n^{(1)})_{n\geq 1}, \cdots, (X_n^{(m)})_{n\geq 1}$ be $m$ independent copies of the sequence $(X_n)_{n\geq 1}$, and put for $q \in \{1, \cdots, m\}^m$

$$W(\mathbf{a}, \mathbf{b}; q) = \sum h(X_{t_1}^{(q_1)}, \cdots, X_{t_m}^{(q_m)}),$$

where the summation extends over the same index-set as in (13.1.5). Let

$$\mathrm{I}_n = \{(\mathbf{a},\ \mathbf{b}) \in \mathbb{N}^{2m} : b_i = a_i + n - 1, a_i = a_j \text{ or } |a_i - a_j| \geq n \text{ for all } i, j\}$$

and define

$$\tau(n) = \sup\{E(W(\mathbf{a}, \mathbf{b}; q))^2 : (\mathbf{a},\ \mathbf{b}) \in \mathrm{I}_n, q \in \{1, \cdots, m\}^m\}.$$

Consider fixed non-negative integers $k, l, p, n$ with $n = kl + p$. For each $(\mathbf{a},\ \mathbf{b}) \in \mathrm{I}_n$ and $q \in \{1, \cdots, m\}^m$ we have by the Hölder inequality and the triangle inequality:

$$\begin{aligned} &|E(W(\mathbf{a}, \mathbf{b}; q))^2 - E(W(\mathbf{a}, \mathbf{b} - p\mathbf{1}; q))^2| \\ &\qquad \leq (\tau(n)^{1/2} + k^m \tau(l)^{1/2}) mpn^{m-1}s, \end{aligned} \tag{13.1.7}$$

where we have tacitly assumed that

$$\sup_{1\leq t_1<\cdots<t_m} E(h(X_{t_1}^{(q_1)},\cdots,X_{t_m}^{(q_m)}))^2 \leq s^2 \tag{13.1.8}$$

for all $q \in \{1,\cdots,m\}^m$, what can easily be proved using repeatedly Lemma 13.1.2.

Now decompose $W(\mathbf{a},\mathbf{b}-p\mathbf{1};q)^2$ as

$$\begin{aligned}W(\mathbf{a},\mathbf{b}-p\mathbf{1};q)^2 = \sum_{u,v\in\{0,\cdots,k-1\}^m} & W(\mathbf{a}+l\mathbf{u},\mathbf{a}+l(\mathbf{u}+\mathbf{1})-\mathbf{1};q)\\ &\cdot W(\mathbf{a}+l\mathbf{v},\mathbf{a}+l(\mathbf{v}+\mathbf{1})-\mathbf{1};q).\end{aligned}$$

Each $\Gamma(\mathbf{u}) := W(\mathbf{a}+l\mathbf{u},\mathbf{a}+l(\mathbf{u}+\mathbf{1})-\mathbf{1};q)$ is determined by $m$ blocks of time coordinates of length $l$ (possibly counting a block several times). Denote for a fixed pair $(\mathbf{u},\ \mathbf{v})$ these $2m$ blocks by $B_1,\cdots,B_{2m}$ and assume $\inf B_i \leq \inf B_{i+1} (i=1,\cdots,2m-1)$. With this convention it is not hard to see that

$$\begin{aligned}&\mathrm{Card}\{(\mathbf{u},\ \mathbf{v}) : \sup B_1 + l \geq \inf B_2 \text{ and } \inf B_{2m} - l \leq \sup B_{2m-1}\}\\ &\qquad \leq C_m k^{2m-2},\end{aligned} \tag{13.1.9}$$

where $C_m$ is a combinatorial constant depending only on $m$. For all pairs $(\mathbf{u},\ \mathbf{v})$ not belonging to the set described in (13.1.9) we can apply Lemma 13.1.1 in the following way: assume that $\sup B_1 + l < \inf B_2$ and that $B_1$ is a block stemming from $\Gamma(\mathbf{u})$. (The remaining three cases are treated exactly in the same way.)

Choose $f_1 = \Gamma(\mathbf{u}), f_2 = \Gamma(\mathbf{v}), \mathcal{B}_1 = \sigma(X_t, t\in B_1), \mathcal{B}_2$ trivial and

$$\mathcal{A} = \sigma(X_t, t \in B_1 \cup \cdots \cup B_{2m}).$$

Observing (13.1.8) we then get

$$|E\Gamma(\mathbf{u})\Gamma(\mathbf{v})| \leq (4+2\sqrt{2})\varphi^*(l)^{1/2}\tau(l),$$

because $h$ is degenerate. It is for this application of Lemma 13.1.1 that we have to introduce independent copies of the original process, and since in two of above mentioned four cases, we have to single out the block $B_{2m}$ (instead of $B_1$), we need $\varphi$-mixing in both directions of time. Combining the last estimate with (13.1.7) and (13.1.9) we obtain

$$\begin{aligned}W(\mathbf{a},\mathbf{b};q)^2 \leq\ & C_m k^{2m-2}\tau(h) + k^{2m}(4+2\sqrt{2})\varphi^*(h)^{1/2}\tau(l)\\ &+ (\tau(n)^{1/2} + k^m\tau(l)^{1/2})mpn^{m-1}s\end{aligned}$$

and taking the supremum over $(\mathbf{a}, \mathbf{b}) \in \mathbf{I}_n$ and $q \in \{1, \cdots, m\}^m$

$$\tau(n) \leq k^{2m-2}\tau(l)(C_m + (4+2\sqrt{2})k^2\varphi^*(l)^{1/2}) \\ + (\tau(n)^{1/2} + k^m\tau(l)^{1/2})mpn^{m-1}s. \tag{13.1.10}$$

Take $k$ to be large enough such that $(C_m + (4+2\sqrt{2}))k^{-\varepsilon} \leq 1/4$. Then choose $n_0 = \min\{s : k^2\varphi^*(s)^{1/2} \leq 1\}$. For $l \geq n_0$ and $p \leq k$, (13.1.10) implies

$$(\tau(n)^{1/2} - \frac{1}{2}mkn^{m-1}s)^2 \leq (\frac{1}{2}k^{m-1+\varepsilon/2}\tau(l)^{1/2} + k^{1-\varepsilon/2}mkn^{m-1}s)^2,$$

and hence

$$\tau(n)^{1/2} \leq \frac{1}{2}k^{m-1+\varepsilon/2}\tau(l)^{1/2} + (\frac{1}{2} + k^{1-\varepsilon/2})mkn^{m-1}s. \tag{13.1.11}$$

Given $n$ choose $l_0, l_1, \cdots, l_r$ such that $l_0 = n, l_{i-1} = kl_i + p_i$ for some $0 \leq p_i < k (i = 1, \cdots, r)$ and $n_0 \leq l_r < kn_0$. Apply (13.1.11) to each pair $(l_{i-1}, l_i)$ to obtain

$$\tau(l_{i-1})^{1/2} \leq \frac{1}{2}k^{m-1+\varepsilon/2}\tau(l_i)^{1/2} + Al_{i-1}^{m-1}s,$$

where $A = (\frac{1}{2} + k^{1-\varepsilon/2})mk$. By induction this leads to

$$\begin{aligned}\tau(n)^{1/2} &\leq 2^{-r}k^{r(m-1+\varepsilon/2)}\tau(l_r)^{1/2} + An^{m-1+\varepsilon/2}s\sum_{j=0}^{r-1}2^{-j} \\ &\leq n^{m-1+\varepsilon/2}\Big(\frac{1}{2^r}\frac{l_r^m s}{l_r^{m-1+\varepsilon/2}} + 2As\Big) \\ &\leq n^{m-1+\varepsilon/2}((kn_0)^{1-\varepsilon/2} + 2A)s,\end{aligned}$$

since $n \geq k^r l_r$ and $l_r \leq kn_0$. If we put $C = ((kn_0)^{1-\varepsilon/2} + 2A)^2(m!)^{-2}$, we get

$$\tau(n) \leq Cs^2n^{2m-2+\varepsilon}(m!)^2 \tag{13.1.12}$$

and (13.1.4) follows from (13.1.6).

**Lemma 13.1.4.** *Assume that condition (13.1.3) is satisfied. Then for any $\varepsilon > 0$ and $c_N > 0$ we have*

$$R_n = O(n^{-3/4+\varepsilon}) \quad a.s.$$

*and*

$$P\Big\{\max_{1\le n\le N} n|R_n| \ge c_N\Big\} = O(N^{1/2+\varepsilon}c_N^{-2}).$$

**Proof.** By (13.1.2) it clearly suffices to prove the lemma for a degenerate $U$-statistic with kernel $h$. Put

$$Z(p,q) = W(\mathbf{1},(p+q)\mathbf{1}) - W(\mathbf{1},p\mathbf{1}) \quad (p,q\in\mathbb{N}).$$

If $2^{r-1}\le n < 2^r$ and $n=\sum_{i=1}^r d_i 2^{r-i}$ denotes the dyadic expansion of $n$,

$$W(\mathbf{1},n\mathbf{1}) = Z(0,n) = \sum_{k=1}^{r} Z\Big(\sum_{i=1}^{k-1} d_i 2^{r-i}, d_k 2^{r-k}\Big).$$

For $r,u\in\mathbb{N}, l=1,\cdots,r$ and $j=1,\cdots,2^l$ consider the sets

$$E_{j,l}^{r,u} = \{|Z((j-1)2^{r-l},2^{r-l})| \ge \alpha_{r,u}\},$$

where $\alpha_{r,u}$ are constants to be chosen later. We shall show below that

$$E(Z(p,q))^2 = O(q(p+q)^{b-1}), \quad b = 2m-3/2+\varepsilon. \tag{13.1.13}$$

By the Chebyshev inequality $P(E_{j,l}^{r,u}) = O(\alpha_{r,u}^{-2}2^{b(r-l)}j^{b-1})$.

The a.s. bound for $R_n$ follows now from the Borel-Cantelli lemma, because for $\alpha_{r,1} = 2^{b(r-1)/2}(r-1)^3/r$ we have

$$\begin{aligned}P\Big\{&\max_{2^{r-1}\le n\le 2^r} |Z(0,n)| \ge n^{b/2}(\log n)^3\Big\}\\ &\le \sum_{l=1}^{r}\sum_{j=1}^{2^l} P(E_{j,l}^{r,1}) + O(r^{-3}).\end{aligned}$$

To prove the maximal inequality, let $R,N$ be given such that $2^{R-1}\le N<2^R$. Putting $\alpha_{r,N} = 2^{(r-1)(m-1)r^{-1}}c_N$ for $r\le R$, it follows that

$$\begin{aligned}P\Big\{&\max_{1\le n\le N} n^{-m+1}|Z(0,n)| \ge c_N\Big\}\\ &\le P\Big\{\bigcup_{r=1}^{R}\bigcup_{n=2^{r-1}}^{2^r-1}\{|Z(0,n)|\ge n^{m-1}c_N\}\Big\}\\ &\le \sum_{r=1}^{R}\sum_{l=1}^{r}\sum_{j=1}^{2^l} P(E_{j,l}^{r,N}) = O(N^{1/2+\varepsilon}(\log N)^3 c_N^{-2}).\end{aligned}$$

We still have to prove (13.1.13). If $q>p$, it follows from (13.1.12) that

$$\begin{aligned}E(Z(p,q))^2 &\le \{(EZ(0,p)^2)^{1/2} + (EZ(0,p+q)^2)^{1/2}\}^2\\ &= O((p+q)^{2m-2+\varepsilon}) = O(q(p+q)^{b-3/2}).\end{aligned}$$

If $p \geq q \geq p^{1/2}$ we obtain similarly

$$E(Z(p+q))^2 = O((p+q)^{2m-2+\varepsilon}) = O(q(p+q)^{b-1}).$$

Now consider the case of $q < p^{1/2}$. Put $\widetilde{I}_n = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{N}^{2m} : b_i = a_i+n-1$ for exact $m-1$ coordinates and $a_i = b_i$ else; $a_i = a_j$; or $|a_i - a_j| \geq n$ for all $i, j\}$ and define

$$\widetilde{\tau}(n) = \sup\{E(W(\mathbf{a}, \mathbf{b}; q))^2 : (\mathbf{a}, \mathbf{b}) \in \widetilde{I}_n, q \in \{1, \cdots, m\}^m\}.$$

Then similarly to (13.1.12), we can show

$$\widetilde{\tau}(n) \leq Cn^{2m-3+\varepsilon}s^2$$

for some $C > 0$. Hence we obtain

$$\begin{aligned} E(Z(p,q))^2 &= E\Big(\sum_{k=0}^{q-1} Z(p+k,1)\Big)^2 \\ &\leq \Big(\sum_{k=0}^{q-1}(E(Z(p+k,1))^2)^{1/2}\Big)^2 \\ &= O(q^2(p+q)^{2m-3+\varepsilon}) = O(q(p+q)^{b-1}). \end{aligned}$$

This finishes the proof of Lemma 13.1.4.

### *13.1.2 WIP for $U_n$*

Let

$$\sigma_n^2 = E\Big(\sum_{i=1}^{n} \widetilde{h}_1(X_i)\Big)^2,$$

$$W_n(t) = \frac{nt}{m\sigma_n}(U_{[nt]} - \theta), \quad 0 \leq t \leq 1.$$

**Theorem 13.1.1.** *Let $h$ be a non-degenerate kernel. Assume that condition (13.1.3) is satisfied and $\sigma_n^2 \to \infty$. Moreover, assume that for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{n}{\sigma_n^2} E\widetilde{h}_1(X_1)^2 I(|\widetilde{h}_1(X_1)| > \varepsilon\sigma_n) = 0.$$

*Then $W_n \Longrightarrow W$ as $n \to \infty$.*

**Proof.** Put

$$\widehat{W}_n(t) = \frac{nt}{m\sigma_n}(\widehat{U}_{[nt]} - \theta), \quad 0 \leq t \leq 1.$$

By Corollary 5.1.4, $\widehat{W}_n \Longrightarrow W$ as $n \to \infty$. Theorem 2.1.2 implies that $\sigma_n^2/n^{1-\varepsilon} \to \infty$ as $n \to \infty$ for any $\varepsilon > 0$. Hence by the maximal inequality of Lemma 13.1.4 we obtain

$$\begin{aligned} &P\Big\{\sup_{0\le t\le 1} \frac{nt}{m\sigma_n}|R_{[nt]}| \ge \varepsilon\Big\} \\ &\le P\Big\{\sup_{0\le t\le 1} nt|R_{[nt]}| \ge \varepsilon m n^{(1-\varepsilon)/2}\Big\} \\ &= O(n^{-1/2+2\varepsilon}). \end{aligned}$$

Hence the result follows from (13.1.2).

**Remark 13.1.1.** Denker and Keller (1983) and Zhang (1989) discussed the Berry-Esseen inequality for $U$-statistics with a $\varphi^*$-mixing sample. We write the Zhang's result without any proof:

Let $\{X_n, n \ge 1\}$ be a strictly stationary $\varphi^*$-mixing sequence, $U_n$ be a $U$-statistic with kernel $h(x_1, x_2)$. Denote

$$\sigma^2 = Eh_1^2(X_1) - \theta^2 + 2\sum_{k=1}^{\infty}\{Eh_1(X_1)h_1(X_{k+1}) - \theta^2\}.$$

Suppose that there exist constants $C > 0, \beta > 0$ such that

$$\varphi^*(n) \le Ce^{-\beta n}.$$

If $\sigma^2 > 0$ and

$$\sup_{1\le i<j\le n} E|h(X_i, X_j)|^3 < \infty,$$

then for any $\varepsilon > 0$ we have

$$\sup_x\Big|P\Big\{\frac{U_n - \theta}{\sqrt{VarU_n}} \le x\Big\} - \Phi(x)\Big| \le cn^{-1/2+\varepsilon}.$$

### *13.1.3 Strong approximation by a Wiener process for $U_n$*

**Theorem 13.1.2.** *Let $h$ be a non-degenerate kernel. Assume that*

$$\sup_{1\le t_1<\cdots<t_m} E|h(X_{t_1}, \cdots, X_{t_m})|^{2+\delta} < \infty \quad \textit{for some} \quad \delta > 0. \tag{13.1.14}$$

*Moreover, assume that*

*(i) $\sigma_n^2 \ge c_1 n$ for some $c_1 > 0$,*

*(ii) $\varphi^*(n) \le c_2 n^{-\alpha}$ for some $c_2 > 0$ and $\alpha > 0$.*

*Then one can redefine $\{X_n, n \geq 1\}$ without changing its distribution on a richer probability space together with a Wiener process $W(\cdot)$ such that*

$$\frac{n}{m}(U_n - \theta) - W(\sigma_n^2) = O(\sigma_n^{2/(2+\delta)}(\log \sigma_n)^{1+\varepsilon+(1+\lambda)/(2+\delta)}) \quad a.s.$$

*for any $\varepsilon > 0$, where $\lambda = 2(\log 3)/\log \tau^{-1}$ and $\tau = 1 - 2(\alpha - 1)/\alpha(2 + \delta)$.*

**Proof.** By Lemma 13.1.2, condition (13.1.14) implies that $E|\widetilde{h}_1(X_1)|^{2+\delta} < \infty$. Therefore from Remark 9.1.1, we conclude that we can redefine the sequence $\{\widetilde{h}_1(X_n), n \geq 1\}$ on a new probability space together with a Wiener process $W(\cdot)$ such that

$$\sum_{i=1}^{n} \widetilde{h}_1(X_i) - W(\sigma_n^2) = O(\sigma_n^{2/(2+\delta)}(\log \sigma_n)^{1+\varepsilon+(1+\lambda)/(2+\delta)}) \quad \text{a.s.}$$

In fact, it is not hard to see that on the new probabilty space we also can redefine $\{X_n\}$ itself, for example by considering strong invariance principles for $R^2$-valued random vectors. By Lemma 13.1.4, $nR_n = O(n^{1/4+\varepsilon})$ a.s., hence the theorem is proved.

### *13.1.4 SLLN for $U_n$*

Wang (1994) proved a SLLN for $U$-statistics with a $\varphi^*$-mixing sample. We consider only the case of $m = 2$. In order to prove the theorem, We need the following lemma, which was proved in the proof of Theorem 3 in Babbel (1989):

**Lemma 13.1.5.** *Let $h$ be a degenerate kernel. Suppose that condition (13.1.3) is satisfied and*

$$\varphi^*(n) = O(n^{-(4+\delta)}) \quad \text{for some } \delta > 0. \tag{13.1.15}$$

*Then*

$$E\Big(\max_{1\leq i\leq n} U_i\Big)^2 \leq cn^{-2}s^2.$$

**Theorem 13.1.3.** *Assume that condition (13.1.15) is satisfied and*

$$\sup_{n\geq 2} E|h(X_1, X_n)| < \infty. \tag{13.1.16}$$

*Then we have*

$$U_n \longrightarrow \theta \quad a.s. \quad as \quad n \to \infty.$$

*where $\theta = \int\int h(x_1, x_2)dF(x_1)dF(x_2)$.*

**Proof.** For $k \in \mathbb{N}$, put

$$\begin{aligned}
h^{(k)}(x_1, x_2) &= h(x_1, x_2) I(|h(x_1, x_2)| \le 2^{2k}), \\
\theta^{(k)} &= \int\int h^{(k)}(x_1, x_2) dF(x_1) dF(x_2), \\
\widetilde{h}_1^{(k)}(x) &= \int h^{(k)}(x, y) dF(y), \\
H^{(k)}(x_1, x_2) &= h^{(k)}(x_1, x_2) - \widetilde{h}_1^{(k)}(x_1) - \widetilde{h}_1^{(k)}(x_2) + \theta^{(k)}
\end{aligned}$$

and

$$\begin{aligned}
U_n^{(k)} &= \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} h^{(k)}(X_i, X_j), \\
\widetilde{U}_n^{(k)} &= \frac{2}{n} \sum_{j=1}^{n} \widetilde{h}_n^{(k)}(X_j), \\
\Delta_n^{(k)} &= \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} H^{(k)}(X_i, X_j).
\end{aligned}$$

The last one is a $U$-statistic with a degenerate kernel $H^{(k)}$. It is easy to see that

$$U_n^{(k)} = \widetilde{U}_n^{(k)} + \Delta_n^{(k)} - \theta^{(k)}$$

and hence we can write

$$\begin{aligned}
&\limsup_{n \to \infty} |U_n - \theta| \\
&\quad \le \limsup_{k \to \infty} \max_{2^k \le n < 2^{k+1}} \{ |\widetilde{U}_n^{(k)} - 2\theta^{(k)}| + |\Delta_n^{(k)}| \\
&\qquad + |U_n - U_n^{(k)}| + |\theta - \theta^{(k)}| \}. \qquad (13.1.17)
\end{aligned}$$

Obviously, condition (13.1.16) implies that

$$\theta - \theta^{(k)} \to 0 \quad \text{as } k \to \infty. \qquad (13.1.18)$$

Applying Lemma 13.1.5 and condition (13.1.16), we have for any $\varepsilon > 0$

$$\begin{aligned}
&\sum_{k=1}^{\infty} P\Big\{\max_{2^k \le n < 2^{k+1}} |\Delta_n^{(k)}| \ge \varepsilon\Big\} \\
&\le c\varepsilon^{-2} \sum_{k=1}^{\infty} 2^{-2k} \sup_{n\ge 2} Eh^2(X_1, X_n) I(|h(X_1, X_n)| \le 2^{2k}) \\
&\le c\varepsilon^{-2} \sup_{n\ge 2} \sum_{k=1}^{\infty} 2^{-2k} \sum_{j=1}^{k} Eh^2(X_1, X_n) I(2^{2(j-1)} < |h(X_1, X_n)| \le 2^{2j}) \\
&\le c\varepsilon^{-2} \sup_{n\ge 2} \sum_{k=1}^{\infty} Eh^2(X_1, X_n) I(2^{2(j-1)} < |h(X_1, X_n)| \le 2^{2j}) \sum_{k=j}^{\infty} 2^{-2k} \\
&\le c\varepsilon^{-2} \sup_{n\ge 2} E|h(X_1, X_n)| < \infty,
\end{aligned}$$

which implies

$$\limsup_{k\to\infty} \max_{2^k \le n < 2^{k+1}} |\Delta_n^{(k)}| = 0 \quad \text{a.s.} \tag{13.1.19}$$

Moreover

$$\begin{aligned}
&\sum_{k=1}^{\infty} P\Big\{\bigcup_{2^k \le n < 2^{k+1}} \{U_n \ne U_n^{(k)}\}\Big\} \\
&\le \sum_{k=1}^{\infty} P\Big\{\bigcup_{2^k \le n < 2^{k+1}} \Big(\bigcup_{1\le i<j\le n} \{|h(X_i, X_j)| \ge 2^{2k}\}\Big)\Big\} \\
&\le \sum_{k=1}^{\infty} 2^{2(k+1)} \sup_{n\ge 2} P\{|h(X_1, X_n)| \ge 2^{2k}\} \\
&\le 4 \sup_{n\ge 2} \sum_{k=1}^{\infty} 2^{2k} \sum_{j=k}^{\infty} P\{2^{2j} \le |h(X_1, X_n)| < 2^{2(j+1)}\} \\
&\le c \sup_{n\ge 2} \sum_{j=1}^{\infty} E|h(X_1, X_n)| I(2^{2j} \le |h(X_1, X_n)| < 2^{2(j+1)}) \\
&\le c \sup_{n\ge 2} E|h(X_1, X_n)| < \infty,
\end{aligned}$$

which implies

$$\limsup_{k\to\infty} \max_{2^k \le n < 2^{k+1}} |U_n - U_n^{(k)}| = 0 \quad \text{a.s.} \tag{13.1.20}$$

Estimate the first term of the right hand side of (13.1.17). Put

$$\widehat{h}_1^{(k)}(x) = \widetilde{h}_1^{(k)}(x) I(|\widetilde{h}_1^{(k)}(x)| \le 2^k)$$

and write

$$
\begin{aligned}
|\widetilde{U}_n^{(k)} - 2\theta^{(k)}| \le & \Big|\frac{2}{n}\sum_{i=1}^n (\widehat{h}_1^{(k)}(X_i) - E\widehat{h}_1^{(k)}(X_i))\Big| \\
& + \Big|\frac{2}{n}\sum_{i=1}^n (\widetilde{h}_1^{(k)}(X_i) - \widehat{h}_1^{(k)}(X_i))\Big| \\
& + 2|\theta^{(k)} - E\widehat{h}_1^{(k)}(X_1)|. \qquad (13.1.21)
\end{aligned}
$$

Consider $|\theta^{(k)} - E\widehat{h}_1^{(k)}(X_1)|$ first. We have

$$
|\theta^{(k)} - E\widehat{h}_1^{(k)}(X_1)| = |E\widetilde{h}_1^{(k)}(X_1)I(|\widetilde{h}_1^{(k)}(X_1)| > 2^k)| \to 0 \qquad (13.1.22)
$$

as $k \to \infty$. Similarly to (13.1.20), we have

$$
\sum_{k=1}^\infty P\Big\{\bigcup_{2^k \le n < 2^{k+1}} \Big\{\sum_{i=1}^n \widetilde{h}_1^{(k)}(X_i) \ne \sum_{i=1}^n \widehat{h}_1^{(k)}(X_i)\Big\}\Big\} < \infty,
$$

which implies

$$
\limsup_{k\to\infty} \max_{2^k \le n < 2^{k+1}} \Big|\frac{2}{n}\sum_{i=1}^n (\widetilde{h}_1^{(k)}(X_i) - \widehat{h}_1^{(k)}(X_i))\Big| = 0 \quad \text{a.s.} \qquad (13.1.23)
$$

Furthermore, $\{\widehat{h}_1^{(k)}(X_n) - E\widehat{h}_1^{(k)}(X_n), n \ge 1\}$ is a strictly stationary and $\varphi^*$-mixing sequence with mixing coefficients $\varphi^*(n) = O(n^{-(4+\delta)})$. Hence using Lemma 2.2.10 we obtain

$$
\begin{aligned}
& \sum_{k=1}^\infty P\Big\{\max_{2^k \le n < 2^{k+1}} \Big|\frac{2}{n}\sum_{i=1}^n (\widehat{h}_1^{(k)}(X_i) - E\widehat{h}_1^{(k)}(X_i))\Big| \ge \varepsilon\Big\} \\
& \le c\sum_{k=1}^\infty 2^{-k} E(\widehat{h}_1^{(k)}(X_1))^2 \\
& \le c\sum_{k=1}^\infty E|\widetilde{h}_1^{(k)}(X_1)| < \infty,
\end{aligned}
$$

which implies

$$
\limsup_{k\to\infty} \max_{2^k \le n < 2^{k+1}} \frac{2}{n}\sum_{i=1}^n (\widehat{h}_1^{(k)}(X_i) - E\widehat{h}_1^{(k)}(X_i)) = 0 \quad \text{a.s.} \qquad (13.1.24)
$$

Combining (13.1.22)-(13.1.24) with (13.1.21) yields

$$
\limsup_{k\to\infty} \max_{2^k \le n < 2^{k+1}} |\widetilde{U}_n^{(k)} - 2\theta^{(k)}| = 0 \quad \text{a.s.} \qquad (13.1.25)
$$

(13.1.17)-(13.1.20) and (13.1.25) together imply the conclusion of Theorem 13.1.3.

## 13.2 Error variance estimations in linear models

Consider a linear regression model

$$Y_i = x_i'\beta + e_i, \quad i = 1, 2, \cdots,$$

where $\{x_i\}$ is a known $p$-demensional design sequence, $\{Y_i\}$ is a sequence of observed response, $\beta$ is an unknown $p$-dimensional vector, and $\{e_i\}$ is a random error sequence which is strictly stationary, with

$$Ee_1 = 0, \quad \sigma^2 := Ee_1^2 > 0, \quad \nu := Ee_1^4 < \infty. \tag{13.2.1}$$

On the basis of the residual sum of squares, estimation of $\sigma^2$ is

$$\widehat{\sigma}^2(n) = \frac{1}{n - r_n}\Big\{\sum_{j=1}^{n} e_j^2 - \sum_{i=1}^{r_n}\Big(\sum_{j=1}^{n} a_{ij}^{(n)} e_j\Big)^2\Big\},$$

where $r_n$ is the rank of matrix $X_n = (x_1, \cdots, x_n)'$ and stable to $r_n \le p$ when $n$ is large enough, and $(a_{ij}^{(n)})$ is an $n$-th real orthogonal matrix decided by design matrix $X_n$. Put

$$\tau = \nu - \sigma^4 + 2\sum_{j=2}^{\infty} E(e_1^2 - \sigma^2)(e_j^2 - \sigma^2).$$

If $0 < \tau < \infty$, define random functions $Z_n(\cdot)$ of $C[0,1]$ as follows:

$$Z_n(0) = 0, \quad Z_n(i/n) = (i - r_i)(\widehat{\sigma}_i^2 - \sigma^2)/\sqrt{n\tau}, \quad i = 1, \cdots, n,$$

$$Z_n(\cdot) \text{ is linear in } \Big[\frac{i-1}{n}, \frac{i}{n}\Big].$$

Moreover define the random function $Z(\cdot)$ of $C[0,\infty]$ by

$$Z(t) = ([t] - r_{[t]})(\widehat{\sigma}^2([t]) - \sigma^2).$$

Lin (1984) and Lu (1986) studied weak invariance principle and strong approximation for $Z_n(\cdot)$ and $Z(\cdot)$ respectively. We shall concentrate our attention on the sequence $\{e_i\}$ which is $\varphi$-mixing. Similar methods can be used to study other kinds of mixing error sequences.

*13.2.1 Weak invariance principle*

For any $q > 0$, define

$$a(q) = \inf\Big\{a : \sup_{A:P(A)\le\nu/q^4} \sup_i Ee_i^4 I_A \le a\Big\}. \tag{13.2.2}$$

It is clear that $a(q) \downarrow 0$ as $q \uparrow \infty$. Denote the inverse function of $a = a(q)$ by $q = q(a)$. Obviously, it is non-increasing. Let $a = d(t)$ be the solution of the equation $a/q(a)^5 = t$. Putting $g(b) = d(2^{1/4}/b)$, we have $g(b) \downarrow 0$ as $b \to \infty$. Let $t_n(\uparrow \infty)$ be maximum integers satisfying $t_n^2 g(t_n^{-1/2} n^{1/4}) = o(1)$. It is easy to see the existence of such $t_n$. Lin (1984) showed:

**Theorem 13.2.1.** *Let the random error sequence $\{e_i\}$ with (13.2.1) be strictly stationary and $\varphi$-mixing. Suppose that mixing coefficients $\varphi(n)$ satisfy*

*(i)* $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty.$

*Then $\tau < \infty$. If, in addition, $\tau > 0$ and*

*(ii)* $n t_n^{-1}\varphi(t_n) = o(1),$

*then*

$$Z_n \Longrightarrow W \quad \textit{as } n \to \infty.$$

In order to prove the WIP, we need some lemmas.

**Lemma 13.2.1.** *Let $\{\alpha_i^{(n)}, i = 1, \cdots, n\}$ be a sequence of series of random variables, which are independent within each series, satisfying*

$$E{\alpha_i^{(n)}}^4 I(|\alpha_i^{(n)}| \ge q) \le a(q), \quad i = 1, \cdots, n, \tag{13.2.3}$$

*where $a(q)$ is defined in (13.2.2). Then we have*

$$b^4 P(|X| \ge b) = O(g(b)) \quad \textit{as} \quad b \to \infty \tag{13.2.4}$$

*uniformly in the class $\mathcal{F}$ of the random variables with the form of $X = \sum_{i=1}^{n} \cdot a_i \alpha_i^{(n)}$ where $a_k$ satisfy $\sum_{k=1}^{n} a_k^2 \le 1$.*

**Proof.** Let $f_i := f_{in}$ and $F_i := F_{in}$ be the characteristic function and the distribution function of $\alpha_i^{(n)}$ respectively, $f_i^{(k)}$ $k$-th derivative of $f_i$. We first show that

$$f_i(t) = \sum_{k=0}^{4} \frac{f_i^{(k)}(0)}{k!} t^k + o(t^4) \quad \text{as } t \to 0 \tag{13.2.5}$$

uniformly in $i = 1, \cdots, n$. To this end, write

$$f_i(t) - \sum_{k=0}^{4} \frac{f_i^{(k)}(0)}{k!} t^4 = -\frac{t^4}{4!} \int_{-\infty}^{\infty} (1 - e^{it\theta x}) x^4 dF_i(x), \tag{13.2.6}$$

where $|\theta| \leq 1$. Put $t = \pm a(q)/q^5$. For $|x| \leq q$,

$$|1 - e^{it\theta x}| \leq |t\theta x| \leq a(q)/q^4.$$

Hence (13.2.6) and (13.2.3) imply that

$$\begin{aligned}\Big| f_i(t) - \sum_{k=0}^{4} \frac{f_i^{(k)}(0)}{k!} t^4 \Big| \leq \frac{t^4}{24} \Big\{ \int_{|x|<q} |1 - e^{it\theta x}| x^4 dF_i(x) \\ + \int_{|x|\geq q} 2x^4 dF_i(x) \Big\} \leq t^4 a(q)/8\end{aligned}$$

for $i = 1, \cdots, n$. Note that $a(q)$ is independent of $i$ and $a(q) \downarrow 0$ as $q \to \infty$. Uniformity of (13.2.5) is proved. (13.2.5) can be rewritten as

$$\log f_i(t) = \sum_{j=1}^{4} b_{ij} t^j + g_i(t), \quad i = 1, \cdots, n, \tag{13.2.7}$$

where $g_i(t)/t^4 = O(a(q))$ as $t \to 0 (q \to \infty)$ uniformly in $i = 1, \cdots, n$. Let $f$ and $F$ be the characteristic function and the distribution function of $X = \sum_{i=1}^{n} a_i \alpha_i^{(n)}$ respectively. Then (13.2.7) implies

$$\log f(t) = \sum_{i=1}^{n} \log f_i(a_i t) = \sum_{j=1}^{4} \Big( \sum_{i=1}^{n} b_{ij} a_i^j \Big) t^j + \sum_{i=1}^{n} g_i(a_i t)$$

where

$$\sum_{i=1}^{n} g_i(a_i t) \leq c a(q) \sum_{i=1}^{n} (a_i t)^4 \leq c d(t) t^4.$$

And further

$$\begin{aligned} f(t) &= \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} t^k + O(t^5 + d(t) t^4) \\ &= \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} t^k + O(d(t) t^4), \end{aligned}$$

since $t/d(t) \to 0$ as $t \to 0$ by the definition of $d(\cdot)$. Then, uniformly in $f$

$$\begin{aligned} O(d(t)) &= f^{(4)}(0) - \{f(4t) - 4f(2t) + 6f(0) \\ &\quad - 4f(-2t) + f(-4t)\}/(2t)^4 \\ &= \int x^4 dF(x) - \int \Big(\frac{e^{itx} - e^{-itx}}{2t}\Big)^4 dF(x) \\ &= \int \Big(1 - \Big(\frac{\sin tx}{tx}\Big)^4\Big) x^4 dF(x) \\ &\geq \int_{|x|\geq b} \Big(1 - \Big(\frac{\sin tx}{tx}\Big)^4\Big) x^4 dF(x) \\ &\geq \frac{1}{2}\int_{|x|\geq b} x^4 dF(x) \geq \frac{1}{2} b^4 P(|X| \geq b) \end{aligned}$$

provided $(bt)^4 \geq 2$, which implies (13.2.4).

**Remark 13.2.1.** Similarly, if we assume that for some integer $m > 0$ and $\delta$, $0 \leq \delta < 1$,

$$\max_{1\leq i\leq n} E|\alpha_i^{(n)}|^{m+\delta} I(|\alpha_i^{(n)}| > q) \longrightarrow 0 \quad \text{as } q \to \infty$$

instead of condition (13.2.3), we have

$$P\{|X| \geq b\} = o(b^{-m-\delta}) \quad \text{as } b \to \infty$$

uniformly in $\mathcal{F}$ defined in Lemma 13.2.1.

**Lemma 13.2.2.** *Under the conditions (i) and (ii) in Theorem 13.2.1, for any $\varepsilon > 0$*

$$nP(|X| \geq n^{1/4}\varepsilon) = o(1)$$

*uniformly in random variables with form $X = \sum_{k=1}^n a_k e_k$ where $\sum_{k=1}^n a_k^2 \leq 1$.*

**Proof.** Put

$$h_n = [nt_n^{-1}/2], \quad \xi_j = \sum_{k=2jt_n+1}^{(2j+1)t_n} a_k e_k,$$

$$\eta_j = \sum_{k=(2j+1)t_n+1}^{2(j+1)t_n} a_k e_k, \quad j = 0, 1, \cdots, h_n - 1,$$

$$\xi_{h_n} = \sum_{k=2h_nt_n+1}^{n} a_k e_k.$$

For any $q > 0$, we are going to estimate

$$E\xi_j^4 I\Big(|\xi_j| > 5^{1/4} t_n^{1/2}\Big(\sum_{k=2jt_n+1}^{(2j+1)t_n} a_k^2\Big)^{1/2} q\Big).$$

For brevity, we only consider the case when $j = 0$, and put the event

$$B = \Big\{|\xi_0| > 5^{1/4} t_n^{1/2}\Big(\sum_{k=1}^{t_n} a_k^2\Big)^{1/2} q\Big\}.$$

We have

$$\begin{aligned} E\xi_0^4 I_B = &\sum_{k=1}^{t_n} a_k^4 E e_k^4 I_B + \sum_{p\neq q} a_p^2 a_q^2 E e_p^2 e_q^2 I_B \\ &+ \sum_{p\neq q} a_p^3 a_q E e_p^3 e_q I_B + \sum_{p\neq q\neq r} a_p^2 a_q a_r E e_p^2 e_q e_r I_B \\ &+ \sum_{p\neq q\neq r\neq s} a_p a_q a_r a_s E e_p e_q e_r e_s I_B. \end{aligned} \tag{13.2.8}$$

Obviously, none of the first two sums in the right hand side of (13.2.8) is exceeded by

$$\Big(\sum_{k=1}^{t_n} a_k^2\Big)^2 \max_{1\leq k\leq t_n} E e_k^4 I_B.$$

Put $M = \max_{1\leq k\leq t_n} E e_k^4 I_B$. For the third sum, there is

$$\Big|\sum_{p\neq q} a_p^3 a_q E e_p^3 e_q I_B\Big| \leq M \sum_{p=1}^{t_n} |a_p^3| \sum_{q=1}^{t_n} |a_q| \leq M t_n^{1/2}\Big(\sum_{k=1}^{t_n} a_k^2\Big)^2.$$

Similarly the absolute value of the fourth sum has the upper bound $Mt_n(\sum_{k=1}^{t_n} a_k^2)^2$. For the fifth sum, its absolute value has the upper bound $Mt_n^2(\sum_{k=1}^{t_n} a_k^2)^2$. Therefore we obtain

$$E\xi_0^4 I_B \leq 5Mt_n^2\Big(\sum_{k=1}^{t_n} a_k^2\Big)^2. \tag{13.2.9}$$

Referring to the estimation of $E\xi_0^4 I_B$, it is easy to see that $E\xi_0^4$ has upper bound $5\nu t_n^2\left(\sum_{k=1}^{t_n} a_k^2\right)^2$, which implies $P(B) \leq \nu/q^4$. From (13.2.2), we have $M \leq a(q)$. Inserting it into (13.2.9) yields

$$\frac{1}{5} t_n^{-2}\Big(\sum_{k=1}^{t_n} a_k^2\Big)^{-2} E\xi_0^4 I_B \leq a(q). \tag{13.2.10}$$

For $\xi_j, j = 1, \cdots, h_n$, we have almost the same conclusions (for $\xi_{h_n}$, there may be a difference of a constant). Let $\{\xi'_j, j = 0, 1, \cdots, h_n\}$ be independent random variables such that $\xi'_j$ obeys the same distribution as $\xi_j$. By (13.2.10),

$$\Big\{5^{-1/4} t_n^{-1/2} \Big( \sum_{k=2jt_n+1}^{(2j+1)t_n} a_k^2 \Big)^{-1/2} \xi'_j, j = 0, 1, \cdots, h_n \Big\}$$

satisfies the conditions given in Lemma 13.2.1. Choosing $\left(\sum_{k=2jt_n+1}^{(2j+1)t_n} a_k^2\right)^{1/2}$ as $a_k$ and $t_n^{-1/2} n^{1/4} \varepsilon$ as $b$ in Lemma 13.2.1 ($\varepsilon > 0$ is given arbitrarily), we obtain

$$t_n^{-2} n P\Big\{ t_n^{-1/2} \Big| \sum_{j=0}^{h_n} \xi'_j \Big| \geq t_n^{-1/2} n^{1/4} \varepsilon \Big\} = O(g(t_n^{-1/2} n^{1/4} \varepsilon)).$$

Because of the choice of $t_n$, we have

$$nP\Big\{ \Big| \sum_{j=0}^{h_n} \xi'_j \Big| \geq n^{1/4} \varepsilon \Big\} = O(t_n^2 g(t_n^{-1/2} n^{1/4})) = o(1). \tag{13.2.11}$$

Furthermore by Lemma 1.2.9

$$\Big| E \exp\Big( itn^{-1/4} \sum_{j=0}^{h_n} \xi'_j \Big) - E \exp\Big( itn^{-1/4} \sum_{j=0}^{h_n} \xi_j \Big) \Big|$$
$$\leq (h_n + 1) \varphi(t_n) \leq \frac{1}{2} n t_n^{-1} \varphi(t_n).$$

By condition (ii) in Theorem 13.2.1, $n^{-1/4} \sum_{j=0}^{h_n} \xi'_j$ has the same limit distribution as $n^{-1/4} \sum_{j=0}^{h_n} \xi_j$. Thus from (13.2.11)

$$nP\Big\{ \Big| \sum_{j=0}^{h_n} \xi_j \Big| \geq n^{1/4} \varepsilon \Big\} = o(1).$$

For $\eta_j$, we have the same relation. Combining these two results implies the conclusion of the lemma.

**Proof of Theorem 13.2.1.**

Obviously it is enough to prove the WIP. Define $U_n(\cdot)$ and $V_n(\cdot)$ by $U_n(0) = 0, V_n(0) = 0$,

$$U_n(i/n) = \sum_{j=1}^{n}\Big(e_j^2 - \frac{i - r_i}{i}\sigma^2\Big)/\sqrt{n\tau},$$

$$V_n(i/n) = \sum_{j=1}^{r_i}\Big(\sum_{k=1}^{i} a_{jk}^{(i)} e_k\Big)^2/\sqrt{n\tau}, \quad i = 1, \cdots, n,$$

both $U_n$ and $V_n$ are linear in $\Big[\frac{i-1}{n}, \frac{i}{n}\Big]$.

We have

$$Z_n = U_n - V_n.$$

By Theorem 5.1.1

$$U_n \Longrightarrow W \quad \text{as } n \to \infty.$$

Hence in order to prove the theorem, it suffices to show that for any $\varepsilon > 0$

$$P\Big\{\sup_{0\le t\le 1} |V_n(t)| > \varepsilon\Big\} \to 0, \quad \text{as} \quad n \to \infty. \tag{13.2.12}$$

Since $r_i \le p$, (13.2.12) is equivalent to

$$P\Big\{\max_{1\le i\le n}\Big|\sum_{k=1}^{i} a_k^{(i)} e_k\Big| \ge (n\tau)^{1/4}\varepsilon^{1/2}\Big\} \to 0 \quad \text{as} \quad n \to \infty$$

for any $\{a_k^{(n)}\}$ satisfying $\sum_{k=1}^{n} {a_k^{(n)}}^2 \le 1$. By Lemma 13.2.2, we have

$$P\Big\{\max_{1\le i\le n}\Big|\sum_{k=1}^{i} a_k^{(i)} e_k\Big| \ge (n\tau)^{1/4}\varepsilon^{1/2}\Big\}$$
$$\le n \max_{1\le i\le n} P\Big\{\Big|\sum_{k=1}^{i} a_k^{(i)} e_k\Big| \ge (n\tau)^{1/4}\varepsilon^{1/2}\Big\} \to 0 \quad \text{as} \quad n \to \infty.$$

The proof of the theorem is complete.

**Remark 13.2.2.** When $\{e_i\}$ is a strictly stationary $m$-dependent sequence, one can take $t_n$ to be a constant, and hence condition (ii) is satisfied. When $\{e_i\}$ is a bounded sequence, we can take $a(q) \equiv 0$ and $t_n$, for example, to be $[n^{2/3}]$. Hence condition (ii) is also satisfied.

**Remark 13.2.3.** Lin (1984) also gave the WIP when $\{e_i\}$ is $\alpha$-mixing.

### *13.2.2 Strong approximation*

Using the strong appximation result for a $\varphi$-mixing sequence (cf. Theorem 9.1.1), Lu (1986) showed the following theorem.

**Theorem 13.2.2.** *Let the random error sequence $\{e_i\}$ with (13.2.1) be strictly stationary and $\varphi$-mixing. Suppose that $E|e_1|^{8+\delta} < \infty$ for some $0 < \delta \le 1$ and*

$$\varphi(n) = o(n^{1-4(2+\varepsilon)(10-\delta)/\delta\theta}) \quad \text{as} \quad n \to \infty \tag{13.2.13}$$

*where $0 < \theta < 1$ and $\varepsilon > 0$. Then*

$$Z(t) - W(t) = O(t^{1/4}(\log t)^{9/4+\varepsilon}) \quad a.s.$$

In order to prove the theorem, we need a lemma:

**Lemma 13.2.3.** *Under the conditions of Theorem 13.2.2,*

$$X = o(n^{1/8}) \quad a.s.$$

*where $X = \sum_{k=1}^n a_k e_k$ with $\sum_{k=1}^n a_k^2 \le 1$.*

**Proof.** Put

$$d_n = [n^{\delta\theta/4(10-\delta)}], \qquad h_n = [n/2d_n],$$

$$\xi_j = \sum_{k=2jd_n+1}^{(2j+1)d_n} a_k e_k, \qquad \eta_j = \sum_{k=(2j+1)d_n+1}^{2(j+1)d_n} a_k e_k, \quad j = 0, 1, \cdots, h_n - 1,$$

$$\xi_{h_n} = \sum_{k=2h_n d_n+1}^{n} a_k e_k.$$

Similarly to (13.2.9),

$$\begin{aligned}
E|\xi_0|^{8+\delta} &\le E\Big(\xi_0^8 \sum_{k=1}^{d_n} |a_k e_k|^\delta\Big) \\
&\le \sum_k |a_k|^{8+\delta} E|e_k|^{8+\delta} + \sum_{k \ne j}\Big(\sum_{i=1}^{8} |a_k|^i |a_j|^{8-i+\delta} E|e_k|^i |e_j|^{8-i+\delta}|\Big) \\
&\quad + \cdots + \sum_{k_i \ne k_j, i \ne j} |a_{k_1} \cdots a_{k_8}||a_{k_9}|^\delta |E|e_{k_1} \cdots e_{k_8}||e_{k_9}|^\delta| \\
&\le c d_n^{(10-\delta)/2}\Big(\sum_{k=1}^{d_n} a_k^2\Big)^{(8+\delta)/2}.
\end{aligned}$$

Let $\{\xi'_j, j = 0, 1, \cdots, h_n\}$ be such independent random variables that $\xi'_j$ obeys the same distribution as $\xi_j$. Put

$$\xi''_j = d_n^{-\frac{10-\delta}{2(8+\delta)}} \Big( \sum_{k=2jd_n+1}^{(2j+1)d_n} a_k^2 \Big)^{-1/2} \xi'_j$$

and

$$a'_j = \Big( \sum_{k=2jd_n+1}^{(2j+1)d_n} a_k^2 \Big)^{1/2}.$$

Then

$$\sum_{j=0}^{h_n} a'^2_j \le \sum_{k=1}^{n} a_k^2 \le 1.$$

Applying Remark 13.2.1 to $X = \sum_{j=0}^{h_n} a'_j \xi''_j$ and $b = \varepsilon d_n^{-(10-\delta)/2(8+\delta)} n^{1/8}$, we obtain

$$\begin{aligned} &P\Big\{\Big|\sum_{j=0}^{h_n} \xi'_j\Big| \ge \varepsilon n^{1/8}\Big\} \\ &= P\Big\{d_n^{-\frac{10-\delta}{2(8+\delta)}} \Big|\sum_{j=0}^{h_n} \xi'_j\Big| \ge \varepsilon d_n^{-\frac{10-\delta}{2(8+\delta)}} n^{\frac{1}{8}}\Big\} \\ &= o(d_n^{5-\delta/2} n^{-1-\delta/8}) = o(n^{-1-\delta(1-\theta)/8}). \end{aligned} \tag{13.2.14}$$

By $\varphi$-mixing property,

$$\begin{aligned} &|P\{\xi_1 + \xi_2 \le x\} - P\{\xi'_1 + \xi'_2 \le x\}| \\ &\quad \le \int |P\{\xi_1 \le x - u | \xi_2 = u\} - P\{\xi_1 \le x - u\}| dP\{\xi_2 \le x\} \\ &\quad \le \varphi(d_n), \end{aligned}$$

and hence

$$\begin{aligned} &\Big|P\Big\{\sum_{j=0}^{h_n} \xi_j \le x\Big\} - P\Big\{\sum_{j=0}^{h_n} \xi'_j \le x\Big\}\Big| \\ &\qquad \le O(h_n \varphi(d_n)) = O(n^{-1-\varepsilon}). \end{aligned} \tag{13.2.15}$$

Combining (13.2.14) with (13.2.15) implies

$$P\Big\{\Big|\sum_{j=1}^{h_n} \xi_j\Big| \ge \varepsilon n^{1/8}\Big\} = O(n^{-1-\varepsilon}).$$

By the Borel-Cantelli lemma

$$\Big|\sum_{j=0}^{h_n} \xi_j\Big| = o(n^{1/8}) \quad \text{a.s.}$$

Similarly

$$\Big|\sum_{j=0}^{h_n-1} \eta_j\Big| = o(n^{1/8}) \quad \text{a.s.}$$

The lemma is proved.

**Proof of Theorem 13.2.2.**
Write

$$\begin{aligned} Z(t) &= \sum_{k=1}^{[t]}(e_k^2 - \sigma^2) + r_{[t]}\sigma^2 - \sum_{i=1}^{r_{[t]}}\Big(\sum_{j=1}^{[t]} a_{ij}^{([t])} e_j\Big)^2 \\ &=: X(t) + r_{[t]}\sigma^2 - \sum_{i=1}^{r_{[t]}}\Big(\sum_{j=1}^{[t]} a_{ij}^{([t])} e_j\Big)^2. \end{aligned}$$

From Lemma 13.2.3, we have

$$\Big(\sum_{j=1}^{[t]} a_{ij}^{([t])} e_j\Big)^2 = o(t^{1/4}) \quad \text{a.s.} \quad \text{as } t \to \infty. \tag{13.2.16}$$

By Theorem 9.1.1, there exists a Wiener process $\{W(t), t \geq 0\}$ such that for any $\varepsilon > 0$

$$X(t) - W(t) = O(t^{1/4}(\log t)^{9/4+\varepsilon}) \quad \text{a.s.} \quad \text{as } t \to \infty. \tag{13.2.17}$$

Combining (13.2.16) with (13.2.17) implies the conclusion of the theorem.

## 13.3. Density estimations

Let $\{X_n, n \geq 1\}$ be a sequence of $R^d$-valued random variables with a common density function $f(x)$. In general, there are two kinds of estimations for $f(x)$. The first is the so-called kernel estimation, which is defined by

$$f_n(x) = (nh_n^d)^{-1} \sum_{i=1}^{n} K\Big(\frac{x - X_i}{h_n}\Big), \tag{13.3.1}$$

where the window width $h_n \downarrow 0$ as $n \to \infty$. Another kind is the so-called nearest neighbor estimation, which is defined by

$$\widehat{f}_n(x) = k_n/\{n|S(x, a_n(x))|\}, \tag{13.3.2}$$

where $k_n, 1 \le k_n \le n$, is the given integer, $a_n(x)$ is the distance from $x$ to the $k_n$-th closest $X_i$ (in $X_1, \cdots, X_n$), $S(x, a)$ is the hypersphere of center $x$ having the radius $a$ and $|S(x, a)| = L(S(x, a))$, where $L$ denotes the Lebesgue measure in $R^d$.

In this section we always assume $\{X_n\}$ is $\varphi$-mixing. Other kinds of mixing sequences can be studied similarly.

*13.3.1 Kernel estimation*

Many authors, such as Lin(1983), Masry and Györfi (1987), Shao (1990), Cai (1991), Peligrad (1992), Fan and Xue (1993) have studied limit behavior of the kernel estimation (KE) of the density function for a mixing sequence.

Let $\{X_n, n \ge 1\}$ be a $R^d$-valued $\varphi$-mixing sequence with a common unknown density function $f(x) = f(x_1, \cdots, x_d)$. Consider the KN $f_n(x)$ defined by (13.3.1). Peligrad (1992) showed the following result:

**Theorem 13.3.1.** *Suppose that $D$ is a compact subset of $R^d$ and $f$ is continuous on an $\varepsilon$-neighborhood of $D$. Suppose that $K$ satisfies the following conditions:*

*(1) $K(\cdot)$ is a density on $R^d$,*
*(2) $K(x) \le K_1 < \infty$ for any $x \in R^d$,*
*(3) $\|x\|^{d+1}K(x) \to 0$ as $x \to \infty$,*
*(4) $\int \|x\|K(x)\,dx = K_2 < \infty$,*
*(5) $K(\cdot)$ is Lipschitz of order $\gamma$ on $R^d$.*

*Then*

$$\sup_{x \in D} |f_n(x) - f(x)| = O(h_n + d_n^{1/2} \log n/(nh_n^d)^{1/2}) \quad a.s. \tag{13.3.3}$$

*where* $d_n = \exp\Big(2\sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i)\Big)$. *If, in addition, the condition*

$$\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty \tag{13.3.4}$$

*is satisfied, then for* $h_n = O\Big((\log^2 n/n)^{1/(d+2)}\Big)$

$$\sup_{x \in D} |f_n(x) - f(x)| = O((\log^2 n/n)^{1/(d+2)}) \quad a.s. \tag{13.3.5}$$

**Remark 13.3.1.** If condition 3) is weakened into

(3)′ $\|x\|^d K(x) \to 0$ as $x \to \infty$,

condition (4) is dropped, and condition (13.3.4) is replaced by

$$\lim_{n\to\infty} \varphi(n) < 1/2$$

and

$$nh_n^d/(d_n \log^2 n) \to \infty \quad \text{as } n \to \infty,$$

we have

$$\sup_{x\in D} |f_n(x) - f(x)| \to 0 \quad \text{a.s. as } n \to \infty.$$

**Remark 13.3.2.** Under independence assumption, the rate of the maximal deviation of the estimation from the true density is due to Kuelbs (1976) and its size is $O((\log\log n)^{1/2}/n^{1/(2(d+1))})$. (13.3.5) improves the speed of this convergence to $O((\log^2 n/n)^{1/(d+2)})$.

**Remark 13.3.3.** Shao (1990) further improved the speed to $O((\log n/n)^{1/(d+2)})$, but a stronger mixing condition, i.e. $\varphi(n) = O(n^{-(2+d)})$, is required.

**Proof of Theorem 13.3.1.**

Obviously by Lemma 2.2.2, under condition (13.3.4), (13.3.3) with $h_n = O((\log^2 n/n)^{1/(d+2)})$ implies (13.3.5). We prove (13.3.3). Write

$$\begin{aligned}\sup_{x\in D}&|f_n(x) - f(x)| \\ &\le \sup_{x\in D} |f_n(x) - Ef_n(x)| + \sup_{x\in D} |Ef_n(x) - f(x)| \\ &=: \Delta_1 + \Delta_2.\end{aligned} \tag{13.3.6}$$

Estimate $\Delta_1$ first. Because $D$ is compact, we can choose a covering of $D$ with $l(n)$ balls $B_1, \cdots, B_{l(n)}$ of centers $t_1, \cdots, t_{l(n)}$ having the radius

$$R_n = h_n(n^{-1}h_n^d \log^2 n)^{1/2\gamma};$$

the number $l(n)$ can be chosen less than $O(h_n^{-d}(n/(h_n^d \log^2 n))^{d/(2\gamma)})$. Without loss of generality, we can assume $h_n > n^{-2/d}$. Hence

$$l(n) = O(n^{2+3d/(2\gamma)}). \tag{13.3.7}$$

Let $x$ be in $D$ and define

$$S_j(x) = h_n^{-d/2} \sum_{i=1}^{j} \Big(K\Big(\frac{x - X_i}{h_n}\Big) - EK\Big(\frac{x - X_i}{h_n}\Big)\Big).$$

So

$$\Delta_1 = \sup_{x\in D}(nh_n^{d/2})^{-1}S_n(x).$$

By condition 5), for every $x \in B_k$, we have

$$|S_n(x) - S_n(t_k)| \le CnR_n^{\gamma}h_n^{-d/2-\gamma} \le C(n\log^2 n)^{1/2} \tag{13.3.8}$$

for some $C > 0$. And by Lemma 2.2.2, there exists $G > 0$ such that

$$ES_n^2(x) \le Gnd_n. \tag{13.3.9}$$

Moreover by condition 2)

$$h_n^{-d/2}\Big|K\Big(\frac{x-X_i}{h_n}\Big) - EK\Big(\frac{x-X_i}{h_n}\Big)\Big|/(Gnd_n)^{1/2}$$
$$\le 2K_1(Gnd_nh_n^d)^{-1/2} =: C_n.$$

Now let $A(> C)$ be a positive number specified later on. By (13.3.8) we have

$$\begin{aligned}
&P\Big\{\sup_{x\in D}|S_n(x)| \ge 2A(n\log^2 n)^{1/2}\Big\}\\
&\le \sum_{k=1}^{l(n)}\Big(P\Big\{\sup_{x\in B_k}|S_n(x) - S_n(t_k)| \ge A(n\log^2 n)^{1/2}\Big\}\\
&\quad + P\{|S_n(t_k)| \ge A(n\log^2 n)^{1/2}\}\Big)\\
&\le l(n)\max_{1\le k\le l(n)} P\{|S_n(t_k)| \ge A(n\log^2 n)^{1/2}\}.
\end{aligned} \tag{13.3.10}$$

Estimate the probability of the right hand side of (13.3.10). It is clear that for any $\eta > 0$, $0 < \eta < 1/2$, there are $p \ge 1$ and $A > 0$ such that for $n \ge p$

$$\varphi(p) + \max_{1\le i\le n} P\{|S_n(t_k) - S_i(t_k)|/(Gnd_n)^{1/2} \ge A\} \le \eta.$$

By a combination of (2.2.18) and (2.2.19) in Lemma 2.2.7, we have

$$\begin{aligned}
&P\Big\{\max_{1\le j\le n}|S_j(t_k)|/(Gnd_n)^{1/2} \ge x + 2A + 2pC_n\Big\}\\
&\le \frac{\eta}{1-\eta}P\Big\{\max_{1\le j\le n}|S_j(t_k)|/(Gnd_n)^{1/2} \ge x\Big\}.
\end{aligned} \tag{13.3.11}$$

Let $B_n = 2A + 2pC_n$ and $M_n = \max_{1\le j\le n}|S_j(t_k)|/(Gnd_n)^{1/2}$. Obviously, for any $\alpha_n > 0$

$$E\exp(\alpha_n M_n) \le \exp(\alpha_n B_n) + \alpha_n\int_{B_n}^{\infty}\exp(\alpha_n x)P(M_n > x)\,dx.$$

After changing $x$ to $x + B_n$, by (13.3.11) we get

$$E\exp(\alpha_n M_n) \le \exp(\alpha_n B_n) + \frac{\eta}{1-\eta} E\exp\{\alpha_n(M_n + B_n)\}.$$

Letting $\alpha_n = (2B_n)^{-1}\log(\eta^{-1} - 1)$ yields

$$E\exp(\alpha_n M_n) \le ((\eta^{-1} - 1)^{-1/2} - (\eta^{-1} - 1)^{-1})^{-1} =: g(\eta).$$

Put $\alpha = \inf_n \alpha_n = (4A)^{-1}\log(\eta^{-1} - 1)$. Then

$$\begin{aligned} &P\{\alpha(\log n)^{-1}(Gnd_n)^{-1/2}|S_n(t_k)| \ge 3d/(2\gamma) + 4\} \\ &\quad \le g(\eta)\exp\{-(3d/(2\gamma) + 4)\log n\} \\ &\quad = O(n^{-(3d/(2\gamma)+4)}) \end{aligned} \tag{13.3.12}$$

uniformly in $k$ as $n \to \infty$. Putting $A = 2((3d/2\gamma + 4)G^{1/2}/\log(\eta^{-1} - 1))^{1/2}$, which implies $A = (3d/2\gamma + 4)G^{1/2}/\alpha$. From (13.3.7), (13.3.10) and (13.3.12) we obtain

$$\sum_{n=1}^{\infty} P\Big\{\sup_{x\in D}|S_n(x)| \ge 2A(nd_n \log^2 n)^{1/2}\Big\} < \infty. \tag{13.3.13}$$

Therefore

$$\Delta_1 = O\Big(\Big(\frac{d_n \log^2 n}{nh_n^d}\Big)^{1/2}\Big) \quad \text{a.s.} \quad \text{as } n \to \infty.$$

As for $\Delta_2$, by the well-known Bochner-Parzen theorem (cf. Parzen 1962), under the conditions of the theorem, we have

$$\Delta_2 = O(h_n) \quad \text{as } n \to \infty.$$

The proof of the theorem is completed.

*13.3.2 Nearest neighbor estimation*

Chai (1984) studied strong consistency of nearest neighbor estimation (NNE) of the density function for a mixing sequence. Using the improved Bernstein inequality, Lemma 12.4.1, Chai's theorems hold under the weaker mixing condition.

**Theorem 13.3.2.** *Suppose that condition (13.3.4) is satisfied and $k_n$ in (13.3.2) satisfy*

$$k_n \to \infty, \quad k_n/n \to 0 \quad \textit{as } n \to \infty. \tag{13.3.14}$$

*Then* $k_n/\sqrt{n} \to \infty$ *implies*

$$\widehat{f}_n(x) \xrightarrow{P} f(x) \qquad a.s.\ x \in R^d(L); \tag{13.3.15}$$

*and* $\sum_{n=1}^{\infty} \exp(-ck_n^2/n) < \infty$ *for any* $c > 0$ *implies*

$$\widehat{f}_n(x) \to f(x) \qquad a.s., \qquad a.s.\ x \in R^d(L). \tag{13.3.16}$$

**Proof.** Denote $V_d$ the volume of an unit ball in $R^d$, $\mu$ and $\mu_n$ the distribution of $X_1$ and the empirical distribution of $X_1, \cdots, X_n$ respectively. For any given $\varepsilon > 0$ put

$$\begin{aligned}
b_n(x) &= (f(x)+\varepsilon)V_d n/k_n, \\
b'_n(x) &= (f(x)-\varepsilon)V_d n/k_n, \\
S_n(x,b) &= S(x, b_n^{-1/d}(x)), \\
S_n(x,b') &= S(x, b'^{-1/d}_n(x)).
\end{aligned}$$

Then

$$\begin{aligned}
&P\{|\widehat{f}_n(x) - f(x)| > \varepsilon\} \\
&\quad \le P\{\widehat{f}_n(x) - f(x) > \varepsilon\} + P\{\widehat{f}_n(x) - f(x) < -\varepsilon\} \\
&\quad \le P\{\mu_n(S_n(x,b)) - \mu(S_n(x,b)) \ge \frac{k_n}{n} \\
&\qquad - \mu(S_n(x,b))\} + P\{\mu_n(S_n(x,b')) \\
&\qquad - \mu(S_n(x,b')) \ge \frac{k_n}{n} - \mu(S_n(x,b'))\}
\end{aligned} \tag{13.3.17}$$

(If $f(x) \le \varepsilon$, the second term of the right hand side of the first and the second inequality sign disappears). By the well-known Lebesgue density theorem we have that as $n \to \infty$

$$\begin{aligned}
&\mu(S_n(x,b))/|S_n(x,b)| \to f(x) \quad \text{a.s. } x \in R^d(L), \\
&\mu(S_n(x,b'))/|S_n(x,b')| \to f(x) \quad \text{a.s. } x \in R^d(L).
\end{aligned}$$

Let the exceptional sets be denoted by $D$ and $D'$ respectively. Put $E = D^c \bigcap D'^c$. Then for any $x \in E$ and $n$ large enough,

$$\begin{aligned}
\mu(S_n(x,b)) &\le \frac{k_n}{n}(f(x) + \frac{\varepsilon}{2})/(f(x)+\varepsilon), \\
\mu(S_n(x,b')) &\ge \frac{k_n}{n}(f(x) - \frac{\varepsilon}{2})/(f(x)-\varepsilon).
\end{aligned}$$

Put $a(x) = \varepsilon/(2(f(x)+\varepsilon))$ and $a'(x) = \varepsilon/(2(f(x)-\varepsilon))$. For any $x \in E$

$$\begin{aligned} &P\{|\widehat{f}_n(x) - f(x)| > \varepsilon\} \\ &\quad \leq P\{|\mu_n(S_n(x,b)) - \mu(S_n(x,b))| \geq \frac{k_n}{n}a(x)\} \\ &\qquad + P\{|\mu_n(S_n(x,b')) - \mu(S_n(x,b'))| \geq \frac{k_n}{n}a'(x)\} \\ &\quad =: I_{n1} + I_{n2}. \end{aligned}$$

Let $\xi_i = I(X_i \in S_n(x,b)) - \mu(S_n(x,b))$, $i = 1, \cdots, n$. Then using Lemma 12.4.1 we have

$$I_{n1} \leq 2\exp\Big\{-c\frac{k_n^2}{n}a^2(x)\Big\},$$

and similarly

$$I_{n2} \leq 2\exp\Big\{-c\frac{k_n^2}{n}a'^2(x)\Big\}.$$

They imply (13.3.15) if $k_n/\sqrt{n} \to \infty$ and (13.3.16) if $\sum_{n=1}^{\infty}\exp(-ck_n^2/n) < \infty$ for any $c > 0$.

**Remark 13.3.4.** By a finer analysis, we can obtain a rate of a.s. consistency, i.e.

$$\widehat{f}_n(x) - f(x) = o(r_n^{-1}) \quad \text{a.s.}$$

where $r_n = n^{\beta}, 0 < \beta < 1/(2(d+1))$, if $f(\cdot)$ satisfies the local Lipschitz condition on $x$ and $f(x) > 0$ and conditions (13.3.4) and (13.3.14) are satisfied.

Next we consider uniform strong consistency in the case of $d = 1$. At this time, $\widehat{f}_n(x) = k_n/(2na_n(x))$, $x \in R$. We need some lemmas. Let $F$ and $F_n$ be the distribution function of $X_1$ and the empirical distribution function of $X_1, \cdots, X_n$ respectively. Define the empirical process of $\{X_n, n \geq 1\}$:

$$R(s,t) = [t](F_{[t]}(s) - F(s)), \quad s \in R, t \geq 0.$$

**Lemma 13.3.1.** (Berkes and Philipp 1977) *Suppose that*

$$\varphi(n) = O(n^{-5-\delta}) \quad \textit{for some } \delta \in (0, 1/4). \tag{13.3.18}$$

*Then there exists a version $K(s,t)$ of a Kiefer process such that*

$$\sup_{t \leq T}\sup_{s \in R}|R(s,t) - K(F(s),t)| = O(T^{1/2}(\log T)^{-\lambda}) \quad \textit{a.s.}$$

*for some $\lambda > 0$.*

**Lemma 13.3.2.**(Csörgő and Révész 1981) *For a Kiefer process $K(s,t)$,*

$$\limsup_{t\to\infty} \sup_{0\le s\le 1} |K(s,t)|/(t\log\log t)^{1/2} = 1/\sqrt{2} \quad a.s.$$

**Theorem 13.3.3.** *Suppose that (13.3.18) is satisfied and $\{k_n\}$ satisfies*

$$k_n/n \to 0 \quad \text{and} \quad k_n/(n\log\log n)^{1/2} \to \infty. \tag{13.3.19}$$

*And suppose that $f$ is uniformly continuous in $R$. Then*

$$\sup_x |\widehat{f}_n(x) - f(x)| \to 0 \quad a.s.$$

**proof.** By Lemma 13.3.1, for any $\varepsilon > 0$ there is a constant $a > 0$ such that

$$P(A_n, i.o.) \le \varepsilon, \tag{13.3.20}$$

where $A_n = \left\{\sup_x |R(x,n) - K(F(x),n)| \ge an^{1/2}(\log n)^{-\lambda}\right\}$. Similarly, by Lemma 13.3.2 for any $\varepsilon > 0$ there is a constant $b > 0$ such that

$$P(B_n, i.o.) \le \varepsilon, \tag{13.3.21}$$

where $B_n = \left\{\sup_{0\le s\le 1} |K(s,n)| \ge b(n\log\log n)^{1/2}\right\}$. Put

$$d_n(x) = \frac{k_n}{2n(f(x)+\varepsilon)}, \quad d_n'(x) = \frac{k_n}{2n(f(x)-\varepsilon)}$$

and

$$S_n(x,d_n) = (x - d_n(x), x + d_n(x)),$$
$$S_n(x,d_n') = (x - d_n'(x), x + d_n'(x)).$$

Then similarly to (13.3.17)

$$\begin{aligned} P\Big\{\bigcup_{n\ge m} \Big\{\sup_x|\widehat{f}_n(x) - f(x)| > \varepsilon\Big\}\Big\} &\le P\Big\{\bigcup_{n\ge m}\bigcup_x\{\mu_n(S_n(x,d_n)) \\ &\quad - \mu(S_n(x,d_n)) \ge \frac{k_n}{n} - \mu(S_n(x,d_n))\}\Big\} \\ &\quad + P\Big\{\bigcup_{n\ge m}\bigcup_{x:f(x)>\varepsilon}\{\mu_n(S_n(x,d_n')) \\ &\quad - \mu(S_n(x,d_n')) \ge \frac{k_n}{n} - \mu(S_n(x,d_n'))\}\Big\} \\ &=: J_{m1} + J_{m2}. \end{aligned} \tag{13.3.22}$$

By uniform continuity of $f$, $M := \sup_x f(x) < \infty$, and further for any $x$ and large $n$,

$$\begin{aligned}
\mu(S_n(x,d_n)) &\le \frac{k_n}{n}(f(x)+\frac{\varepsilon}{2})/(f(x)+\varepsilon),\\
\frac{k_n}{n} - \mu(S_n(x,d_n)) &\ge \frac{k_n}{n}\cdot\frac{\varepsilon}{2(f(x)+\varepsilon)} \ge \frac{k_n}{n}\cdot\frac{\varepsilon}{2(M+\varepsilon)} =: p_n,\\
\mu(S_n(x,d'_n)) &\ge \frac{k_n}{n}(f(x)-\frac{\varepsilon}{2})/(f(x)-\varepsilon),\\
\mu(S_n(x,d'_n)) - \frac{k_n}{n} &\ge \frac{k_n}{n}\cdot\frac{\varepsilon}{2(f(x)-\varepsilon)}\\
&\ge \frac{k_n}{n}\cdot\frac{\varepsilon}{2(M-\varepsilon)} =: q_n, \quad \text{as } f(x) > \varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
J_{m1} &\le P\Big\{\bigcup_{n\ge m}\Big\{\sup_x |\mu_n(S_n(x,d_n)) - \mu(S_n(x,d_n))| \ge p_n\Big\}\Big\}\\
&\le 2P\Big\{\bigcup_{n\ge m}\Big\{\sup_x |F_n(x) - F(x)| \ge \frac{p_n}{2}\Big\}\Big\}\\
&\le 2P\Big\{\bigcup_{n\ge m}\Big\{\sup_x \Big|\frac{R(x,n)-K(F(x),n)}{n}\Big| \ge \frac{p_n}{4}\Big\}\Big\}\\
&\quad + 2P\Big\{\bigcup_{n\ge m}\Big\{\sup_x \frac{|K(F(x),n\ )|}{n} \ge \frac{p_n}{4}\Big\}\Big\}\\
&=: 2J_{m1}^{(1)} + 2J_{m1}^{(2)}.
\end{aligned}$$

By condition (13.3.19) we have

$$\sqrt{n}p_n(\log n)^{\lambda} \to \infty \quad \text{as } n \to \infty.$$

Hence it follows from (13.3.20) that

$$\begin{aligned}
J_{m1}^{(1)} &\le P\Big\{\bigcup_{n\ge m}\Big\{\sup_x \Big|\frac{R(x,n)-K(F(x),n)}{n^{1/2}(\log n)^{-\lambda}}\Big| \ge \frac{n^{1/2}p_n(\log n)^{\lambda}}{4}\Big\}\Big\}\\
&\le P\Big\{\bigcup_{n\ge m} A_n\Big\} \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Similarly (13.3.21) implies

$$J_{m1}^{(2)} \le P\Big\{\bigcup_{n\ge m} B_n\Big\} \to 0 \quad \text{as } m \to \infty.$$

Thus $J_{m1} \to 0$ as $m \to \infty$. Moreover

$$J_{m2} \le P\Big\{ \bigcup_{n\ge m} \Big\{ \sup_x |\mu_n(S_n(x, d_n')) - \mu(S_n(x, d_n'))| \ge q_n \Big\}\Big\}.$$

In the same way as for $J_{m1}$, we have $J_{m2} \to 0$ as $m \to \infty$. Then it follows from (13.3.22) that for any $\varepsilon > 0$

$$P\Big\{ \bigcup_{n\ge m} \Big\{ \sup_x |\widehat{f}_n(x) - f(x)| > \varepsilon \Big\}\Big\} \to 0 \quad \text{as } m \to \infty.$$

This completes the proof of the theorem.

**Remark 13.3.5.** If $f$ possesses a bounded second derivative then for $k_n = [n^{7/10}]$ and any $C_n \to \infty$ we have

$$\sup_x |\widehat{f}_n(x) - f(x)| = o(n^{-1/5}(\log\log n)^{1/2} C_n) \quad \text{a.s.}$$

**Remark 13.3.6.** In Yu (1993) a simple and useful nonparametric estimator of a density $f(x)$ based on a sample $X_1, \cdots, X_n$ has been defined. If $m = m_n$ is a positive integer, the nonparametric estimator $f_n(x)$ of $f(x)$ is defined by dividing $2m_n$ by $n$ times the length of the smallest interval containing $x$ which consists of $2m_n$ of the $n$ observations in which the half of them lie on the left side of $x$ and the other half on the right side. Formally,

$$f_n(x) = \begin{cases} \dfrac{2m_n}{n(X_{(2m_n+j)} - X_{(j+1)})}, & x \in [X_{(m_n+j)}, X_{(m_n+j+1)}), \\ & \qquad \text{for } j = 0, 1, \cdots, , n - 2m_n; \\ 0 & x < X_{(m_n)} \text{ or } x \ge X_{(n-m_n+1)}, \end{cases}$$

where $(X_{(1)}, \cdots, X_{(n)})$ is the order statistic of $(X_1, \cdots, X_n)$. Yu (1993) showed that in the case of i.i.d. observations $f_n(x)$ converges to $f(x)$ (in probability and a.s.) under some conditions. Yu (1995) studied that the rate of strong uniform convergence for the estimator defined as above when the observations satisfy a $\varphi$-mixing or an $\alpha$-mixing condition.

# Chapter 14 Strong Approximations for Other Kinds of Dependent Random Variables

The almost sure invariance principle for sums of weakly dependent random variables has been given by Philipp and Stout (1975), such as lacunary trigonometric series, $\varphi$-mixing or $\alpha$-mixing sequence of random variables, Gaussian sequence and additive functional of Markov chains . The strong approximations of sums of $\varphi$-mixing and $\alpha$-mixing sequences have been studied in Chapter 9. In this chapter we shall investigate the strong approximations of sums of other kinds of dependent random variables, including lacunary trigonometric series with weights, a class of Gaussian sequences and additive functional of Markov processes. All of these improve essentially and comprehensively the results in Philipp and Stout's monograph (1975).

## 14.1 Lacunary trigonometric series with weights

Let $\{n_k, k \geq 1\}$ be a sequence of positive real numbers with

$$n_{k+1}/n_k \geq 1 + q/k^r \tag{14.1.1}$$

for all $k$ and some $q > 0$, $0 \leq r < 1/2$. It is said to be a *lacunary sequence* when $r = 0$. Let $\{a_k, k \geq 1\}$ be a sequence of non-zero real numbers. Put

$$A_n^2 = \frac{1}{2}\sum_{k=1}^{n} a_k^2, \qquad B_n^2 = \frac{1}{2}\sum_{k=1}^{n} a_k^4. \tag{14.1.2}$$

Suppose that $A_n \to \infty$ and that there exist constants $\delta, \beta$ with $0 < \delta \leq$

$1, \beta > 0$ such that

$$a_k = O(A_k^{1-\delta}), \tag{14.1.3}$$
$$A_{2k} = O(A_k), \tag{14.1.4}$$
$$k^\beta = O(A_k). \tag{14.1.5}$$

In this section, $([0,1), \mathcal{B}, P)$ denotes a probability space, where $\mathcal{B}$ consists of the Lebesgue measurable sets of [0,1), and $P$ is the Lebesgue measure on $\mathcal{B}$. We consider the trigonometric series

$$S(A_n^2, \omega) = \sum_{k=1}^{n} a_k \cos 2\pi n_k \omega, \quad \omega \in [0,1). \tag{14.1.6}$$

For $t \geq 0$, put

$$S(t, \omega) = S(A_n^2, \omega), \quad \text{if } A_n^2 \leq t < A_{n+1}^2, \tag{14.1.7}$$

where $A_0 = 0$.

The strong approximations of $S(t)$ by a Wiener process $W(t)$ were first discussed by Gaposhkin (1966). Later on, in the case of $r = 0$, Philipp and Stout (1975) proved the almost sure invariance principle for lacunary trigonometric series with weights under condition (14.1.3) and obtained the approximation order of $1/2 - \lambda$ for each $\lambda < \delta/32$. In a special case of unweighted summands, they obtained the approximation order of $5/12 + \lambda$ for each $\lambda > 0$ and conjectured that the constant $5/12$ would be replaced by $1/3$. Sun (1984) showed this fact. Shao (1987) improved all these results comprehensively. He studied the general case of lacunary trigonometric series with weights and pointed out that Sun's order is not the best possible, and obtained, when the case of $r = 0$, the order of $1/4$ which is the best if one were only to use the Skorohod embedding scheme. Furthermore, in some particular cases, the logarithmic order is obtained

**Theorem 14.1.1.** *Suppose that conditions (14.1.3)–(14.1.5) are satisfied and that $\delta\beta > r$. Then without changing the distribution of $\{S(t), t \geq 0\}$, we can redefine the process $\{S(t), t \geq 0\}$ on a richer probability space together with a Wiener process $\{W(t), t \geq 0\}$ such that*

$$S(t) - W(t) = O\left(t^{(2-\delta)/4 + r/(4\beta)} \log^2 t\right) \quad a.s. \tag{14.1.8}$$

**Theorem 14.1.2.** *Under the conditions of Theorem 14.1.1 and the assumption of $|a_k|$ non-increasing, we have*

$$S(A_n^2) - W(A_n^2) = O\Big(A_n^{r/(2\beta)}(B_n^{1/2} \vee A_n^{r/(2\beta)}) \log^2 A_n\Big) \qquad a.s. \quad (14.1.9)$$

We immediately obtain the approximation order for lacunary trigonometric series with unweighted summands from the above general theorems.

**Corollary 14.1.1.** *If $a_k \equiv 1$, then we have*

$$S(t) - W(t) = O\Big(t^{(1+2r)/4} \log^2 t\Big) \qquad a.s.$$

In the case of $r = 0$ the result (14.1.10) is an essential improvement of Theorem 3.1 in Philipp and Stout (1975), the order of 1/4 is the best possible provided $S(t)$ is constructed via the Skorohod embedding method.

**Corollary 14.1.2.** *Suppose that the condition of Theorem 14.1.2 and condition (14.1.5) are satisfied (for some $\beta > 0$), and $B_N = O(1)$, $|a_k| \downarrow 0$. Then when $r = 0$ we have*

$$S(t) - W(t) = O(\log^2 t) \qquad a.s.$$

The proofs of Theorems need the following lemmas.

**Lemma 14.1.1.** *Let $\xi_1, \cdots, \xi_n$ be a sequence of random variables. Put*

$$S_k = \sum_{j=1}^{k} \xi_j, \quad M_k = \max_{1 \le j \le k} |S_j|.$$

*Suppose that there exists a sequence $\{c_k\}$ such that for all $0 \le i < j \le n$*

$$E|S_j - S_i|^2 \le \sum_{i<k\le j} c_k,$$

*then*

$$EM_k^2 \le \Big(\sum_{i=1}^{k} c_i\Big)\Big(\frac{\log 2k}{\log 2}\Big)^2$$

*for each $k \le n$.*

The proof refers to Theorem 2.4.1 in Stout (1974).

**Lemma 14.1.2.** *For any $v \geq 0$,*

$$\sum_{j \geq k} \frac{A_j^v}{n_j} = O\Big(\frac{A_k^v}{n_k} k^r\Big), \tag{14.1.10}$$

$$\sum_{j > k} \frac{A_j^v}{n_j - n_k} = O\Big(\frac{A_k^v}{n_k} k^{2r}\Big). \tag{14.1.11}$$

**Proof.** (14.1.4) implies that there exists a constant $C > 0$ such that

$$A_{k+j} = O\Big(\Big(\frac{k+j}{j}\Big)^C A_j\Big)$$

for all $k, j \geq 1$. And by $n_{k+1}/n_k \geq 1 + q/k^r$, we have

$$\begin{aligned} \frac{n_{k+j}}{n_k} &\geq \prod_{i=1}^{j} \Big(1 + \frac{q}{(k+i)^r}\Big) \\ &\geq c \exp\Big(\frac{q((k+j)^{1-r} - k^{1-r})}{2(1-r)}\Big), \end{aligned} \tag{14.1.12}$$

where we have used the well-known formula

$$\sum_{k=1}^{n} k^{-r} = \frac{n^{1-r}}{1-r} + u + O(n^{-r}), \quad 0 \leq r < 1/2,$$

where $u = u(r)$ is a constant. Hence

$$\begin{aligned} \sum_{j \geq k} \frac{A_j^v}{n_j} &= O\Big(\frac{A_k^v}{n_k k^c} \Big(\sum_{j \geq k} j^c \exp\Big(-\frac{q\, j^{1-r}}{2(1-r)}\Big)\Big) \exp\Big(\frac{q\, k^{1-r}}{2(1-r)}\Big)\Big) \\ &= O\Big(\frac{A_k^v}{n_k} k^r\Big). \end{aligned}$$

Thus (14.1.10) is proved. The proof of (14.1.11) is similar.

**Lemma 14.1.3.** *Let $W(t)$ be a Wiener process and $\{t_n\}$ be a sequence of random variables. Suppose that there exists a sequence $\{b_n\}$ of real numbers with $b_n = o(n)$, such that*

$$t_n - n = O(b_n) \quad a.s., \tag{14.1.13}$$

*then*

$$W(t_n) - W(n) = O(b_n^{1/2} \log^{1/2} n) \quad a.s. \tag{14.1.14}$$

**Proof.** (14.1.13) implies that there exists a constant $C$ such that

$$|t_n - n| \le Cb_n \quad \text{a.s.},$$

hence

$$|W(t_n) - W(n)| \le \sup_{0\le t\le n-Cb_n} \sup_{0\le s\le 2Cb_n} |W(t+s) - W(t)| \quad \text{a.s.}$$

In terms of a well-known result (cf. Theorem 3.2B in Hanson and Russo 1983), we have

$$\begin{aligned}&\sup_{0\le t\le n-Cb_n} \sup_{0\le t\le 2Cb_n} |W(t+s) - W(t)| \\ &= O\Big(\Big(b_n\Big(\log\frac{n+Cb_n}{2Cb_n} + \log\log(n+Cb_n)\Big)\Big)^{1/2}\Big) \quad a.s.\end{aligned}$$

(14.1.14) follows from the above relations.

We now define an increasing sequence $\mathcal{F}_k$ of $\sigma$-fields as follows. Put $p = 2/\beta + 4$. For each integer $k$, let $r_k$ be the largest integer $i$ such that

$$2^i \le A_k^p n_k. \tag{14.1.15}$$

We define $\mathcal{F}_k$ to be the $\sigma$-field generated by the intervals of the form

$$U_{vk} = [v2^{-r_k}, (v+1)2^{-r_k}), \quad 0 \le v < 2^{r_k}.$$

Putting $\xi_k(\omega) = a_k cos 2\pi n_k \omega, \quad X_k(\omega) = E(\xi_k|\mathcal{F}_k)$, we obtain by (14.1.15), for each $k,\ j \ge 1$

$$E(\xi_{k+j}|\mathcal{F}_j) = O(a_{k+j}(1 \wedge A_j^p n_j/n_{k+j})). \tag{14.1.16}$$

**Lemma 14.1.4.** *We have*

$$\sum_{k=1}^{\infty} |\xi_k - X_k| = O(1). \tag{14.1.17}$$

**Proof.** We have from (14.1.15)

$$|\xi_k - X_k| = O(A_k^{-p} a_k) = O(A_k^{-p+1}).$$

Hence (14.1.17) follows from (14.1.5).

**Lemma 14.1.5.** *We have*

$$\sum_{1\le j\le N} EX_j^2 - A_N^2 = O(1). \tag{14.1.18}$$

**Proof.** Noting that

$$E\xi_j^2 - a_j^2/2 = a_j^2 \sin 2\pi n_j/2\pi n_j,$$
$$\sum_{j\le N} E\xi_j^2 - A_N^2 = O(\sum_{j\le N} a_j^2/n_j) = O(1)$$

and

$$\sum_{j\le N}(EX_j^2 - E\xi_j^2) = O(\sum_{j\le N} a_j^2 A_j^{-p}) = O(1),$$

we get (14.1.18).

We can now represent $X_j$ as

$$X_j = Y_j + u_j - u_{j+1}, \tag{14.1.19}$$

where $\{Y_j, \mathcal{F}_j\}$ is a martingale difference sequence and $u_1 = 0$

$$u_j = \sum_{k=0}^{\infty} E(X_{j+k}|\mathcal{F}_{j-1}), \quad j \ge 2. \tag{14.1.20}$$

**Lemma 14.1.6.** *We have for all $j$, $1 \le j < n-1$,*

$$\sum_{j<i<k\le n} |E(X_i X_k|\mathcal{F}_j)| = Cn^{2r} A_n^{2-2\delta} \log^2 A_n, \tag{14.1.21}$$

*where constant $C$ does not depend on $j$ and $n$.*

**Proof.** By the definition of $X_k$, we admit

$$E(X_i X_k|\mathcal{F}_j) = E(X_i \xi_k|\mathcal{F}_j).$$

Write

$$E(\xi_k X_i|\mathcal{F}_j) = \sum_{v=0}^{2^{r_j}-1} I(U_{vj}) b_v.$$

We have

$$\begin{aligned}\frac{2^{-r_j}}{a_k a_i} b_v = & \sum_{l=0}^{2^{r_i-r_j}-1} 2^{r_i} \int_{v2^{-r_j}+l2^{-r_i}}^{v2^{-r_j}+(l+1)2^{-r_i}} \cos 2\pi n_k t \, dt \\ & \cdot \int_{v2^{-r_j}+l2^{-r_i}}^{v2^{-r_j}+(l+1)2^{-r_i}} \cos 2\pi n_i t \, dt \\ = & \, 2^{r_i+1} \frac{\sin 2\pi n_i 2^{-r_i-1} \sin 2\pi n_k 2^{-r_i-1}}{(2\pi)^2 n_i n_k} \\ & \cdot \sum_{l=0}^{2^{r_i-r_j}-1} \Big( \cos 2\pi (n_k - n_i) \Big( v2^{-r_j} + \Big(l + \frac{1}{2}\Big) 2^{-r_i} \Big) \\ & + \cos 2\pi (n_k + n_i) \Big( v2^{-r_j} + \Big(l + \frac{1}{2}\Big) 2^{-r_i} \Big) \Big).\end{aligned}$$

Using the equality

$$\sum_{v=0}^{n-1} \cos(av + b) = \frac{\sin an/2}{\sin a/2} \cos(b + a(n-1)/2),$$

where $a$ and $b$ are any real numbers with $\sin(a/2) \neq 0$, we obtain

$$\begin{aligned}\frac{2^{-r_j} b_v}{a_k a_i} = & \frac{2^{r_i+1} \sin 2\pi n_i 2^{-r_i-1} \sin 2\pi n_k 2^{-r_i-1}}{(2\pi)^2 n_i n_k} \\ & \cdot \Big( \frac{\sin 2\pi (n_k - n_i) 2^{-r_j-1}}{\sin 2\pi (n_k - n_i) 2^{-r_i-1}} \cos 2\pi (n_k - n_i) \Big( v + \frac{1}{2} \Big) 2^{-r_j} \\ & + \frac{\sin 2\pi (n_k + n_i) 2^{-r_j-1}}{\sin 2\pi (n_k + n_i) 2^{-r_i-1}} \cos 2\pi (n_k + n_i) \Big( v + \frac{1}{2} \Big) 2^{-r_j} \Big).\end{aligned}$$

Hence

$$b_v = O(a_k a_i 2^{r_j} (1 + 2^{-r_j} n_i)/n_k)$$

for all $i, k$. And we also have for $i, k$ with $n_k + n_i \leq 2^{r_i-1}$,

$$b_v = O(a_k a_i (1 \wedge 2^{r_j}/(n_k - n_i)).$$

For each $i : j < i < n$, take $k_0(i) = \max\{k : (n_k + n_i) \leq 2^{r_i-1}\} \wedge (n-1)$. (14.1.12) and (14.1.15) imply that $k_0(i) - i = O(i^r \log A_i)$, hence

$$\begin{aligned}& \sum_{j<i<k\leq n} E(X_i X_k | \mathcal{F}_j) \\ & = O\Big( \sum_{i=j+1}^{n-1} \sum_{k>k_0(i)}^{n} |a_k a_i| \Big( \frac{2^{r_j}}{n_k} + \frac{n_i}{n_k} \Big) \\ & \quad + \sum_{i=j+1}^{n-1} \sum_{k=i+1}^{k_0(i)} |a_k a_i| \Big( 1 \wedge \frac{2^{r_j}}{n_k - n_i} \Big) \Big). \qquad (14.1.22)\end{aligned}$$

From Lemma 14.1.2 and (14.1.15), the first part of the right hand side of (14.1.22) is bounded by

$$\begin{aligned}A_n^{2-2\delta}\sum_{i=j+1}^{n-1}\frac{(2^{r_j}+n_i)}{n_{k_0(i)}}n^r &= O\Big(A_n^{2-2\delta}\sum_{i=j+1}^{n-1}\frac{(2^{r_j}+n_i)}{n_iA_i^p}n^r\Big)\\ &= O(A_n^{2-2\delta}n^{2r}).\end{aligned}$$

Take $i(j)=\max\{i:\ 2^{r_j}\ge n_i i^{-3r}\}\wedge(n-1)$. Then (14.1.12) and (14.1.15) imply that $i(j)-j=O(j^r\log A_j)$. Again from Lemma 14.1.2, the second part of (14.1.22) is bounded by

$$\begin{aligned}&A_n^{2-2\delta}\Big(\sum_{i=j}^{i(j)}\sum_{i<k\le k_0(i)}1+\sum_{i>i(j)}^{n-1}\sum_{k>i}^{k_0(i)}\frac{2^{r_j}}{n_k-n_i}\Big)\\ &= O\Big(A_n^{2-2\delta}\Big(n^{2r}\log^2 A_n+\sum_{i>i(j)}^{n-1}\frac{2^{r_j}}{n_{i+1}}i^{2r}\Big)\Big)\\ &= O\Big(A_n^{2-2\delta}\Big(n^{2r}\log^2 A_n+\frac{2^{r_j}(i(j))^{3r}}{n_{i(j)+1}}\Big)\Big)\\ &= O(n^{2r}A_n^{2-2\delta}\log^2 A_n).\end{aligned}$$

The lemma is proved.

**Lemma 14.1.7.** *We have*

$$u_j=O(j^rA_j^{1-\delta}\log A_j). \tag{14.1.23}$$

**Proof.** Using (14.1.16), (14.1.20) and Lemma 14.1.2, we find

$$\begin{aligned}u_j &= \sum_{k=0}^{\infty}E(\xi_{k+j}|\mathcal{F}_{j-1})\\ &= O\Big(\sum_{k=0}^{\infty}A_{k+j}^{1-\delta}(1\wedge A_j^pn_j/n_{k+j})\Big)\\ &= O(j^rA_j^{1-\delta}\log A_j).\end{aligned}$$

**Lemma 14.1.8.** *For each $k,n(k<n)$,*

$$\sum_{k<i\le n}E(X_iu_{n+1}|\mathcal{F}_i)=O(A_n^{2-2\delta}n^{2r}\log^2 A_n); \tag{14.1.24}$$

$$\sum_{k<i\le n}E(X_iu_k|\mathcal{F}_k)=O(A_n^{2-2\delta}n^{r}\log A_n). \tag{14.1.25}$$

**Proof.** By (14.1.20) we have

$$E(u_{n+1}|\mathcal{F}_i) = \sum_{j=0}^{\infty} E(\xi_{j+n+1}|\mathcal{F}_i)$$
$$= O\Big(\sum_{j=0}^{\infty} A_{j+n+1}^{1-\delta}(1 \wedge A_i^p n_i/n_{j+n+1})\Big).$$

Hence

$$\sum_{i=k+1}^{n} E(X_i u_{n+1}|\mathcal{F}_i) = O\Big(\sum_{i=k+1}^{n}\sum_{j=0}^{\infty}(1 \wedge A_i^p n_i/n_{j+n+1})A_{j+n+1}^{1-\delta}A_n^{1-\delta}\Big).$$

Taking $k_0 = [c_0 n^r \log A_n]$, where $c_0$ will be specified later, we have

$$\sum_{i=k+1}^{n}\sum_{j=0}^{\infty}(1 \wedge A_i^p n_i/n_{j+n+1})A_{j+n+1}^{1-\delta}$$
$$\le \sum_{i=n-k_0+1}^{n}\sum_{j=0}^{k_0} A_{j+n+1}^{1-\delta} + \sum_{i=k+1}^{n-k_0}\sum_{j=0}^{\infty}\frac{A_i^p n_i}{n_{j+n+1}}A_{j+n+1}^{1-\delta}$$
$$+ \sum_{i=n-k_0+1}^{n}\sum_{j=k_0+1}^{\infty}\frac{A_i^p n_i}{n_{j+n+1}}A_{j+n+1}^{1-\delta}$$
$$= O\Big(\sum_{i=k+1}^{n-k_0}\frac{A_i^p n^r n_i}{n_{n+1}}A_{n+1}^{1-\delta} + k_0^2 A_n^{1-\delta}$$
$$+ \sum_{i=n-k_0+1}^{n}\frac{A_i^p n_i}{n_{k_0+n+1}}(k_0+n+1)^r A_{k_0+n+1}^{1-\delta}\Big)$$
$$= O\Big(n^{3r}A_n^{p+1-\delta}\exp\Big(\frac{(n-k_0)^{1-r}-n^{1-r}}{2(1-r)}q\Big) + k_0^2 A_n^{1-\delta}$$
$$+ A_n^p n^r k_0 \exp\Big(\frac{n^{1-r}-(n+k_0)^{1-r}}{2(1-r)}q\Big)\Big)$$
$$= O(A_n^{1-\delta}n^{2r}\log^2 A_n).$$

The last equality holds so long as we take $c_0$ sufficiently large. This proves (14.1.24); the proof of (14.1.25) is similar.

**Lemma 14.1.9.** *We have*

$$\sum_{j=1}^{n} EY_j^2 - A_n^2 = O(n^{2r}A_n^{2-2\delta}\log^2 A_n). \tag{14.1.26}$$

**Proof.** Since $\{Y_j\}$ is a martingale difference sequence, we have

$$\begin{aligned}\sum_{j=1}^{n} EY_j^2 &= E\Big(\sum_{j=1}^{n} Y_j\Big)^2 \\ &= E\Big(\sum_{j=1}^{n} X_j\Big)^2 + 2Eu_{n+1}\sum_{j=1}^{n} X_j + Eu_{n+1}^2.\end{aligned}$$

Hence (14.1.26) follows from Lemma 14.1.5 - Lemma 14.1.8.

**Lemma 14.1.10.** *Under the assumption $\delta\beta > r$, we have*

$$\sum_{j=1}^{n} Y_j^2 - A_n^2 = O(A_n^{2-\delta} n^r \log^3 A_n) \quad a.s. \tag{14.1.27}$$

**Proof.** Note that (14.1.5) and $\delta\beta > r$ imply $n^r = O(A_n^\delta)$. Applying Lemma 14.1.1, the Borel-Cantelli lemma and the subsequence method, we need only to show that for any $0 \le m \le n$

$$E\Big(\sum_{m<j\le n} (Y_j^2 - EY_j^2)\Big)^2 = O\Big(\Big(\sum_{m<j\le n} EY_j^2\Big) A_n^{2-2\delta} n^{2r} \log^2 A_n\Big), \tag{14.1.28}$$

where the constant implied by $O$ does not depend on $m$, $n$. Observe that

$$\begin{aligned}&E\Big(\sum_{m<j\le n} (Y_j^2 - EY_j^2)\Big)^2 \\ &= \sum_{m<j\le n} E(Y_j^2 - EY_j^2)^2 + 2\sum_{m<k\le n} E(Y_k^2 - EY_k^2)\Big(\sum_{k<i\le n} Y_i^2\Big),\end{aligned}$$

where

$$\begin{aligned}&\sum_{m<k\le n} E(Y_k^2 - EY_k^2)\Big(\sum_{k<i\le n} Y_i^2\Big) \\ &= \sum_{m<k\le n} E(Y_k^2 - EY_k^2)\Big(\sum_{k<i\le n} Y_i\Big)^2 \\ &= \sum_{m<k\le n} E(Y_k^2 - EY_k^2)\Big(\sum_{k<i\le n} X_i^2 + 2\sum_{k<i<j\le n} X_i X_j \\ &\quad + 2\sum_{k<i\le n} X_i(u_{n+1} - u_{k+1}) + (u_{n+1} - u_{k+1})^2\Big) \\ &=: \sum_{m<k\le n} (I_1(k) + I_2(k) + I_3(k) + I_4(k)).\end{aligned}$$

By Lemma 14.1.6 – Lemma 14.1.8, it follows that

$$\max_{2\le i\le 4} I_i(k) = O(n^{2r} A_n^{2-2\delta} EY_k^2 \log^2 A_n). \tag{14.1.29}$$

We now show that (14.1.29) holds for $I_1(k)$ as well. Since

$$\Big| \sum_{k<i\le n} X_i^2 - \sum_{k<i\le n} \xi_i^2 \Big| = O(1),$$

it suffices to show that (14.1.29) holds for $E(Y_k^2 - EY_k^2)\big(\sum_{k<i\le n} \xi_i^2\big)$. Noting that $Y_k$ is $\mathcal{F}_k$-measurable and writing

$$Y_k = \sum_{i=0}^{2^{r_k}-1} d_i I(U_{ik}),$$

we have by the definition of $\mathcal{F}_k$

$$\begin{aligned}
&E(Y_k^2 - EY_k^2)\Big(\sum_{j=k+1}^{n} \xi_j^2\Big)\\
&= \sum_{i=0}^{2^{r_k}-1} d_i^2 \sum_{j=k+1}^{n} \int_{i2^{-r_k}}^{(i+1)2^{-r_k}} a_j^2 \cos^2 2\pi n_j t\, dt\\
&\quad - 2^{-r_k}\Big(\sum_{i=0}^{2^{r_k}-1} d_i^2\Big) \sum_{j=k+1}^{n} \int_0^1 a_j^2 \cos^2 2\pi n_j t\, dt\\
&= \sum_{i=0}^{2^{r_k}-1} d_i^2 \sum_{j=k+1}^{n} a_j^2 \frac{\sin 2\pi n_j 2^{-r_k}}{\pi n_j} \cos 4\pi n_j \Big(i + \frac{1}{2}\Big) 2^{-r_k}\\
&\quad - 2^{-r_k}\Big(\sum_{i=0}^{2^{r_k}-1} d_i^2\Big) \sum_{j=k+1}^{n} a_j^2 \frac{\sin 4\pi n_j}{2\pi n_j}\\
&= O\Big(2^{-r_k}\Big(\sum_{i=0}^{2^{r_k}-1} d_i^2\Big)\Big(\sum_{j=k+1}^{n} \frac{a_j^2}{n_j} + \sum_{j=k+1}^{n} a_j^2 (1 \wedge A_k^p n_k/n_j)\Big)\Big)\\
&= O(A_n^{2-2\delta} n^r EY_k^2 \log A_n).
\end{aligned}$$

This proves that (14.1.29) holds for $I_1(k)$. For $EY_k^4$, we have

$$EY_k^4 = O(EY_k^2 A_k^{2-2\delta} k^{2r} \log^2 A_k).$$

This proves (14.1.28), and the lemma follows.

**Lemma 14.1.11.** *We have*

$$\sum_{j=1}^{n}(E(Y_j^2|\mathcal{F}_{j-1}) - Y_j^2) = O(n^r A_n^{2-\delta} \log^3 A_n) \quad a.s. \tag{14.1.30}$$

**Proof.** Put $R_j = Y_j^2 - E(Y_j^2|\mathcal{F}_{j-1})$. Then $\{R_j, \mathcal{F}_j\}$ is a martingale defference sequence, and we have

$$ER_k^2 = O(EY_k^4) = O(EY_k^2 A_k^{2-2\delta} k^{2r} \log^2 A_k).$$

The proof is verbatim as that of Lemma 14.1.10.

By a martingale version of the Skorohod representation theorem, there exists a probability space, on which a Wiener process and a sequence of non-negative random variables $T_i$ are defined such that

$$\Big\{W\Big(\Big(\sum_{j\le m} T_j\Big), m \ge 1\Big\} \quad \text{and} \quad \Big\{\sum_{j\le m} Y_j, m \ge 1\Big\}$$

have the same distribution. Hence on the new probability space, without loss of generality we can redefine $\{Y_j\}$ by

$$Y_j = W\Big(\sum_{i\le j} T_i\Big) - W\Big(\sum_{i<j} T_i\Big)$$

and can keep the same notation. Write

$$\mathcal{L}_m = \sigma\Big\{W\Big(\sum_{j=1}^{k} T_i\Big), k \le m\Big\},$$

$$\mathcal{A}_m = \sigma\Big\{W(t), 0 \le t \le \sum_{j=1}^{m} T_j\Big\}.$$

It is clear that $\mathcal{L}_m \subset \mathcal{A}_m, m \ge 1$, and $T_j$ is $\mathcal{A}_j$- measurable. By the embedding theorem, for every $j \ge 1$, $ET_j = EY_j^2$ , and

$$E(T_j|\mathcal{A}_{j-1}) = E(Y_j^2|\mathcal{A}_{j-1}) = E(Y_j^2|\mathcal{L}_{j-1}) \quad \text{a.s.} \tag{14.1.31}$$

Moreover, for any $v > 1$ there is a $C_v$ such that

$$E|T_j|^v = C_v E|Y_j|^{2v}. \tag{14.1.32}$$

**Lemma 14.1.12.** *We have under the conditions of Theorem 14.1.1*

$$\sum_{j\le n} T_j - A_n^2 = O\Big(A_n^{2-\delta+r/\beta} \log^3 A_n\Big). \tag{14.1.33}$$

**Proof.** Write

$$\begin{aligned}\sum_{j\le n} T_j - A_n^2 &= \sum_{j\le n}(T_j - E(T_j|\mathcal{F}_{j-1})) + \sum_{j\le n}(E(Y_j^2|\mathcal{F}_{j-1}) - Y_j^2) \\ &\quad + \sum_{j\le n} Y_j^2 - A_n^2 =: I_1 + I_2 + I_3.\end{aligned}$$

Put $Z_j = T_j - E(T_j|\mathcal{A}_{j-1})$. Then $\{(Z_j, \mathcal{A}_j)\}$ is a martingale difference sequence and $EZ_j^2 = O(EY_j^4)$. By the proof analogous to that of Lemma 14.1.11 we obtain

$$I_1 = O(A_n^{2-\delta} n^r \log^3 A_n) \quad \text{a.s.} \tag{14.1.34}$$

The lemma follows from (14.1.5), (14.1.27) (14.1.30) and (14.1.34).

**Proof of Theorem 14.1.1.**
Note that

$$\sum_{j\le n} Y_j - \sum_{j\le n} X_j = u_{n+1} = O(A_n^{1-\delta} n^r \log A_n).$$

The theorem follows from Lemmas 14.1.3 and 14.1.12.

**Proof of Theorem 14.1.2.**
Note that $\delta = 1$ now. By the proof analogous to that of all the above lemmas under the conditions of Theorem 14.1.2, we also have

$$\sum_{j\le n} T_j - A_n^2 = O(n^r(n^r \vee \overline{B}_n)\log^3 A_n) \quad \text{a.s.}, \tag{14.1.35}$$

where $\overline{B}_n = \sum_{k\le n} a_k^2 EY_k^2$.
Since

$$\sum_{j\le k} EY_j^2 = O\Big(\sum_{j\le k} a_j^2\Big),$$

and $a_j^2$ is non-increasing, we have

$$\overline{B}_n = O\Big(\sum_{k\le n} a_k^4\Big)$$

by using the Abel transformation. Hence

$$\sum_{j\le n} T_j - A_n^2 = O\Big(n^r(n^r \vee \overline{B}_n)\log^3 A_n\Big) \quad \text{a.s.} \tag{14.1.36}$$

The theorem follows from (14.1.36) and Lemma 14.1.3.

**Remark 14.1.1.** If $k^\beta = O(A_k)$ for some $\beta > 0$, $|a_k| \downarrow$, and $|a_k| = O(k^{-\theta})$ for some $r \le \theta < 1/2$, then by the proof similar to that of Theorem 14.1.1, we have

$$S(A_n^2) - W(A_n^2) = O(C_n^{1/4} \log^2 A_n) \quad \text{a.s.}$$

where $C_n = \sum_{k\le n} |a_k|^{4-2r/\theta}$.

Let $\overline{S}_n(\omega) = \sum_{k\le n} \sqrt{2} \cos 2\pi n_k \omega$, $\omega \in [0,1)$. Combining Corollary 14.1.1 with the results of the increments of a Wiener process, we can obtain a.s. limiting behavior of increments of the sums $\overline{S}_n$.

## 14.2 A class of Gaussian sequences

Let $\{X_n, n \ge 1\}$ be a centered sequence of Gaussian random variables. Under some conditions, including $E\Big(\sum_{k=m+1}^{m+n} X_k\Big)^2 = n\sigma^2 + O(n^{1-\varepsilon})$ for some $\varepsilon > 0$, $\sigma^2 > 0$ and $EX_mX_{m+n} = O(n^{-2})$, Philipp and Stout (1975) established strong approximations of partial sums

$$S(t) = \sum_{k\le t} X_k \qquad t \ge 0$$

by a Wiener process with order $O(t^{1/2-\lambda})$, where $0 < \lambda < \frac{1}{60} \wedge \frac{4\varepsilon}{15}$. By applying the property of a symmetric matrix and the circle-plate theorem about eigenvalues of a matrix, Shao (1985) proved an ideal result as follows:

**Theorem 14.2.1.** *Suppose that there exist $C_i > 0, i = 1,2,3,$ such that for every $n$*

$$E\Big(\sum_{k=m+1}^{m+n} X_k\Big)^2 \ge C_1 n \quad \textit{uniformly in } m \tag{14.2.1}$$

$$EX_n^2 \le C_2 \tag{14.2.2}$$

*and*

$$\gamma(n) := \sup_m |EX_mX_{m+n}| \le C_3 n^{-3/2-\lambda} \tag{14.2.3}$$

*for some $\lambda > 0$. Then without changing the distribution of $\{S(t), t \ge 0\}$, we can redefine the process $\{S(t), t \ge 0\}$ on a richer probability space together with a Wiener process $\{W(t), t \ge 0\}$ such that*

$$S(t) - W(b_t) = O(\log^{1/2} t) \qquad a.s. \tag{14.2.4}$$

*where*

$$b_t = b_{[t]} = a_t + r_t, \quad a_t = ES^2(t), \quad r_t = O\Big(\sum_{k=1}^{[t]} k^{-1/2-\lambda}\Big). \tag{14.2.5}$$

**Corollary 14.2.1.** *If the condition (14.2.3) is strengthened by*

$$\gamma(n) \le Cn^{-2}(\log n)^{-1-\varepsilon} \tag{14.2.3'}$$

*for some* $\varepsilon > 0$, *then we have*

$$S(t) - W(a_t) = O(\log^{1/2} t) \qquad a.s. \tag{14.2.4'}$$

**Corollary 14.2.2.** *Suppose that* $\{X_n, n \ge 1\}$ *is a centered stationary Gaussian sequence and (14.2.3′) is satisfied, then*

$$\sigma^2 := EX_1^2 + 2\sum_{k=2}^{\infty} EX_1X_k$$

*converges absolutely. If* $\sigma^2 > 0$, *then, assuming* $\sigma^2 = 1$, *we have*

$$S(t) - W(t) = O(\log^{1/2} t) \qquad a.s.$$

The proof of the theorem needs the following lemmas.

**Lemma 14.2.1.** *Let* $A$ *be a real symmetric matrix of order* $n$ *with eigenvalues* $\lambda_1, \cdots, \lambda_n$. *Denote* $\lambda = \max_{1\le i\le n} |\lambda_i|$. *Then for any row vector* $\mathbf{C}$ *we have*

$$|\mathbf{C}A\mathbf{C}'| \le \lambda\mathbf{C}\mathbf{C}' \tag{14.2.6}$$

*where* $\mathbf{C}'$ *is the transposed vector of* $\mathbf{C}$.

**Proof.** We need only to prove that matrices $\lambda I - A$ and $\lambda I + A$ are non-negative definite. By the well-known property of a matrix, there exists a real orthogonal matrix $U$ such that $U'AU$ is a diagonal matrix $\Lambda$ with the diagonal elements which is just equal to the eigenvalues of the matrix $A$. Therefore we have

$$\lambda I - A = U(\lambda I - \Lambda)U'.$$

It is clear that the eigenvalues of $\lambda I - \Lambda$ are all non-negative, so that the eigenvalues of the real symmetric matrix $\lambda I - A$ are also non-negative.

Thus the matrix $\lambda I - A$ is non-negative definite, similarly, so is the matrix $\lambda I + A$. The lemma is proved.

**Lemma 14.2.2.** *At least one of the following inequalities is satisfied for eigenvalues of any matrix* $A = (a_{ij})_{n\times n}$*:*

$$|\lambda - a_{ii}| \le \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad i = 1, \cdots, n. \tag{14.2.7}$$

This result, the so-called circle-plate theorem, is due to Gerschgorin (cf. Franklin 1968, p.161 Theorem 1).

Denote $\mathcal{F}_n = \sigma\{X_k, 1 \le k \le n\}$. Write

$$\begin{aligned} Y_n &= \sum_{k=0}^{\infty} (E(X_{n+k}|\mathcal{F}_n) - E(X_{n+k}|\mathcal{F}_{n-1})) \\ &= X_n + u_{n+1} - u_n \end{aligned} \tag{14.2.8}$$

where $u_1 = 0$ and

$$u_n = \sum_{k=0}^{\infty} E(X_{n+k}|\mathcal{F}_{n-1}), \quad n = 2, 3, \cdots. \tag{14.2.9}$$

It is clear that $\{Y_n, \mathcal{F}_n\}$ is a martingale difference sequence. We shall prove below that the series in (14.2.9) is convergent under the assumptions of Theorem 14.2.1.

**Lemma 14.2.3.** *If (14.2.1), (14.2.2) and (14.2.3) are satisfied, and for any* $k \ge 1$

$$EX_k^2 \ge \sum_{j=1, j\neq k}^{\infty} |EX_kX_j| + 1, \tag{14.2.10}$$

*then*

$$\|u_n\|_2 = O(1).$$

**Proof.** Let $A$ be the covariance matrix of $(X_1, \cdots, X_j)$ and $\mathbf{C} = (EX_1X_{j+k}, \cdots, EX_jX_{j+k})$. Then, by (5.22) in Philipp and Stout (1975), we have

$$E(E^2(X_{j+k}|\mathcal{F}_j)) = \mathbf{C}A^{-1}\mathbf{C}', \tag{14.2.11}$$

for any $j, k \ge 1$. By condition (14.2.10) and Lemmas 14.2.1 and 14.2.2, we obtain

$$\mathbf{C}A^{-1}\mathbf{C}' \le \mathbf{C}\mathbf{C}'. \tag{14.2.12}$$

Denote

$$u_{nk} = E(X_{n+k}|\mathcal{F}_{n-1}), \quad k = 0, 1, \cdots; n = 1, 2, \cdots. \tag{14.2.13}$$

It follows from (14.2.11), (14.2.12) and (14.2.3) that

$$\begin{aligned} Eu_{nk}^2 &\le \sum_{i=1}^{n-1} (EX_{n+k}X_i)^2 \\ &\le C_3^2 \sum_{i=1}^{n-1} (n+k-i)^{-3-2\lambda} \le C_3^2(k+1)^{-2-2\lambda}. \end{aligned} \tag{14.2.14}$$

Thus we have

$$\|u_n\|_2 \le \sum_{k=0}^{\infty} \|u_{nk}\|_2 = O(1).$$

**Proof of Theorem 14.2.1.**

1) We first prove Theorem 14.2.1 under condition (14.2.10). By a well-known property of a Gaussian sequence (cf. Ibragimov and Rozanov 1978, p.14), $Y_n$, which is defined by (14.2.8), is a linear combination of $X_1, \cdots, X_n$, so that $\{Y_n, n \ge 1\}$ is also a Gaussian sequence. By (14.2.8) $\{Y_n, n \ge 1\}$ is a martingale difference sequence. Therefore $Y_n, n = 1, 2, \cdots$ are independent. Then we can construct a Wiener process $\{W(t), t \ge 0\}$ from $\{Y_n, n \ge 1\}$. Let $b_t = \sum_{k \le t} EY_k^2$. We redefine

$$Y_n = W(b_n) - W(b_{n-1}), \quad n = 1, 2, \cdots$$

and from (14.2.8) we have

$$\sum_{k \le n} X_k - \sum_{k \le n} Y_k = -u_{n+1}.$$

Note that $u_n, n = 1, 2, \cdots$, are normal with uniformly bounded variances, then we have

$$u_n = O(\log^{1/2} n) \qquad \text{a.s.}$$

i.e.

$$\sum_{k \le n} X_k - \sum_{k \le n} Y_k = O(\log^{1/2} n) \quad \text{a.s.}$$

or

$$S(t) - W(b_t) = O(\log^{1/2} t) \quad \text{a.s.} \tag{14.2.15}$$

Furthermore

$$\begin{aligned} b_n - a_n &= 2E\Big(\sum_{k=1}^{n} X_k u_{n+1}\Big) + Eu_{n+1}^2 \\ &= 2E\Big(\sum_{k=1}^{n}\sum_{j=1}^{\infty} X_k X_{n+j}\Big) + Eu_{n+1}^2 \\ &= O\Big(\sum_{k=1}^{n}\sum_{j=1}^{\infty}\gamma(n+j-k)\Big) = O\Big(\sum_{k=1}^{n} k^{-1/2-\lambda}\Big). \end{aligned}$$

Then Theorem 14.2.1 holds true under (14.2.10).

2) Now we prove Theorem 14.2.1 in general case. Put $l = [312C_3^2/C_1^2]$, define

$$X_m^* = \sum_{(m-1)l<k\le ml} X_k \quad m = 1, 2, \cdots, \quad S_t^* = \sum_{k\le t} X_k^*.$$

Then $\{X_n^*\}$ is also a Gaussian sequence satisfying (14.2.1), (14.2.2) and (14.2.3). It follows from (14.2.1) that

$$EX_n^{*2} \ge C_1 l. \tag{14.2.16}$$

We prove that $\{X_n^*\}$ satisfies (14.2.10). In fact

$$\begin{aligned} &\sum_{\substack{j=1\\ j\ne n}}^{\infty} |EX_n^* X_j^*| \\ &\quad= \sum_{\substack{j=1\\ j\ne n}}^{\infty}\Big|E\Big(\sum_{(n-1)l<k\le nl} X_k\Big)\Big(\sum_{(j-1)l<i\le jl} X_i\Big)\Big| \\ &\quad\le C_3 \sum_{\substack{j=1\\ j\ne n}}^{\infty}\ \sum_{(n-1)l<k\le nl}\ \sum_{(j-1)l<i\le jl} |i-k|^{-3/2} \\ &\quad\le 2C_3 \sum_{k=1}^{l}\sum_{m=0}^{\infty}(m+k)^{-3/2} \\ &\quad\le 6C_3 \sum_{k=1}^{l} k^{-1/2} \le 6C_3(1+2l^{1/2}). \end{aligned} \tag{14.2.17}$$

By the definition of $l$, it is easy to verify that the right hand side of (14.2.17) does not exceed $C_1 l - 1$, that is to say, $\{X_n^*\}$ satisfies (14.2.10).

For any fixed $n$, there exists an $m$ such that $(m-1)l \le n < ml$ and hence

$$|S_n - S_m^*| \le \max_{(m-1)l\le k<ml} |X_k|. \tag{14.2.18}$$

Note that $\{X_n, n \geq 1\}$ is a Gaussian sequence with uniformly bounded variances. Then we have

$$\max_{(m-1)l \leq k < ml} |X_k| = O(\log^{1/2} m) \quad \text{a.s.} \tag{14.2.19}$$

and

$$\begin{aligned} a_n - a_m^* &= E\Big(\sum_{k=1}^{n} X_k\Big)^2 - E\Big(\sum_{k=1}^{ml} X_k\Big)^2 \\ &= -2E\Big(\sum_{k=1}^{n} \sum_{j=n+1}^{ml} X_k X_j\Big) - E\Big(\sum_{j=n+1}^{ml} X_j\Big)^2 \\ &= O\Big(\sum_{k=1}^{\infty} \gamma(k)\Big) = O(1), \end{aligned} \tag{14.2.20}$$

where $a_m^* = E(S_m^*)^2$. It follows from 1), (14.2.18)–(14.2.20) that the proof of Theorem 14.2.1 is completed.

## 14.3 The non-negative additive functional of a Markov process

Let $\mathbf{X} = \{X_t, t \geq 0\}$ be a homogeneous, right continuous, strong-Feller Markov process, which is defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a complete, $\sigma$-compact measurable metric space $(E, \rho, \mathbf{B})$, satisfying the following conditions:

(i) for any $x \in E$, $t > 0$ and open set $U \in \mathbf{B}$, the stationary transition function of the Markov process $\mathbf{X}$

$$p(t, x, U) > 0; \tag{14.3.1}$$

(ii) for every $a \in E$ there exists a compact set $K$ such that

$$P_a\{X_t(\omega) \in K \text{ for some } t \geq 0\} = 1, \tag{14.3.2}$$

where $P_a(\cdot) = P(\cdot | X_0 = a)$.

From (i) and (ii) it follows that $\mathbf{X}$ is a recurrent strong Markov process.

For every $U \in \mathbf{B}$, put

$$\tau_U(\omega) = \begin{cases} \inf\{t, t \geq 0, X_t(\omega) \in U\}, & \text{if the set is non-empty,} \\ \infty & \text{otherwise.} \end{cases}$$

Define the exit distribution

$$h^U(a,S) = P_a\{X_{\tau_U}(\omega) \in S, \tau_U < \infty\} \quad \text{for } a \in E,\ S \in \mathbf{B}.$$

Denote

$$h^U f(x) = \int_{\overline{U}} h^U(x,dy) f(y) \qquad \text{for } f(x) \in B(\overline{U}),$$

where $B(\overline{U})$ is the set of all bounded measurable function on $\overline{U}$.

Let $\mathbf{H}= \{K, L\}$ be the collection of all set-pairs satisfying the following conditions:

$(H_1)$ $K$ and $L$ are closed subsets of $E$ and have an interior point at least,

$(H_2)$ $K$ and/or $L$ are compact,

$(H_3)$ (i) $K \subset E \backslash L$, $E\backslash L$ is a connected open set, or

(ii) $L \subset E\backslash K$, $E\backslash K$ is a connected open set.

For any given set-pair $(K, L) \in \mathbf{H}$, $x \in K$, put

$$T^K(x,U) = \int_E h^L(x,dy) h^K(y,U), \tag{14.3.3}$$

where $U \in \mathbf{B}$, $U \subset K$. By recurrence $T^K(x,U)$ is a one-step transition function. Define the transformation

$$T^K f(x) = \int_E T^K(x,dy) f(y) \tag{14.3.4}$$

for $x \in K$, $f \in B(K)$.

Without loss of generality, assume that $K$ is compact, and (i) of $(H_3)$ is satisfied. Then for $T^K$ there exists a unique invariant probability measure $\mu$ on $K$. Moreover, suppose that $\mathbf{X}$ satisfies

(iii) $T^K\{B(K)\} \subset C(K) := \{\text{all continuous function on } K\}$.

Let $\tau$ be a stopping time of the process $\mathbf{X}$, put $\Omega_\tau = \{\tau(\omega) < \infty\}$. Then

$$\mathcal{N}_\tau := \{A : A \subset \Omega_\tau, \forall t \geq 0, A \cap (\tau \leq t) \in \mathcal{N}_t\}$$

is a $\sigma$-field of $\Omega_\tau$, where $\mathcal{N}_t = \sigma\{X_s, 0 \leq s \leq t\}$. Let $\mathcal{N}^*$ be a $\sigma$-field of $\Omega$ which was introduced in Dynkin (1963 (3, 3.5)). Define the shift operators $\theta_t$ from $\mathcal{N}^*$ to $\mathcal{N}^*$ and the shift operator $\theta_\tau$ from $\mathcal{N}^*$ to $\Omega_\tau \mathcal{N}^*$ satisfying

$$\theta_\tau A = \bigcup_{t \geq 0} \{\theta_t A,\ \tau(\omega) = t\} \subset \Omega_\tau \qquad A \in \mathcal{N}^*, \tag{14.3.5}$$

where the operations of union, intersection and complement are keeped by the operator $\theta_\tau$, and we have

$$\theta_\tau\{X_t \in \Gamma\} = \{X_{t+\tau} \in \Gamma\} \qquad \Gamma \in \mathbf{B}. \tag{14.3.6}$$

For given $(K, L) \in \mathbf{H}$, we define random functions on $\Omega$ as follows:

$$\begin{aligned}
\tau_1 &= \tau_1(K, L, \omega) \\
&= \begin{cases} \inf\{t, t \geq 0, X_t(\omega) \in K\} & \text{if the set is non-empty,} \\ \infty & \text{otherwise;} \end{cases} \\
\sigma_n &= \sigma_n(K, L, \omega) \\
&= \begin{cases} \inf\{t, t > \tau_n, X_t(\omega) \in L\} & \text{if the set is non-empty,} \\ \infty & \text{otherwise;} \end{cases} \\
\tau_{n+1} &= \tau_{n+1}(K, L, \omega) \\
&= \begin{cases} \inf\{t, t > \sigma_n, X_t(\omega) \in K\} & \text{if the set is non-empty,} \\ \infty & \text{otherwise,} \end{cases}
\end{aligned}$$

for $n \geq 1$. By Doob (1953), $\tau_n$, $\sigma_n$ $(n \geq 1)$ are the almost sure finite stopping times of the process $\mathbf{X}$. Denote $\xi_n(\omega) = X_{\tau_n}(\omega)$. Its transition function $T^K(x, U)$ satisfies the Doeblin condition and there exists a unique ergodic set, non-periodic. Moreover $T^K(x, U)$ satisfies

$$|(T^K)^n(x, U) - \mu(U)| < C\delta^n, \tag{14.3.7}$$

where $0 < \delta < 1$, $C$ is a constant. Put

$$P_\mu(B) = \int_E P_a(B)\mu(da) \qquad B \in \mathcal{F}. \tag{14.3.8}$$

Then $P_\mu$ is a probability measure on $(\Omega, \mathcal{F})$, and $P_a(B) = 1$ implies $P_\mu(B) = 1$. Put

$$E_\mu f(\cdot) = \int_\Omega f(\cdot) P_\mu(d\omega).$$

At last, let $\phi = \{\phi_t^s(\omega)\}$ be a non-negative, strongly measurable, homogeneous additive functional on $\Omega$, i.e. $\{\phi_t^s, 0 \leq s \leq t\}$ is a family of real valued functions satisfying the following condtions:

($\phi_1$) For any $s \leq t$, $\phi_t^s(\omega)$ is non-negative $\overline{\mathcal{N}}_t$-measurable, where $\overline{\mathcal{N}}_t$ is the completion of $\mathcal{N}_t$ in probability space $(\Omega, \mathcal{F}, P)$,

($\phi_2$) for any $\omega \in \Omega$ and $s \leq t \leq u$, $\phi_u^s = \phi_t^s + \phi_u^t$,

($\phi_3$) for any $\omega \in \Omega$ and $h \geq 0, s \leq t$, $\theta_h \phi_t^s = \phi_{t+h}^{s+h}$,

($\phi_4$) for any $0 \leq u \leq v$, the bivariate function $\phi_t^0(\omega)$ of point $(t, \omega)$ is $\mathbf{B}_{[u,v]} \times \overline{\mathcal{N}}_v^u$-measurable on $[u, v] \times \Omega$, where $\mathbf{B}_{[u,v]}$ is the $\sigma$-field of interval $[u, v]$, $\overline{\mathcal{N}}_v^u$ is the completion of $\mathcal{N}_v^u = \sigma(X_t, u \leq t \leq v)$.

It is easy to check that

$$\theta_{\sigma_1} \tau_n = \tau_{n+1} - \sigma_1. \tag{14.3.9}$$

Denote

$$y_n = \phi_{\tau_{n+1}}^{\tau_n}, \quad z_n = \tau_{n+1} - \tau_n.$$

Then

$$\theta_{\sigma_1} y_n = y_{n+1}, \qquad \theta_{\tau_1} z_n = z_{n+1}. \tag{14.3.10}$$

**Lemma 14.3.1.** *$\{y_n, n \ge 1\}$ is a strictly stationary $\varphi$-mixing sequence of random variables with $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$.*

*Particularly, $\{z_n, n \ge 1\}$ and $\{w_n = y_n + cz_n, n \ge 1\}$ are also the strictly stationary $\varphi$-mixing sequences.*

**Proof.** In order to prove $\{y_n\}$ is a strictly stationary sequence, it needs only to show that $\theta_{\sigma_1}$ on $P_\mu$ is an operator preserving measure. Indeed, from (14.3.8) and Dynkin (1963 Theorem 3.11), for any $B \in \mathcal{N}_0 := \sigma\{X_t, t \ge 0\}$ we have

$$P_\mu(\theta_{\sigma_1} B) = \int_{\Omega_{\sigma_1}} P_{X_{\sigma_1}}(B) P_\mu(dx) = P_\mu(B).$$

In order to prove $\{y_n\}$ is $\varphi$-mixing, we first show the following two facts:

(i) $\sigma\{y_1, \cdots, y_{m-1}\} \subset \sigma\{\xi_1, \cdots, \xi_m\} \subset \mathcal{N}_{\tau_m}$,

(ii) $\sigma\{y_{m+k-1}, y_{m+k}, \cdots\} \subset \sigma\{\xi_{m+k}, \xi_{m+k+1}, \cdots\}$.

Since $\mathcal{N}_{\tau_{m-1}} \subset \mathcal{N}_{\tau_m}$ and $\xi_m = X_{\tau_m}$ is $\mathcal{N}_{\tau_m}$-measurable, $\xi_k$, $1 \le k \le m$ are also $\mathcal{N}_{\tau_m}$-measurable. Therefore

$$\sigma\{\xi_1, \cdots, \xi_m\} \subset \mathcal{N}_{\tau_m}.$$

Thus (i) holds true, if we can prove

$$\sigma\{y_n \in A,\ A \in \mathcal{B}_R\} \subset \sigma\{\xi_{n+1} \in \Gamma,\ \Gamma \in \mathbf{B}\} \tag{14.3.11}$$

where $\mathcal{B}_R$ is Borel $\sigma$-field of $R_+$. Put

$$\Lambda = \{A :\ (y_n \in A) \in \sigma\{\xi_{n+1} \in \Gamma, \Gamma \in \mathbf{B}\}, A \in \mathcal{B}_R\},$$

it is easy to verify that $\Lambda$ is a $\lambda$-system and contains a $\pi$-system

$$\Pi = \{[0,t] :\ (y_n \in [0,t)) \in \sigma\{\xi_{n+1} \in \Gamma, \Gamma \in \mathbf{B}\}\}.$$

Then $\sigma(\Pi) \subset \Lambda$ that implies (14.3.11). The proof of (ii) is similar.

Thus for any $A \in \sigma\{y_k, 1 \le k \le m-1\}$, $B \in \sigma\{y_k, k \ge m+n-1\}$, by the strong Markov property and (14.3.7) we have

$$\begin{aligned} P_\mu(AB) &= E_\mu(E_\mu(I(A)I(B)|\mathcal{N}_{\tau_m})) \\ &= E_\mu(I(A)E_\mu(I(B)|\mathcal{N}_{\tau_m})) = E_\mu(I(A)E_\mu(I(B)|\xi_m)) \end{aligned}$$

and

$$
\begin{aligned}
&|P_\mu(AB) - P_\mu(A)P_\mu(B)| \\
&\quad = |E_\mu(I(A)E_\mu(I(B)|\xi_m)) - E_\mu I(A)E_\mu I(B)| \\
&\quad = \Big|E_\mu\Big\{I(A)\int_E E_\mu(I(B)|\xi_{m+n} = \eta)\Big[(T^K)^n(\xi_m, d\eta) - \mu(d\eta)\Big]\Big\}\Big| \\
&\quad \le E_\mu\Big\{I(A)\int_E E_\mu(I(B)|\xi_{m+n} = \eta)V_n(\xi_m, d\eta)\Big\} \\
&\quad \le E_\mu I(A)V_n(\xi_m, E) \le P_\mu(A)C\delta^n, \qquad (14.3.12)
\end{aligned}
$$

where $V_n(\xi_m, A) = |(T^K)^n(\xi_m, A) - \mu(A)|$. Taking $\varphi(n) = C\delta^n (0 < \delta < 1)$ we obtain $\{y_n\}$ is a $\varphi$-mixing sequence with $\sum_{n=1}^\infty \varphi^{1/2}(n) < \infty$.

Particularly, letting $\phi_t^s = t - s$, we have $y_n = z_n$, so that the result of $\{z_n\}$ holds true. Moreover, from (i) and (ii)

$$
\begin{aligned}
&\sigma\{w_k, 1 \le k \le m-1\} \subset \sigma\{\xi_k, 1 \le k \le m\} \subset \mathcal{N}_{\tau_m}, \\
&\sigma\{w_k, k \ge m+n-1\} \subset \sigma\{\xi_k, k \ge m+n\}.
\end{aligned}
$$

$\{w_n\}$ is also a strictly stationary $\varphi$-mixing sequence. Lemma 14.3.1 is proved.

**Lemma 14.3.2.** *Let $\phi_t^s$ and $\psi_t^s$ be the non-negative, strongly measurable, homogeneous additive functionals of* $\mathbf{X}$ *with the finite $\alpha_\phi = E_\mu y_1$, $\alpha_\psi = E_\mu \psi_{\tau_2}^{\tau_1} \neq 0$. Then*

*(a)*

$$
P_a\Big\{\lim_{t\to\infty} \phi_t^0/\psi_t^0 = \alpha_\phi/\alpha_\psi\Big\} = P\Big\{\lim_{t\to\infty} \phi_t^0/\psi_t^0 = \alpha_\phi/\alpha_\psi\Big\} = 1 \qquad (14.3.13)
$$

*where*

$$
P(A) = \int_E P_a(A)P_0(da), \qquad (14.3.14)
$$

*$P_0$ is an initial distribution.*

*(b) Particularly, letting $l(t)$ to be a positive integer-valued random variable such that*

$$
\tau_{l(t)} \le t < \tau_{l(t)+1} \qquad (14.3.15)
$$

*and $\alpha_z = E_\mu z_n$, we have*

$$
P_a\Big\{\lim_{t\to\infty} \frac{t}{l(t)} = \alpha_z\Big\} = P\Big\{\lim_{t\to\infty} \frac{t}{l(t)} = \alpha_z\Big\} = 1, \qquad (14.3.16)
$$

$$
P_a\Big\{\lim_{t\to\infty} \frac{\tau_{l(t)}}{t} = 1\Big\} = P\Big\{\lim_{t\to\infty} \frac{\tau_{l(t)+1}}{t} = 1\Big\} = 1. \qquad (14.3.17)
$$

**Proof.** By the strict stationarity and $\varphi$-mixing property of $\{y_n\}$, $\{y_n\}$ is an ergodic sequence and its invariant $\sigma$-field $\mathcal{U}$ is trivial. By (14.3.8) for any $A \in \mathcal{U}$

$$P_a(A) = P_\mu(A) = 0 \text{ or } 1.$$

Hence we have

$$P_\mu\Big\{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} y_k = \alpha_\phi\Big\} = 1.$$

And since $\limsup \frac{1}{n}\sum_{k=1}^{n} y_k$ and $\liminf \frac{1}{n}\sum_{k=1}^{n} y_k$ are $\mathcal{U}$-measurable, we have

$$P_a\Big\{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} y_k = \alpha_\phi\Big\} = 1. \tag{14.3.18}$$

Obviously $l(t) \to \infty (t \to \infty)$. Write

$$\phi_t^0 = \phi_{\tau_1}^0 + \sum_{k=1}^{l(t)-1} y_k + \phi_t^{\tau_{l(t)}}. \tag{14.3.19}$$

Since $\tau_1 < \infty$ a.s., for any given $\varepsilon > 0$ and large $t$ we have

$$P_a\{\phi_{\tau_1}^0 / l(t) \geq \varepsilon\} = 0. \tag{14.3.20}$$

By the non-negativity of $\phi_t^s$, for large $t$ we also have

$$P_a\{\phi_t^{\tau_{l(t)}} / l(t) \geq \varepsilon\} = P_a\{\phi_{\tau_{l(t)+1}}^{\tau_{l(t)}} / l(t) \geq \varepsilon\} = 0. \tag{14.3.21}$$

Combining (14.3.18)–(14.3.21), we get

$$P_a\Big\{\lim_{t\to\infty} \phi_t^0 / l(t) = \alpha_\phi\Big\} = 1. \tag{14.3.22}$$

Similarly we have

$$P_a\Big\{\lim_{t\to\infty} \psi_t^0 / l(t) = \alpha_\psi\Big\} = 1. \tag{14.3.23}$$

Putting (14.3.22), (14.3.23) and (14.3.14) together yields (14.3.13).

Note that (14.3.16) is a special case of (14.3.22) with $\phi_t^s = t - s$. Moreover

$$\frac{\tau_{l(t)}}{t} = \frac{\tau_1}{t} + \frac{l(t)-1}{t} \cdot \frac{1}{l(t)-1} \sum_{k=1}^{l(t)-1} z_k.$$

From $\tau_1 < \infty$ a.s., (14.3.18) and (14.3.16), (14.3.17) holds true.

Now let us consider the process $S(t)$ generated by the additive functional $\phi_t^0$ as follows:

$$S(t) = \phi_t^0 - Mt.$$

Put

$$\psi_t^s = t - s, \quad M = \alpha_\phi / \alpha_\psi.$$

Lu (1986) proved the following theorem

**Theorem 14.3.1.** *Let $\phi_t^s$ be the non-negative, strongly measurable, homogeneous additive functional of $\mathbf{X}$, $w_n = y_n - Mz_n$. Suppose that for some $0 < \delta \le 2$,*

$$E_\mu y_n^{2+\delta} < \infty, \qquad E_\mu z_n^{2+\delta} < \infty.$$

*Then $E_\mu w_n^{2+\delta} < \infty$ and*

$$\sigma_w^2 = E_\mu w_1^2 + 2\sum_{k=2}^{\infty} E_\mu w_1 w_k$$

*converges absolutely. Without loss of generality assume that $\sigma_w^2/\alpha_\psi = 1$, then without changing the distribution of $S(t)$, we can redefine the process $\{S(t), t \ge 0\}$ on a richer probability space together with a Wiener process $\{W(t), t \ge 0\}$ such that for any given $\varepsilon > 0$ a.s.*

$$S(t) - W(t) = \begin{cases} O\left(t^{\frac{1}{2+\delta}+\varepsilon}\right) & \text{if } 0 < \delta < 2, \\ O\left(t^{\frac{1}{4}}(\log t)^{\frac{9}{4}+\varepsilon}\right) & \text{if } \delta = 2. \end{cases}$$

**Proof.** We can write

$$S(t) = \phi_t^0 - Mt = \phi_{\tau_1}^0 + \sum_{k=1}^{l(t)-1} w_k + \phi_t^{\tau_{l(t)}} - M(t - \tau_{l(t)} + \tau_1).$$

Denote $Z(t) = \sum_{k=1}^{l(t)-1} w_k$, we have

$$S(t) - Z(t) = \phi_{\tau_1}^0 + \phi_t^{\tau_{l(t)}} - M(t - \tau_{l(t)} + \tau_1). \tag{14.3.24}$$

We prove that the right hand side of (14.3.24) a.s. equals to $O((t \log\log t)^{1/(2+\delta)})$. By Lemma 14.3.1 $\{y_n\}$ is a strictly stationary $\varphi$-mixing sequence with $\varphi(n) = Ce^{-\theta n}$, $\theta > 0$. Moreover $\{y_n^{(2+\delta)/2}, n \ge 1\}$ is also a strictly stationary $\varphi$-mixing sequence with

$$E(y_n^{(2+\delta)/2})^2 = Ey_n^{2+\delta} < \infty.$$

Therefore by the law of iterated logarithm we have

$$\sum_{k=1}^{n}(y_k^{(2+\delta)/2} - Ey_k^{(2+\delta)/2}) = O((n \log\log n)^{1/2}) \quad \text{a.s.}$$

From the non-negativity of additive functional $\phi_t^s$ and Lemma 14.3.2 we obtain

$$\phi_t^{\tau_{l(t)}} \le y_{l(t)} = O((l(t)\log\log l(t))^{1/(2+\delta)}) = O((t\log\log t)^{1/(2+\delta)}) \quad \text{a.s.}$$

Similarly we also have

$$t - \tau_{l(t)} \le z_{l(t)} = O((t\log\log t)^{1/(2+\delta)}) \quad \text{a.s.} \tag{14.3.25}$$

Obviously $\phi_{\tau_0}^0 = O(1)$, $M\tau_1 = O(1)$. Thus we obtain

$$S(t) - Z(t) = O((t\log\log t)^{1/(2+\delta)}) \quad \text{a.s.}$$

For the strictly stationary $\varphi$-mixing sequence $\{w_n\}$, from Theorems 9.1.1 and 9.1.2 we have a.s.

$$\sum_{k=1}^{n} w_k - W(n\sigma_w^2) = \begin{cases} O\left(t^{\frac{1}{2+\delta}+\varepsilon}\right) & \text{if } 0 < \delta < 2 \\ O\left(t^{\frac{1}{4}}(\log t)^{\frac{9}{4}+\varepsilon}\right) & \text{if } \delta = 2. \end{cases} \tag{14.3.26}$$

Now we write

$$\begin{aligned} &Z(t) - W(t) \\ &= \Big(\sum_{k=1}^{l(t)-1} w_k - W((l(t)-1)\sigma_w^2)\Big) + \Big(W((l(t)-1)\sigma_w^2) - W(t)\Big) \\ &=: I_1 + I_2. \end{aligned} \tag{14.3.27}$$

From Lemma 14.3.2 and (14.3.26) we have a.s.

$$I_1 = \begin{cases} O\left(l(t)^{\frac{1}{2+\delta}+\varepsilon}\right) = O(t^{\frac{1}{2+\delta}+\varepsilon}) & \text{if } 0 < \delta < 2 \\ O\left(l(t)^{\frac{1}{4}}(\log l(t))^{\frac{9}{4}+\varepsilon}\right) = O(t^{\frac{1}{4}}(\log t)^{\frac{9}{4}+\varepsilon}) & \text{if } \delta = 2. \end{cases} \tag{14.3.28}$$

On the other hand, by the law of iterated logarithm for the strictly stationary $\varphi$-mixing sequence $\{z_n\}$ we have

$$\sum_{k=1}^{n} z_k - n\alpha_\psi = O((n\log\log n)^{1/2}) \quad \text{a.s.}$$

Hence

$$\tau_{l(t)} - l(t)\alpha_\psi = O((l(t)\log\log l(t))^{1/2}) = O((t\log\log t)^{1/2}) \quad \text{a.s.,}$$

so that

$$t - l(t)\alpha_\psi = t - l(t)\sigma_w^2 = O((t\log\log t)^{1/2}) \quad \text{a.s.}$$

By theorem 1.2.2 of Csörgő and Révész (1981) we have

$$I_2 = W((l(t)-1)\alpha_\psi) - W(t) = O((t\log\log t)^{1/4}) \quad \text{a.s.}$$

Combining it with (14.3.25)–(14.3.27) we have a.s.

$$S(t) - W(t) = \begin{cases} O\left(t^{\frac{1}{2+\delta}+\varepsilon}\right) & \text{if } 0 < \delta < 2 \\ O\left(t^{\frac{1}{4}}(\log t)^{\frac{9}{4}+\varepsilon}\right) & \text{if } \delta = 2. \end{cases}$$

Theorem 14.3.1 is proved.

**Remark 14.3.1.** From Theorem 14.3.1 we can give a weak invariance principle and a law of iterated logarithm for the additive functional of a Markov process.

# Appendix Slowly Varying Function

**Definition A1.** A positive and measurable function $R(x)$ on $[A, \infty)$ for some $A \geq 0$ is called *regularly varying* at infinite point with an exponent $\alpha$, if for any $a > 0$

$$\lim_{x \to \infty} R(ax)/R(x) = a^{\alpha}. \tag{A1}$$

Rewrite a regularly varying function $R(x)$ with an exponent $\alpha$ as

$$R(x) = x^{\alpha} L(x). \tag{A2}$$

Then, by (A1), we have

$$\lim_{x \to \infty} L(ax)/L(x) = 1.$$

**Definition A2.** A regularly varying function $L(x)$ with the exponent $\alpha = 0$ is called a *slowly varying function.*

We shall list some main propositions on a slowly varying function. For their proof, we refer to Seneta (1976) and Ibragimov and Linnik (1971).

There are Karamata's representation theorem for a slowly varying function.

**Theorem A1.** *Let $L(x)$ be a slowly varying function defined on $[A, \infty)$, $A$ 0. Then there exists a positive $B \geq A$ such that for any $x \geq B$*

$$L(x) = \exp\Big\{\eta(x) + \int_B^x \frac{\varepsilon(t)}{t}\, dt\Big\},$$

*where $\eta(x)$ is a bounded measurable function on $[B, \infty)$ with $\eta(x) \to c(|c| < \infty)$ as $x \to \infty$ and $\varepsilon(x)$ is a continuous function on $[B, \infty)$ with $\varepsilon(x) \to 0$ as $x \to \infty$.*

Using this representation theorem, we can derive many useful properties. In the sequel, we always assume that $L(x)$, $L_1(x)$, $L_2(x)$ are slowly varying functions.

**Property A1.** For any $a \geq 0$.

$$\lim_{x \to \infty} L(x+a)/L(x) = 1.$$

**Property A2.** For any $\varepsilon > 0$,

$$\lim_{x \to \infty} x^{\varepsilon} L(x) = \infty, \quad \lim_{x \to \infty} x^{-\varepsilon} L(x) = 0.$$

**Property A3.** $\sup_{2^k \leq t \leq 2^{k+1}} L(t)/L(2^k) \to 1$, as $k \to \infty$.

**Property A4.** Let $a = a(x) \to 0$ as $x \to \infty$. Then for any $\varepsilon > 0$,

$$\lim_{x \to \infty} a^{\varepsilon} \frac{L(ax)}{L(x)} = \lim_{x \to \infty} a^{\varepsilon} \frac{L(x)}{L(ax)} = 0.$$

**Property A5.** $(\log L(x))/\log x \to 0$ as $x \to \infty$.

**Property A6.** For any real number $\alpha$, $L^{\alpha}(x)$, $L_1(x)L_2(x)$ and $L_1(x)+L_2(x)$ all are slowly varying. Moreover, if $L_2(x) \to \infty$ as $x \to \infty$, then $L_1(L_2(x))$ also is slowly varying.

**Property A7.** Define $\overline{L}(x)$ and $\underline{L}(x)$ by

$$x^{\gamma}\overline{L}(x) = \sup_{B \leq t \leq x} t^{\gamma} L(t),$$
$$x^{\gamma}\underline{L}(x) = \inf_{x \leq t} t^{\gamma} L(t),$$

where $\gamma > 0$ is an arbitrary constant. Then $\overline{L} \sim L$ and $\underline{L} \sim L$.

**Remark A1.** As a consequence of Property A7, $x^{\gamma}L(x)$ is equal asymptotically to a non-decreasing regularly varying function with the same exponent $\gamma$.

**Property A8.** For $R_1(x) = x^{\gamma}L_1(x)$, $\gamma > 0$, there exists a regularly varying function $R_2(x) = x^{1/\gamma}L_2(x)$ such that

$$R_1(R_2(x)) \sim x, \; R_2(R_1(x)) \sim x \quad \text{as } x \to \infty.$$

$R_2(x)$ here is defined asymptotically uniquely, i.e. if the above relations hold true with $R_3$ instead of $R_2$ and $R_3(x) \to \infty$ as $x \to \infty$, then $R_3(x) \sim x^{1/\gamma}L_2(x)$.

**Property A9.** Let $L(x)$ is a positive slowly varying function on $[A, \infty)$. Assume that $R(x) = x^\gamma L(x)$ is non-decreasing on $[A, \infty]$ for some $\gamma > 0$. For $x \geq R(A)$ let

$$R^*(x) = \inf\{y : y \in [A, \infty), R(y) \geq x\}.$$

Then $R^*(x) = x^{1/\gamma} L^*(x)$, where $L^*(x)$ is a slowly varying function and $R^*(x)$ is an inverse function of $R(x)$ with the meaning in Property A8.

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